Distributionally Robust Stochastic and Online Optimization

Models/Algorithms for Data-Driven Optimization

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(Joint work with many others ...)

Outline

- Introduction to Distributionally Robust Optimization
- DRO under Moment and Likelihood Bounds
- Price of Correlation of High-Dimension Uncertainty
- Online Linear Optimization and Dynamic Learning

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Develop tractable and provable models and algorithms for optimization with uncertain and online data.

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Introduction to DRO

We start from considering a stochastic optimization problem as follows:

$$\text{maximize}_{\mathbf{x}\in X} \quad \mathbb{E}_{F_\xi}[h(\mathbf{x},\xi)] \tag{1}$$

where $\mathbf{x}$ is the decision variable with feasible region X, ξ represents random variables satisfying joint distribution F_ξ.

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where $\mathbf{x}$ is the decision variable with feasible region X, ξ represents random variables satisfying joint distribution F_ξ.

- Pros: In many cases, the expected value is a good measure of performance
- Cons: One has to know the exact distribution of ξ to perform the stochastic optimization. Deviant from the assumed distribution may result in sub-optimal solutions

Robust Optimization

In order to overcome the lack of knowledge on the distribution, people proposed the following (static) robust optimization approach:

$$\text{maximize}_{\mathbf{x}\in X} \quad \min_{\xi\in\Xi} h(\mathbf{x},\xi) \tag{2}$$

where Ξ is the support of ξ.

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- Pros: Robust to any distribution; only the support of the parameters are needed.
- Cons: Too conservative. The decision that maximizes the worst-case pay-off may perform badly in usual cases; e.g., Ben-Tal and Nemirovski [1998, 2000], etc.

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- In practice, although the exact distribution of the random variables may not be known, people usually know certain observed samples or training data and other statistical information.

- Thus we could choose an intermediate approach between stochastic optimization, which has no robustness in the error of distribution; and the robust optimization, which admits vast unrealistic single-point distribution on the support set of random variables.

Distributionally Robust Optimization

A solution to the above-mentioned question is to take the following Distributionally Robust Optimization (DRO) model:

$$\text{maximize}_{\mathbf{x}\in X} \quad \min_{F_\xi \in \mathcal{D}} \mathbb{E}_{F_\xi}[h(\mathbf{x}, \xi)] \tag{3}$$

In DRO, we consider a set of distributions $\mathcal{D}$ and choose one to maximize the expected value for any given $\mathbf{x} \in X$.

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When choosing $\mathcal{D}$, we need to consider the following:

- Tractability
- Practical (Statistical) Meanings
- Performance (the potential loss comparing to the benchmark cases)

Sample History of DRO

- First introduced by Scarf [1958] in the context of inventory control problem with a single random demand variable.
- Distribution set based on moments: Dupacova [1987], Prekopa [1995], Bertsimas and Popescu [2005], Delage and Y [2007], etc
- Distribution set based on Likelihood/Divergences: Nilim and El Ghaoui [2005], Iyanger [2005], Wang, Glynn and Y [2012], etc
- Distribution set based on Wasserstein ambiguity set: Mohajerin Esfahani and Kuhn [2015], Blanchet, Kang, Murthy [2016], Duchi and Namkoong [2016]
- Axiomatic motivation for DRO: Delage et al. [2017]; Ambiguous Joint Chance Constraints Under Mean and Dispersion Information: Hanasusanto et al. [2017]
- Lagoa and Barmish [2002] and Shapiro [2006] simply considers a set containing unimodal distributions, Kleinberg et al. [1997] and M$'$ohring et al. [1999] considers the product distribution

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DRO with Moment Bounds

Define

$$\mathcal{D} = \left\{ F_\xi \;\middle|\; \begin{array}{l} P(\xi \in \Xi) = 1 \\ (\mathbb{E}[\xi] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \preceq \gamma_2 \Sigma_0 \end{array} \right\}$$

That is, the distribution set is defined based on the support, first and second order moments constraints.

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That is, the distribution set is defined based on the support, first and second order moments constraints.

Theorem

Under mild technical conditions, the DRO model can be solved to any precision ϵ in time polynomial in $\log(1/\epsilon)$ and the sizes of $\mathbf{x}$ and ξ

Delage and Y [2010]

Confidence Region on F_ξ

Does the construction of $\mathcal{D}$ make a statistical sense?

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Consider

$$D(\gamma_1, \gamma_2) = \left\{ F_\xi \;\middle|\; \begin{array}{l} P(\xi \in \Xi) = 1 \\ (\mathbb{E}[\xi] - \mu_0)^T \Sigma_0^{-1} (\mathbb{E}[\xi] - \mu_0) \leq \gamma_1 \\ \mathbb{E}[(\xi - \mu_0)(\xi - \mu_0)^T] \preceq \gamma_2 \Sigma_0 \end{array} \right\}$$

where μ_0 and Σ_0 are point estimates from the empirical data (of size m) and Ξ lies in a ball of radius R such that $||\xi||_2 \leq R$ a.s..

Then for $\gamma_1 = O(\frac{R^2}{m} \log(4/\delta))$ and $\gamma_2 = O(\frac{R^2}{\sqrt{m}} \sqrt{\log(4/\delta)})$,

$$P(F_\xi \in D(\gamma_1, \gamma_2)) \geq 1 - \delta$$

DRO with Likelihood Bounds

Define the distribution set by the constraint on the likelihood ratio. With observed Data: $\xi_1, \xi_2, ...\xi_N$, we define

$$\mathcal{D}_N = \left\{ F_\xi \middle| \begin{array}{l} P(\xi \in \Xi) = 1 \\ L(\xi, F_\xi) \geq \gamma \end{array} \right\}$$

where γ adjusts the level of robustness and N represents the sample size.

