Risk-averse Selfish Routing

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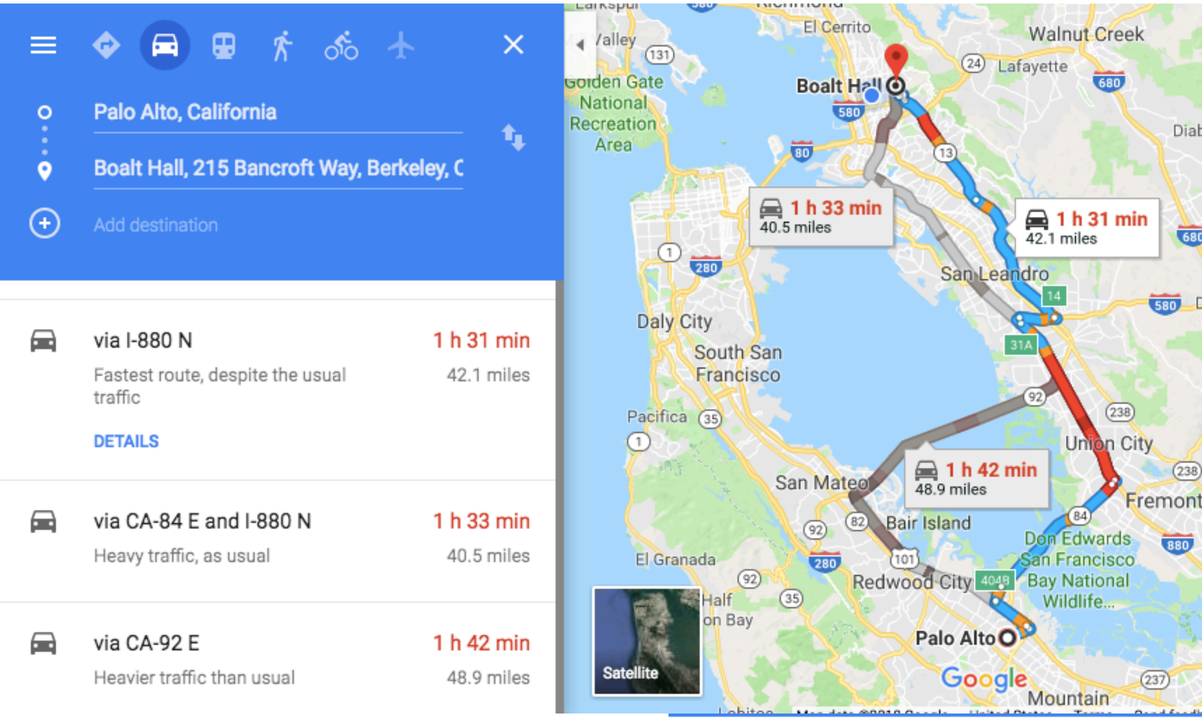
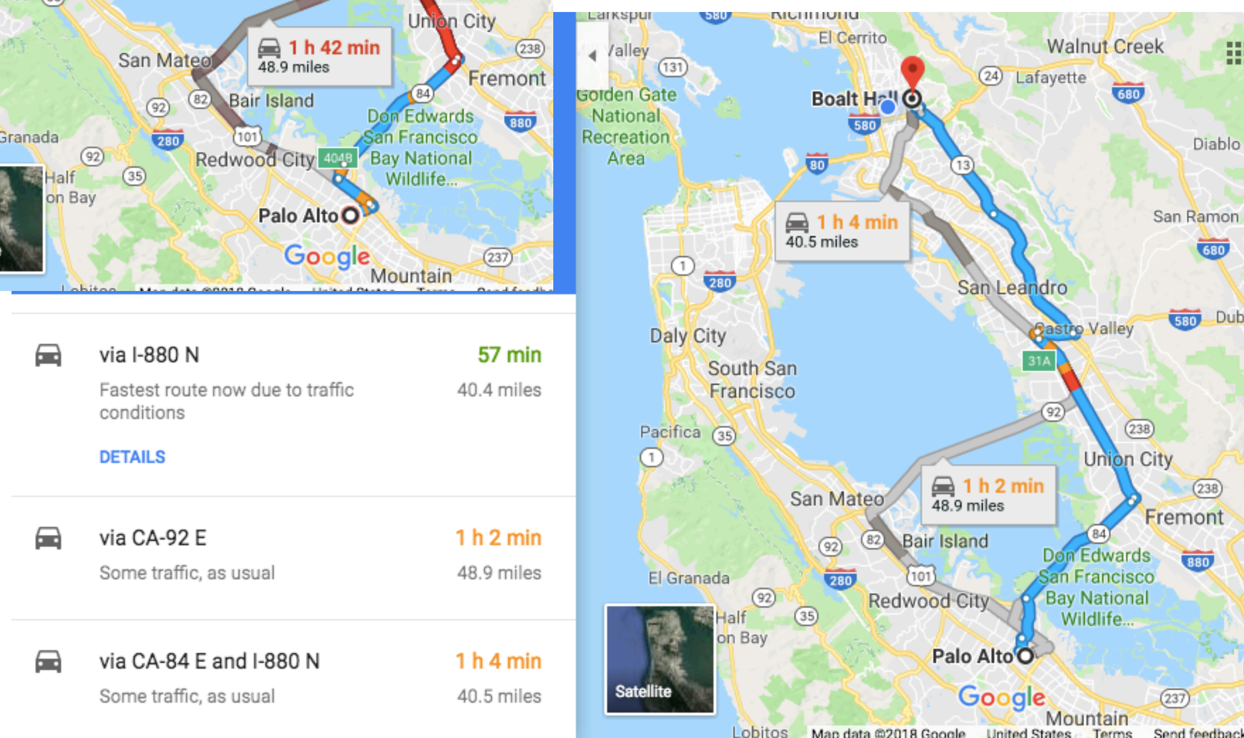
Traffic conditions are uncertain

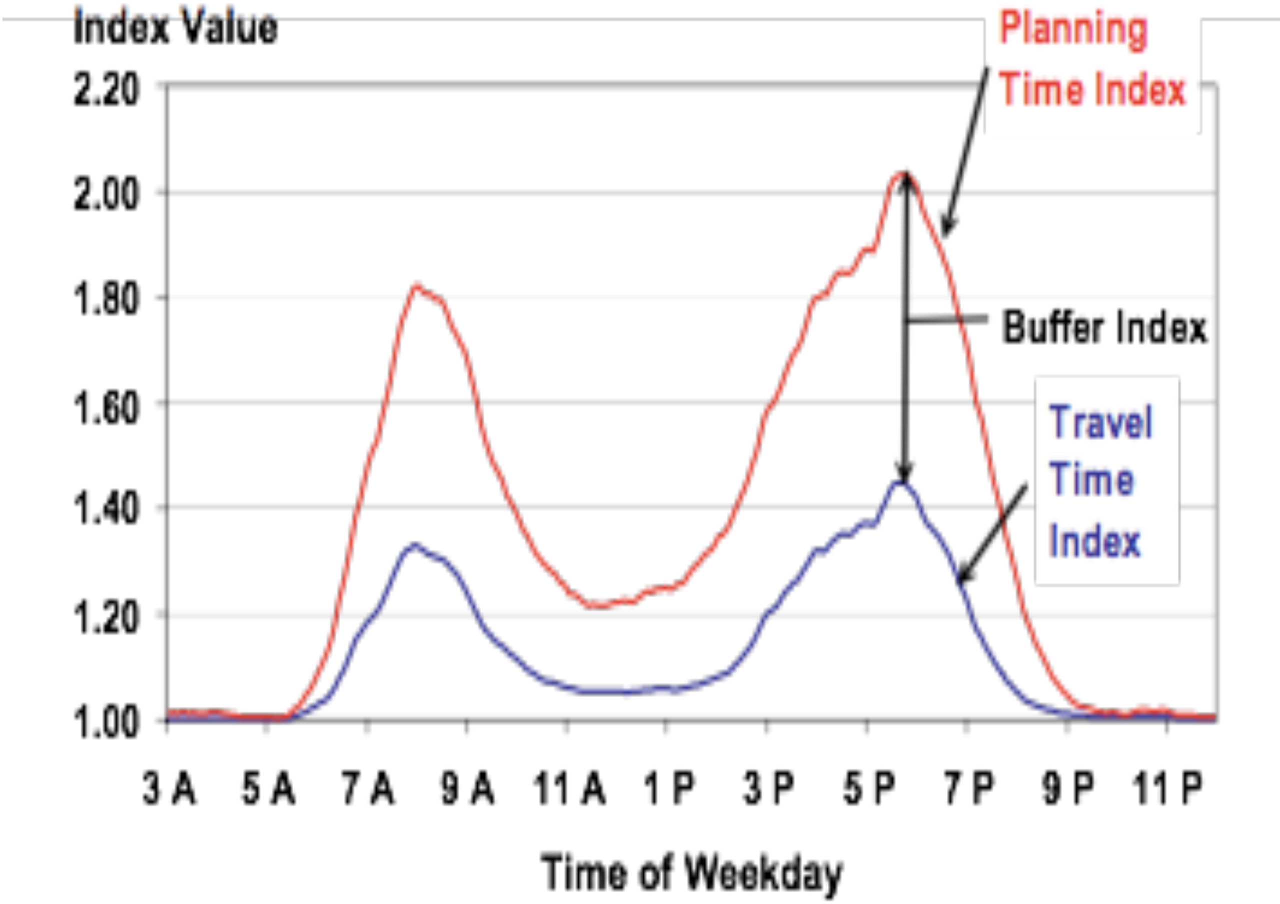




Evdokia Nikolova

Commuters pad travel times

Worst case > twice free flow time



Source: Texas Transportation Institute; ABC News Survey.

Goal

Understand effect of **risk-aversion** on congestion, by studying resulting traffic assignment:

- Uncertain travel times influence users' decisions
- Equilibrium existence, encoding, efficiency*
- Price of Risk Aversion**

* E. Nikolova, N. Stier-Moses. *SAGT 2011 / Operations Research, 2014*

** T. Lianeas, E. Nikolova, N. Stier-Moses. *Math of OR, forthcoming*

Understanding traffic congestion

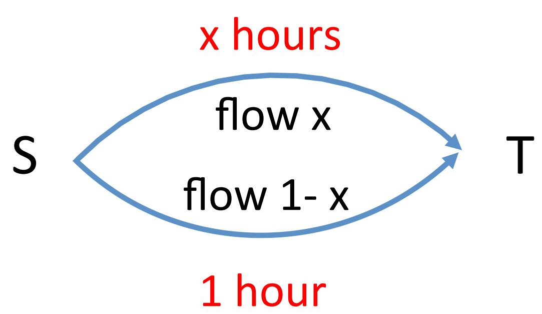
- Price of Anarchy [Koutsoupias, Papadimitriou ’99] measures the degradation of system performance due to free will (selfish behavior)

$$\sup_{\substack{\text{problem}\\\text{instances}}} \frac{\text{Equilibrium Cost}}{\text{Social Optimum Cost}}$$

- 4/3 in general graphs, linear travel times as function of traffic; 2 for quartic travel times (Roughgarden, Tardos ’02; Correa, Schulz, Stier-Moses ‘04, ‘08)

Price of anarchy = 4/3

- Example: One unit of traffic (flow) from S to T