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where γ adjusts the level of robustness and N represents the sample size.

For example, assume the support of the uncertainty is finite

$$\xi_1, \xi_2, ...\xi_n$$

and we observed m_i samples on ξ_i. Then, F_ξ has a finite discrete distribution $p_1, ..., p_n$ and

$$L(\xi, F_\xi) = \sum_{i=1}^{n} m_i \log p_i.$$

Theory on Likelihood Bounds

The model is a convex optimization problem, and connects to many statistical theories:

- Statistical Divergence theory: provide a bound on KL divergence
- Bayesian Statistics with the threshold γ estimated by samples: confidence level on the true distribution
- Non-parametric Empirical Likelihood theory: inference based on empirical likelihood by Owen
- Asymptotic Theory of the likelihood region
- Possible extensions to deal with Continuous Case

Wang, Glynn and Y [2012,2016]

DRO using Wasserstein Ambiguity Set

By the Kantorovich-Rubinstein theorem, the Wasserstein distance between two distributions can be expressed as the minimum cost of moving one to the other, which is a semi-infinite transportation LP.

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Theorem

When using the Wasserstein ambiguity set

$$\mathcal{D}_N := \{F_\xi \mid P(\xi \in \Xi) = 1 \ \& \ d(F_\xi, \hat{F}_N) \leq \varepsilon_N\},$$

where $d(F_1, F_2)$ is the Wasserstein distance function and N is the sample size, the DRO model satisfies the following properties:

- *Finite sample guarantee : the correctness probability $\bar{P}^N$ is high*
- *Asymptotic guarantee : $\bar{P}^\infty(\lim_{N\to\infty} \hat{x}_{\varepsilon_N} = x^*) = 1$*
- *Tractability : DRO is in the same complexity class as SAA*

Mohajerin Esfahani & Kuhn [15, 17], Blanchet, Kang, Murthy [16], Duchi and Namkoong [16]

DRO for Logistic Regression

- Let $\{(\hat{\xi}_i, \hat{\lambda}_i)\}_{i=1}^N$ be a feature-label training set i.i.d. from P, and consider applying logistic regression :

$$\min_x \frac{1}{N} \sum_{i=1}^{N} \ell(x, \hat{\xi}_i, \hat{\lambda}_i) \text{ where } \ell(x, \xi, \lambda) = \ln(1 + \exp(-\lambda x^T \xi))$$

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- DRO suggests solving

$$\min_x \sup_{F \in \mathcal{D}_N} \mathbb{E}_F[\ell(x, \xi_i, \lambda_i)]$$

with the Wasserstein ambiguity set.

- When labels are considered to be error free, DRO with $\mathcal{D}_N$ reduces to regularized logistic regression:

$$\min_x \frac{1}{N} \sum_{i=1}^{N} \ell(x, \hat{\xi}_i, \hat{\lambda}_i) + \varepsilon \|x\|_*$$

Shafieezadeh Abadeh, Mohajerin Esfahani, & Kuhn, NIPS, [2015]

Summary of DRO under Moment, Likelihood or Wasserstein Ambiguity Set

- Therefore, the DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.

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- Therefore, the DRO models yield a solution with a guaranteed confidence level to the possible distributions. Specifically, the confidence region of the distributions can be constructed upon the historical data and sample distributions.
- The DRO models are tractable, and sometimes maintain the same computational complexity as the stochastic optimization models with known distribution.
- This approach can be applied to a wide range of problems, including inventory problems (e.g., newsvendor problem), portfolio selection problems, image reconstruction, machine learning, etc., with reported superior numerical results

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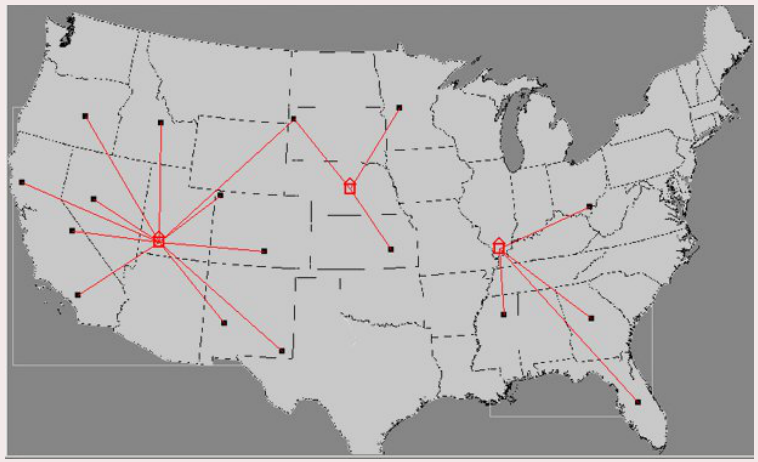
Planning under High-Dimensional Stochastic Data