- Equilibrium: Route all flow on top; cost 1 hour
- Social optimum: Route flow ½ on each link; cost ¾ hour
- Price of anarchy: (Equil. Cost/ Optimum Cost) = 4/3

Risk sensitivity of price of anarchy

- Routing games with uncertain delays resulting from "uniform schedulers"
- Price of anarchy of linear congestion games under risk attitudes:

– Wald's minimax cost	2
– Savage's minimax regret	[4/3, 1]
– Minimizing Expected cost	5/3
– Average case analysis	5/3
– Win-or-Go-Home	unbounded
– Second moment method	unbounded

- **Conclusion: Risk critically affects predictions of system performance**

* G. Piliouras, E. Nikolova, J. Shamma. *EC 2013 / ACM Transactions on Economics and Computation 2016*

Related Work

- Routing Games: Wardrop'52, Beckmann et al. '56, … Surveys in Nisan et al. '07, Correa & Stier-Moses'11
- Stochastic Equilibrium models: Dial '71, Gupta-Stahl-Whinston'97
- Risk-aversion in routing games: a few references in transportation (but not too many), Ordóñez & Stier-Moses'10, Nie'11, Angelidakis-Fotakis-Lianeas'13, Cominetti-Torico'13, Meir-Parkes'15, Kleer-Schäfer'16-'17.

Routing games with stochastic delays

- Directed graph $G = (V,E)$
  Unit demand between source-dest. pair (s,t)
- Nonatomic players (*flow model)* choose feasible s-t paths
  Players' decisions: flow vector $x \in R^{|Paths|}$
- Edge delay functions: $l_e(x_e) + \xi_e(x_e)$

E. Nikolova, N. Stier-Moses. *SAGT 2011 / Operations Research, 2014*

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- Players minimize risk-averse path cost:
  - Mean-stdev $Q_{path}(x) = \sum_{e \in path} l_e(x_e) + r\sqrt{\sum_{e \in path} \sigma_e(x_e)^2}$
  - Mean-var $Q_{path}(x) = \sum_{e \in path} l_e(x_e) + r\sum_{e \in path} \sigma_e(x_e)^2 = \sum_{e \in path} \left(l_e(x_e) + r\sigma_e(x_e)^2\right)$

E. Nikolova, N. Stier-Moses. *SAGT 2011 / Operations Research, 2014*

Risk-averse vs Risk-neutral Equilibrium

- Users select minimum-risk path with risk $Q_{path}(x)$

- Definition: A flow x is at equilibrium if for every source-destination pair k and for every *path* with positive flow

$$Q_{path}(x) \le Q_{path'}(x), \quad \text{for every} \quad path'$$

- We call it a *Risk-Averse Wardrop Equilibrium (RAWE)* if Q is the mean-variance or mean-stdev cost of a path
- We call it a *Risk-Neutral Wardrop Equilibrium (RNWE)* if Q is the mean cost of a path

Equilibrium characterization for mean-stdev risk

Equilibrium characterization	Uncertainty independent of flow (σ constant)	Uncertainty depending on flow (σ depends on flow)
Non-atomic model	Eq. **exists** It solves a **convex program** (exponentially large)	Eq. **exists** It solves **variational ineq.** (also exponent. large)
Atomic model	Eq. **exists** Game is **potential**	**No equilibrium**! (in pure strategies)

E. Nikolova, N. Stier-Moses. *SAGT 2011 / Operations Research, 2014*

Are Risk-Averse Equilibria Efficient?

- **POA**: Impact of selfish behavior by comparing equilibrium to social optimum flow (flow minimizing total user cost)

Theorem*: POA with risk aversion = **POA** in classic routing games when uncertainty does not depend on flow.

- Problem: selfish behavior and risk aversion coupled together. Not clear which causes the inefficiency
- Decouple effects of selfishness and risk by comparing to the risk-neutral equilibrium

* E. Nikolova, N. Stier-Moses. *SAGT 2011 / Operations Research, 2014*

Price of Risk Aversion

Cost of Flow $C(x)$: although users are risk-averse, central planner is risk-neutral.

- Consider the sum of *expected travel times*

Price of Risk Aversion (**PRA**): captures inefficiency introduced by user risk-aversion by comparing with the risk-neutral case

$$\sup_{\substack{\text{problem} \\ \text{instances}}} \frac{C(x^r)}{C(x^0)}$$

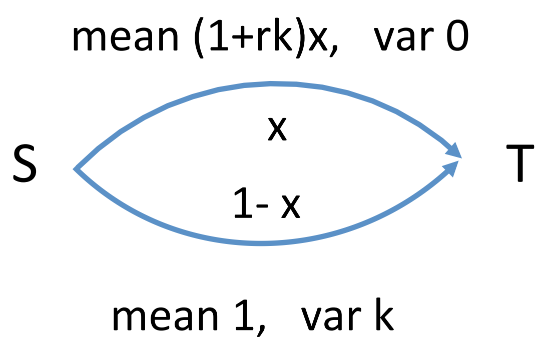
Risk-averse equilibrium ($C(x^r)$)

Risk-neutral equilibrium ($C(x^0)$)

T. Lianeas, E. Nikolova, N. Stier-Moses. *Math of OR, forthcoming*

Risk-averse vs Risk-neutral equilibria

- Example: Send one unit of flow from S to T



- Risk-averse eq.: Route all flow on top; cost (1+rk)
- Risk-neutral eq.: Route flow on both links; cost 1
- Price of risk aversion: (1+rk)

Price of Risk Aversion (PRA)

- Price of Risk Aversion (PRA) is unbounded in general, but uncertainty is not arbitrary in real world
- Consider a bounded variance-to-mean ratio:

$$\sigma_e^2(x_e)/l_e(x_e) \le k$$

- GOAL: Compute **PRA** for fixed k
  - As function of topology, for general edge delays
  - As function of edge delays, for general topologies