Name	Symbol	Shares	Cost basis	Mkt value	Gain	Gain %
Apple Inc.	AAPL	1	122.4	86.29	-36.11	-29.5
Amazon.com, Inc.	AMZN	2	128.18	101.52	-26.66	-20.8
eBay Inc.	EBAY	1.25	32.23	17.45	-14.78	-45.85
Microsoft Corporation	MSFT	3.02	84.17	58.41	-25.76	-30.61
Google Inc.	GOOG	4	1733.4	1212.44	-521	-30.05
Yahoo! Inc.	YHOO	2	58.06	23.94	-34.12	-58.77
News Corporation	NWS	5	93.5	46.85	-46.65	-49.89
Time Warner Inc.	TWX	4	59.44	40.28	-19.16	-32.23
The Walt Disney Company	DIS	3	92.28	67.44	-24.84	-26.92
Bank of America Corporation	BAC	1	36.74	13.24	-23.5	-63.96
Best Buy Co., Inc.	BBY	1	39.87	27.75	-12.12	-30.4
Comcast Corporation	CMCSK	1	19.41	15.79	-3.62	-18.65
AT&T Inc.	T	2	70.02	56.46	-13.56	-19.37
Netflix, Inc.	NFLX	1	31.26	28.66	-2.6	-8.32
JPMorgan Chase & Co.	JPM	1	37.56	31.01	-6.55	-17.44
Dell Inc.	DELL	1	19.35	10.23	-9.12	-47.13
DivX, Inc.	DIVX	1	9.38	5.1	-4.28	-45.63
Wal-Mart Stores, Inc.	WMT	1	49.9	55.05	5.15	10.32
IAC/InterActiveCorp	IACI	1	19.53	15.77	-3.76	-19.25
Target Corporation	TGT	1	52	33.55	-18.45	-35.48
Adobe Systems Incorporated	ADBE	1	32.55	21.05	-11.5	-35.33
Electronic Arts Inc.	ERTS	1	46.31	15.34	-30.97	-66.88
Monster Worldwide, Inc.	MWW	1	24.75	11.55	-13.2	-53.33
Circuit City Stores, Inc.	CCTYQ	1	3.84	0.14	-3.71	-96.48
Gannett Co., Inc.	GCI	0.41	11.85	3.27	-8.58	-72.42
The McClatchy Company	MNI	0.14	1.24	0.11	-1.13	-91.21
					-910.5	-1035.58



Portfolio Optimization

Choice among many investment options with Uncertain returns

Facility Location

Many possible demand locations with different demand distributions

Planning under High-Dimensional Stochastic Data

- Stochastic optimization approach

$$\underset{x \in X}{\text{minimize}} \ \mathbb{E}_p[f(x, \xi)]$$

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  where ξ is a high-dimensional random vector.
- The common solution method is by Sample Average Approximation (SAA).
- However, to sample such a random vector, one suffers from the "curse of dimensionality"

Dimensionality	Required sample size
1	4
2	19
5	786
7	10,700
10	842,000

Price of Correlations

One can also consider the distributionally robust approach:

$$\underset{x \in X}{\text{minimize}} \ \underset{p \in \mathcal{D}}{\text{maximize}} \ \mathbb{E}_p[f(\mathbf{x}, \xi)]$$

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- We precisely quantify this risk as

  Price of Correlations (POC)

- We provide tight bounds on POC for various cost functions.

Price of Correlations

Define

- $\hat{\mathbf{x}}$ be the optimal solution of stochastic program with independent distribution $\hat{p}(\xi) = \prod_i p_i(\xi_i)$.

$$\boxed{\hat{\mathbf{x}} = \arg\min_{\mathbf{x}\in X} \mathbb{E}_{\hat{p}}[f(\mathbf{x}, \xi)]}$$

- $\mathbf{x}^*$ be the optimal solution for the distributionally robust model.

$$\boxed{\mathbf{x}^* = \arg\min_{\mathbf{x}\in X} \max_{p\in\mathcal{D}} \mathbb{E}_p[f(\mathbf{x}, \xi)]}$$

Then, Price of Correlations (POC), or Correlation Gap, is approximation ratio that $\hat{\mathbf{x}}$ achieves for distributionally robust model.

$$\text{POC} = \frac{\max_{p\in\mathcal{D}} \mathbb{E}_p[f(\hat{\mathbf{x}}, \xi)]}{\max_{p\in\mathcal{D}} \mathbb{E}_p[f(\mathbf{x}^*, \xi)]}$$

Price of Correlations

- Approximation of robust model
  - Minimax stochastic program can be replaced by stochastic program with independent distribution to get approximate solution.
  - Often easy to solve either by sampling or by other algorithmic techniques [e.g., Kleinberg et al. (1997), Möhring et al.(1999)]

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- Captures "Value of Information"
  - Small POC means it is not too risky to assume independence.
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Question: What function class has large POC? What function class has small POC?

- Submodularity leads to small POC
- Supermodularity leads to large POC

Submodularity Leads to Small POC

- For any fixed x, function $f(\xi) = f(x, \xi)$ is submodular in random variable ξ
- Decreasing marginal cost, economies of scale

$$f(\max\{\xi, \theta\}) + f(\min\{\xi, \theta\}) \leq f(\xi) + f(\theta)$$

- For continuous functions: $\dfrac{\partial f(\xi)}{\partial \xi_i \partial \xi_j} \leq 0$

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Theorem

If $f(\cdot, \xi)$ is monotone and submodular in ξ, then POC $\leq e/(e-1)$.

Calinescu, Chekuri, Pál, Vondrák [2007] for binary random variables, Agrawal, Ding, Saberi, Y, [2010] for general domains

Supermodularity Leads to Large POC

- For any fixed x, function $f(\xi) = f(x, \xi)$ is supermodular in random variable ξ
- Increasing marginal cost

$$\frac{\partial f(\xi)}{\partial \xi_i \partial \xi_j} \geq 0$$

  e.g., effects of increase in congestion as demand increases.
- In worst case distribution large values of one variable will appear with large values of other variable – highly correlated
- We show example of supermodular set function with POC $= \Omega(2^n)$.