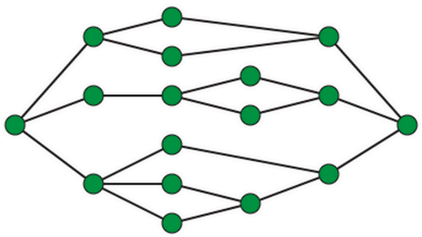
Price of Risk Aversion: Upper Bound for Arbitrary Latency Functions

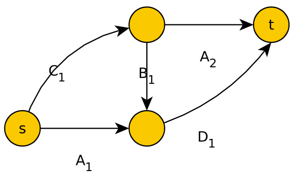
Theorem: In a general graph, **PRA** ≤ 1+ηrk

- Here, η is a graph topology parameter:
  # forward subpaths in an alternating path [η ≤ ½|V|]

Intuition:

- For 2-link networks: **PRA** ≤ 1+1rk
- For series-parallel networks: **PRA** ≤ 1+1rk
- For Braess networks: **PRA** ≤ 1+2rk





T. Lianeas, E. Nikolova, N. Stier-Moses. *Math of OR, forthcoming*

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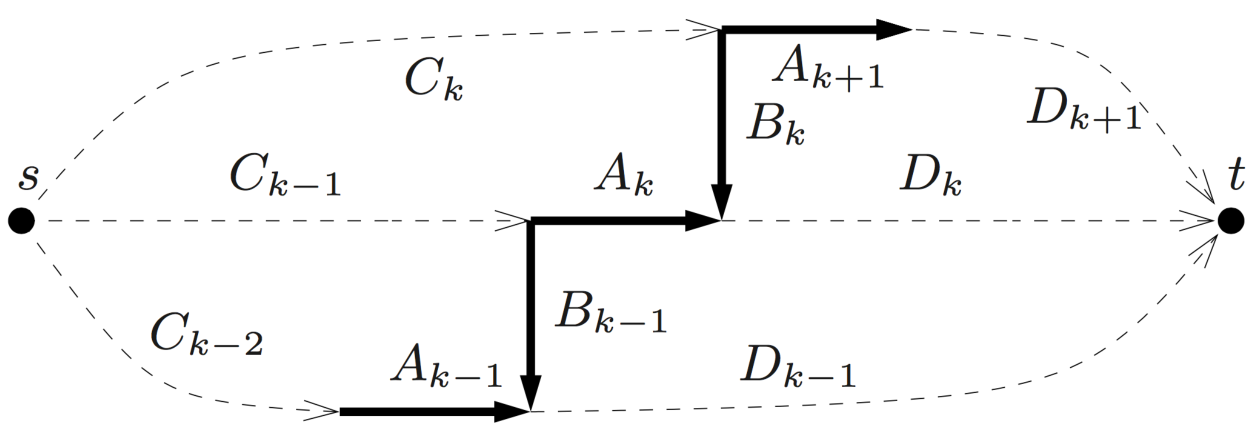
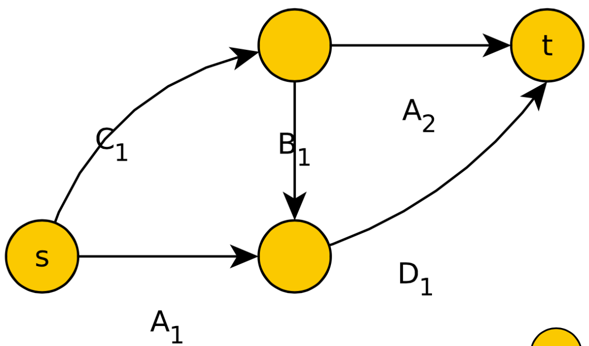
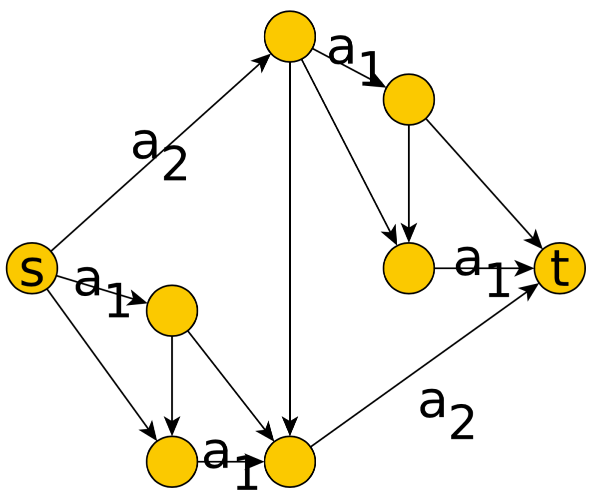
Proof idea: Compare equilibria on an alternating path: forward edges have higher risk-neutral equilibrium flow, and backward edges have higher risk-averse equilibrium flow.