Agrawal, Ding, Saberi, Y, [2010]

Applications: Stochastic Bottleneck Matching

$$\boxed{\text{minimize}_{\mathbf{x}\in X}\ \ \text{maximize}_{p\in\mathcal{P}}\ \ \mathbb{E}_p[\max_i \xi_i x_i]} \Rightarrow$$

$$\boxed{\text{minimize}_{\mathbf{x}\in X}\ \ \mathbb{E}_{\hat{p}}[\max_i \xi_i x_i]}$$

where expected value is under independent distribution $\hat{p}$.

- Monotone submodular function, $e/(e-1) \sim 1.6$ approximation.
- Can be sampled efficiently, Chernoff type concentration bounds hold for monotone submodular functions.
- Reduces to a small convex problem

$$\text{minimize}_{\mathbf{x}\in X} \sum_{s\in\mathcal{S}} \max_i\{s_i x_i\}$$

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Beyond Submodularity?

Monotone Subadditive Functions?

- Preserves economy of scale
- Example with
  $\text{POC} \geq \Omega(\sqrt{n}/\log\log(n))$

Fractionally Subadditive?

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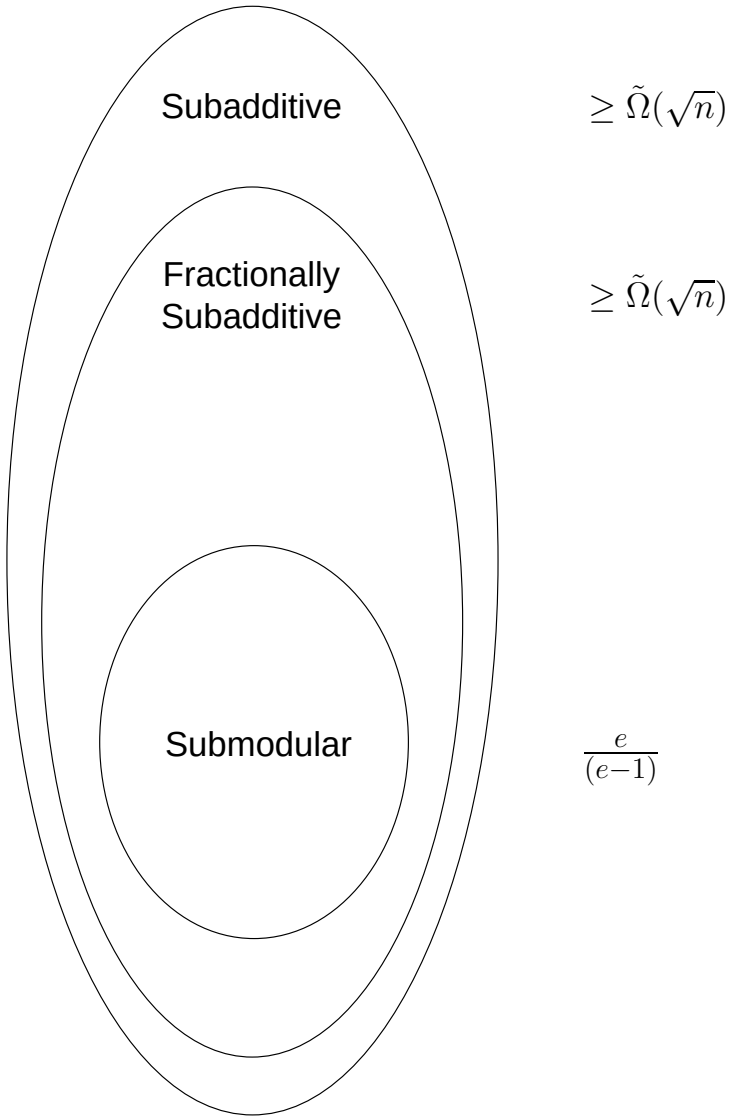

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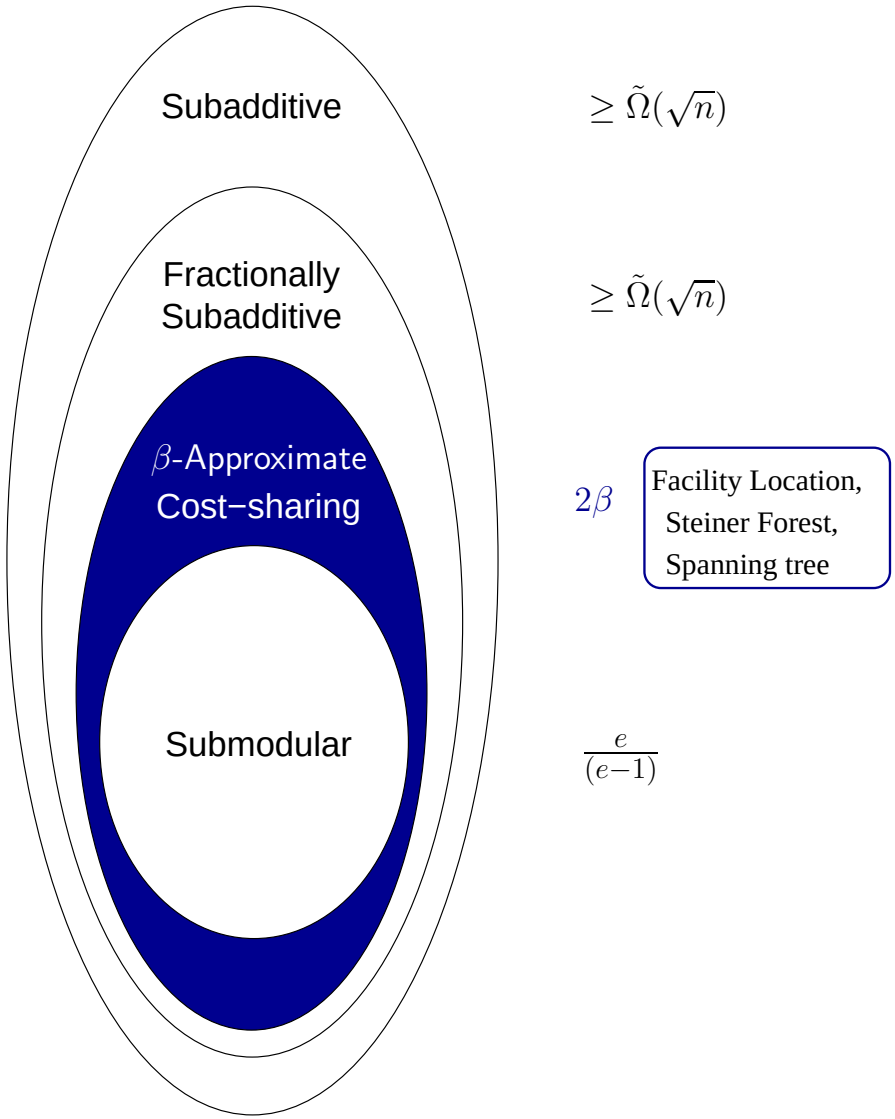
Cost-sharing to the rescue



Cross-Monotone Cost-Sharing

A cooperative game theory concept

- Can cost $f(\xi_1, \ldots, \xi_n)$ be charged to participants $1, \ldots, n$ so that the share charged to participant i decreases as the demands of other participants increase?
  [introduced by Thomson (1983, 1995) in context of bargaining]
- For submodular functions – charge marginal costs.
- β-approximate cost-sharing scheme: total cost charged is within β of the original (expected) function value

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Approximate cost-sharing schemes exist for non-submodular functions

- 3-approximate cost-sharing for facility location cost function
  [Pál, Tardos 2003]
- 2-approximate cost-sharing for Steiner forest cost function
  [Könemann, Leonardi, Schäfer 2005]

Bounding POC via Cost-Sharing

Theorem

If objective function $f(\cdot, \xi)$ is monotone in ξ with β-cost-sharing scheme, POC $\leq 2\beta$.

- POC ≤ 6 for two-stage stochastic facility location
- POC ≤ 4 for two-stage stochastic Steiner forest network design problem.

Agrawal, Ding, Saberi, Y, [2010]

The Cost-Sharing Condition is (near)-Tight

Theorem

If POC for function f is less than β, there exists a cross-monotone cost-sharing scheme with expected β-budget balance.

We show examples of

- Monotone submodular function with POC $\geq \frac{e}{e-1}$.
- Facility location with POC ≥ 3.
- Steiner tree network design with POC ≥ 2.

Agrawal, Ding, Saberi, Y, [2010]

Applications to Deterministic Problems

- Partition of goods among K players to maximize utility
  - Approximation ratio achieved by randomly assigning goods
  - Approximation ratio achieved by independent rounding of the LP solution
  - $e/(e-1)$ is best known approximation ratio for problem with monotone submodular utility

Applications to Deterministic Problems

- Partition of goods among K players to maximize utility
  - Approximation ratio achieved by randomly assigning goods
  - Approximation ratio achieved by independent rounding of the LP solution
  - $e/(e-1)$ is best known approximation ratio for problem with monotone submodular utility
- d-dimensional matching/transportation problem
  - Approximation ratio achieved by random combinations
  - $e/(e-1)$ when weight matrix satisfies monotonicity and Monge property.

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Open questions

- Further characterizations of value of partial information in stochastic optimization problems
- Given partial information about correlations such as Covariance matrix
  - How does worst case distribution compare to maximum entropy distribution?
  - Block-wise independent distributions?

Table of Contents

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- Decision: To sell or not to each individual customer?
- Objective: Maximize the revenue.

An Example

	Bid 1($t=1$)	Bid 2($t=2$)	.....	Inventory($\mathbf{b}$)
Price(π_t)	\$100	\$30	...	
Decision	x_1	x_2	...	
Pants	1	0	...	100
Shoes	1	0	...	50
T-shirts	0	1	...	500
Jackets	0	0	...	200
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Online Linear Programming Model

The classical offline version of the above program can be formulated as a linear (integer) program as all information data would have arrived: compute x_t, $t = 1, ..., n$ and

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- We only know $\mathbf{b}$ and n at the start
- the bidder information is revealed sequentially along with the corresponding objective coefficient.
- an irrevocable decision must be made as soon as an order arrives without observing or knowing the future data.

Model Assumptions

Main Assumptions

- $0 \leq a_{it} \leq 1$, for all (i, t);
- $\pi_t \geq 0$ for all t
- The bids $(\mathbf{a}_t, \pi_t)$ arrive in a random order (rather than from some iid distribution).

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- The bids $(\mathbf{a}_t, \pi_t)$ arrive in a random order (rather than from some iid distribution).