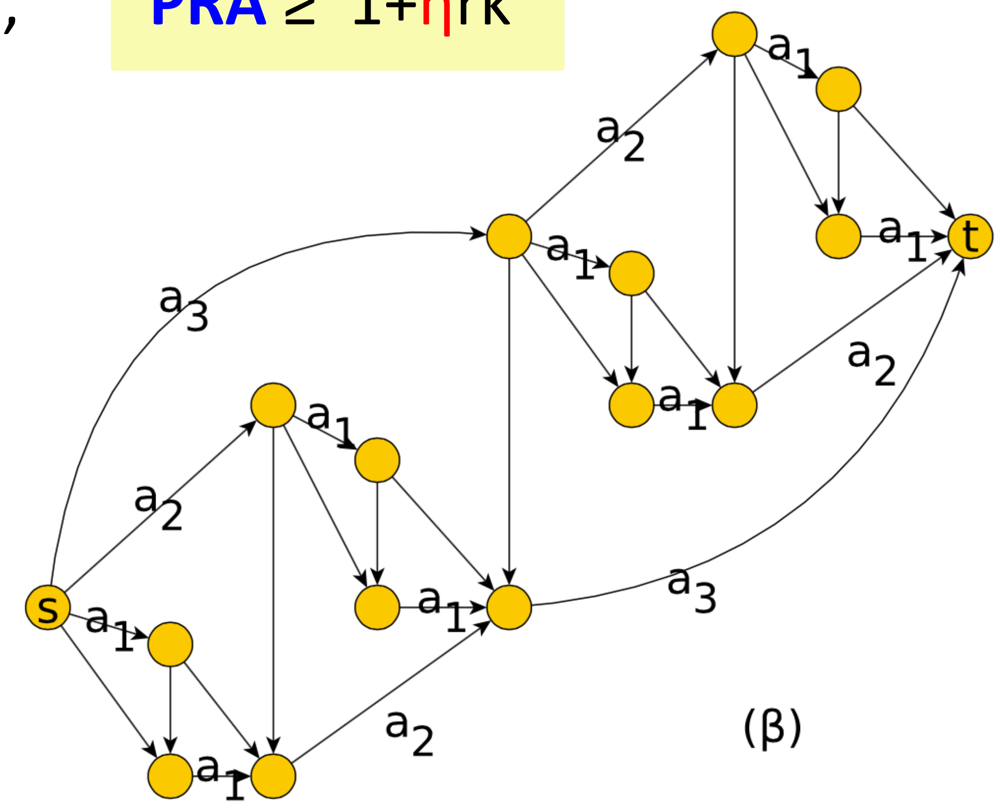


Price of Risk Aversion: Lower Bound for Arbitrary Latency Functions

Theorem: In a general graph, **PRA** ≥ 1+ηrk







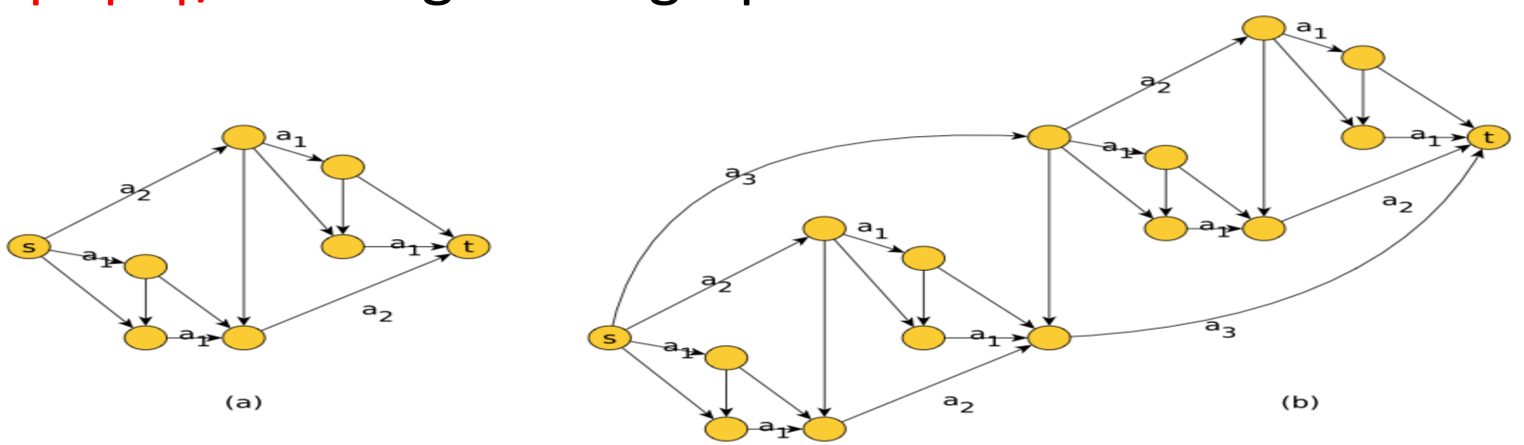
(β)