Denote the offline LP maximal value by $OPT(A, \pi)$. We call an online algorithm $\mathcal{A}$ to be c-competitive if and only if

$$E_\sigma \left[\sum_{t=1}^{n} \pi_t x_t(\sigma, \mathcal{A}) \right] \geq c \cdot OPT(A, \pi) \ \forall (A, \pi),$$

where σ is the permutation of arriving orders.

In what follows, we let

$$B = \min_i \{b_i\} (> 0).$$

Main Results: Necessary and Sufficient Conditions

Theorem

For any fixed $0 < \epsilon < 1$, *there is no online algorithm for solving the linear program with competitive ratio* $1 - \epsilon$ *if*

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Agrawal, Wang and Y [2010, 2014]

Ideas to Prove Negative Result

- Consider $m = 1$ and inventory level B, one can construct an instance where $OPT = B$, and there will be a loss of $\sqrt{B}$ with a high probability, which give an approximation ratio $1 - \frac{1}{\sqrt{B}}$.

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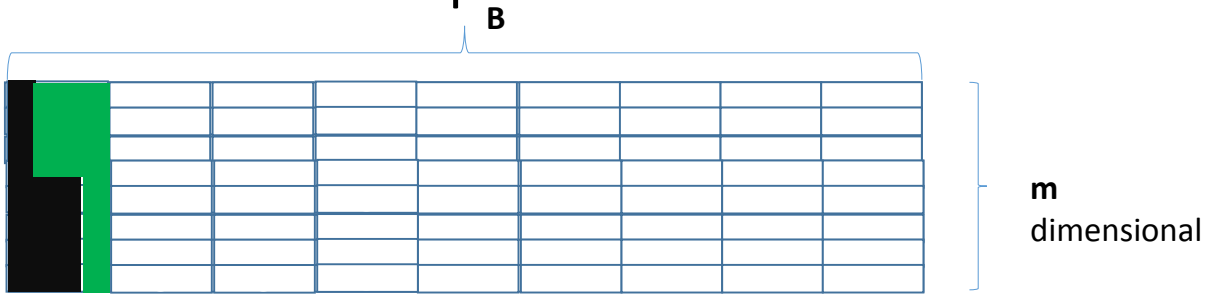
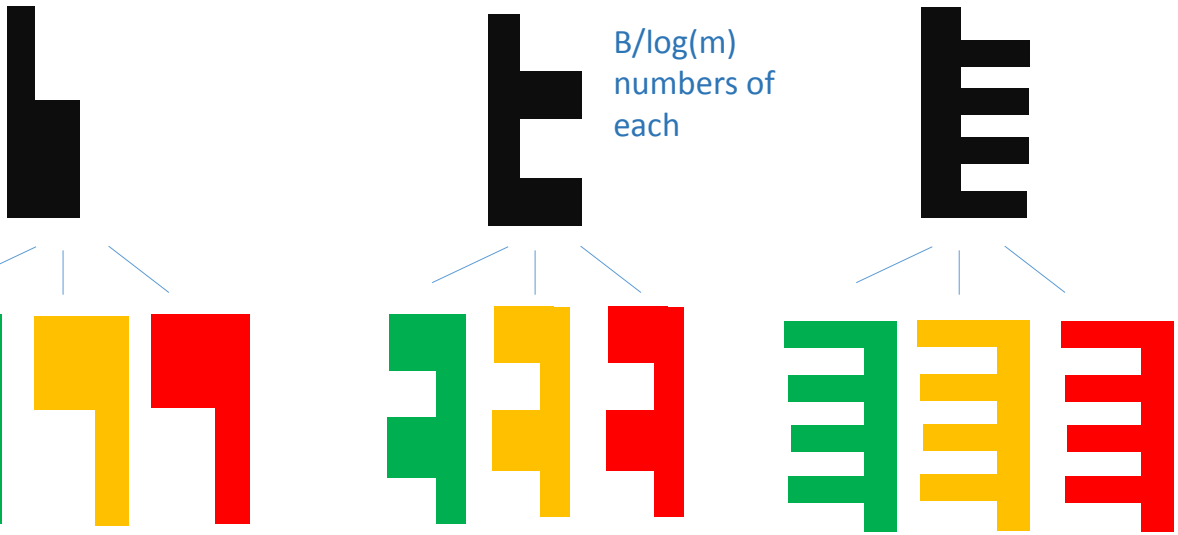
- Then, with hight probability each "single good" case has a loss of $\sqrt{B/\log(m)}$ and the total loss of $\sqrt{B \cdot \log(m)}$. Thus, approximation ratio is at best $1 - \frac{\sqrt{\log(m)}}{\sqrt{B}}$.

Necessary Result I

Multidimensional knapsack



log(m) type of **MUST-HAVE** items (profit = 4)



B/log(m) numbers of each

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Necessary Result II

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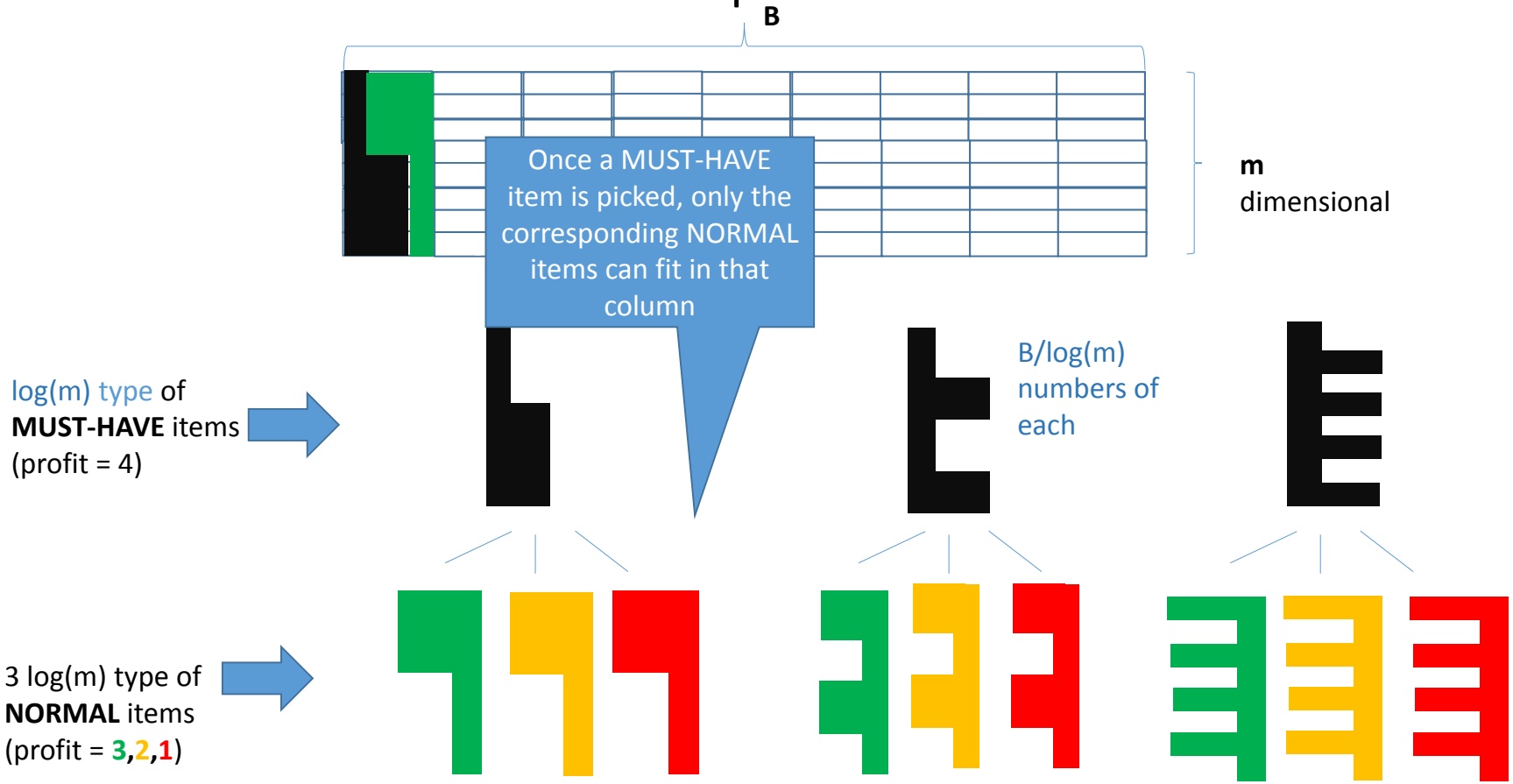

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The proof of the positive result is constructive and based on a learning policy.

- There is no distribution known so that any type of stochastic optimization models is not applicable.
- Unlike dynamic programming, the decision maker does not have full information/data so that a backward recursion can not be carried out to find an optimal sequential decision policy.
- Thus, the online algorithm needs to be learning-based, in particular, learning-while-doing.

Price Observation of Online Learning I

The problem would be easy if there are "ideal prices":

	Bid 1($t=1$)	Bid 2($t=2$)	.....	Inventory($\mathbf{b}$)	$\mathbf{p}^*$
Bid(π_t)	\$100	\$30	...		
Decision	x_1	x_2	...		
Pants	1	0	...	100	\$45
Shoes	1	0	...	50	\$45
T-shirts	0	1	...	500	\$10
Jackets	0	0	...	200	\$55
Hats	1	1	...	1000	\$15

Price Observation of Online Learning II

- Pricing the bid: The optimal dual price vector $\mathbf{p}^*$ of the offline LP problem can play such a role, that is $x_t^* = 1$ if $\pi_t > \mathbf{a}_t^T \mathbf{p}^*$ and $x_t^* = 0$ otherwise, yields a near-optimal solution.

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- One-time learning algorithm: learn the price vector once using the initial ϵn input.
- Dynamic learning algorithm: dynamically update the prices at a carefully chosen pace.

One-Time Learning Algorithm

We illustrate a simple One-Time Learning Algorithm:

- Set $x_t = 0$ for all $1 \leq t \leq \epsilon n$;

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  and get the optimal dual solution $\hat{\mathbf{p}}$;
- Determine the future allocation x_t as:

$$x_t = \begin{cases} 0 & \text{if } \pi_t \le \hat{\mathbf{p}}^T \mathbf{a}_t \\ 1 & \text{if } \pi_t > \hat{\mathbf{p}}^T \mathbf{a}_t \end{cases}$$

  as long as $a_{it} x_t \le b_i - \sum_{j=1}^{t-1} a_{ij} x_j$ for all i; otherwise, set $x_t = 0$.

One-Time Learning Algorithm Result

Theorem

For any fixed $\epsilon > 0$, the one-time learning algorithm is $(1-\epsilon)$ competitive for solving the linear program when

$$B \geq \Omega\left(\frac{m \log(n/\epsilon)}{\epsilon^3}\right)$$

This is one ϵ worse than the optimal bound.