Price of Risk Aversion

- In graphs with general mean, variance functions where users minimize (mean + r*variance):

Cost(Risk-averse eq.) $\leq (1+\eta r k)$ Cost(Risk-neutral eq.)

- $\eta=1$ for series-parallel graphs, $\eta=2$ for Braess graph, $\eta\leq |V|/2$ for a general graph



T. Lianeas, E. Nikolova, N. Stier-Moses. *Mathematics of Operations Research, forthcoming*

Price of Risk Aversion

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- Alternative bound with respect to latency functions:

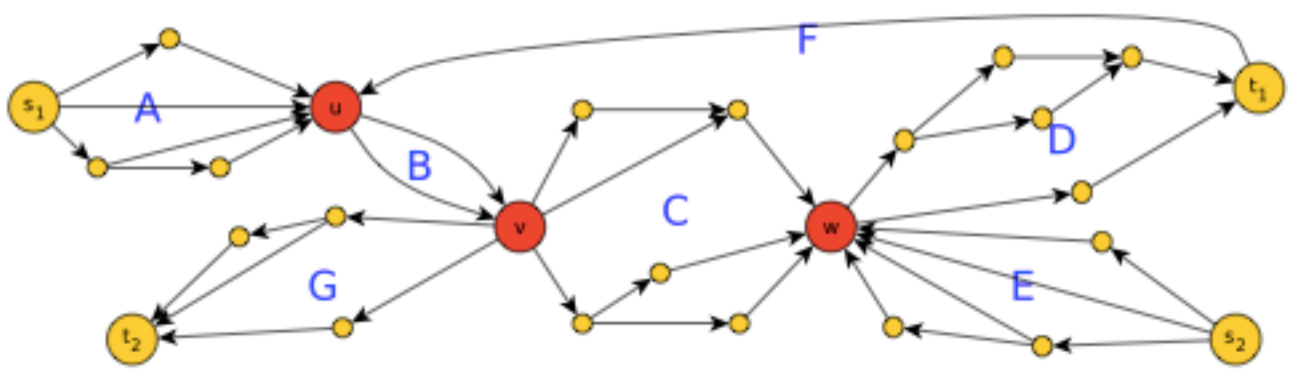
Cost(Risk-averse eq.) ≤ (1+rk) POA Cost(Risk-neutral eq.)

- Open: extend to other risk attitudes.

T. Lianeas, E. Nikolova, N. Stier-Moses. *Mathematics of Operations Research, forthcoming*

Heterogeneous players

- Does heterogeneity (diversity) of users reduce the cost of equilibrium? Users min (delay + α_i cost)
- Diversity helps if and only if the network is series-parallel for single origin-destination.
- Diversity helps if and only if the network is "block-matched" for multiple origin-destination pairs.



R. Cole, T. Lianeas, E. Nikolova, 2017. https://arxiv.org/abs/1702.07806

Summary

- Goal: Develop toolkit of algorithms and game theory techniques for risk mitigation in networks

- Lots of open problems in
  - Algorithms (static, dynamic, online, etc)
  - Algorithmic Game Theory (static, dynamic games, learning)
  - Algorithmic Mechanism Design (what are optimal/simple mechanisms with risk-averse or risk-loving agents?)

- Opportunities for impact in transportation, communications, smart-grid, evacuation from natural disasters, etc.