Outline of the Proof

- With high probability, we clear the market;
- With high probability, the revenue is near-optimal if we include the initial ϵ portion revenue;
- With high probability, the first ϵ portion revenue, a learning cost, doesn't contribute too much.

Then, we prove that the one-time learning algorithm is $(1-\epsilon)$ competitive under condition $B \geq \frac{6m\log(n/\epsilon)}{\epsilon^3}$.

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At time $\ell \in \{\epsilon n, 2\epsilon n, ...\}$, the price vector is the optimal dual solution to the following linear program:

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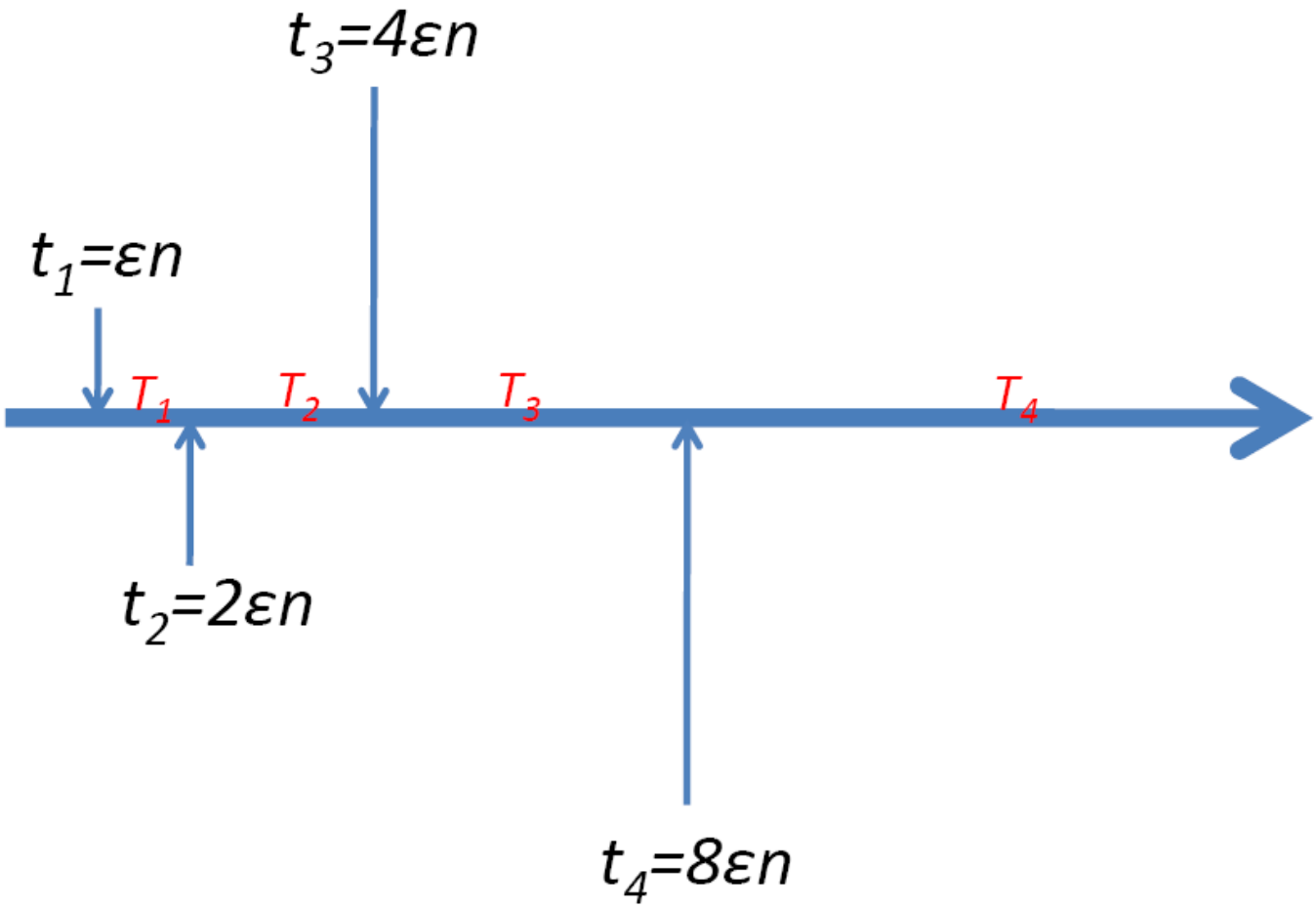
where

$$h_\ell = \epsilon \sqrt{\frac{n}{\ell}};$$

and this price vector is used to determine the allocation for the next immediate period.

Geometric Pace/Grid of Price Updating



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- The numbers h_ℓ play an important role in improving the condition on B in the main theorem. It basically balances the probability that the inventory ever gets violated and the lost of revenue due to the factor $1 - h_\ell$.
- Choosing large h_ℓ (more conservative) at the beginning periods and smaller h_ℓ (more aggressive) at the later periods, one can now control the loss of revenue by an ϵ order while the required size of B can be weakened by an ϵ factor.

Related Work on Random-Permutation

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Kesselheim et al [2014]	$B \geq \frac{\log m}{\epsilon^2}$	Dynamic*
Gupta/Molinaro [2014]	$B \geq \frac{\log m}{\epsilon^2}$	Dynamic*
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Table: Comparison of several existing results

Price-Post Learning

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- Objective is to maximize the expected revenue.

Background: Known Demand Function

Historically, researchers mostly consider the case where the demand function $\lambda(p)$ is known.

- A dynamic programming formula can be established for this problem. When the demand function takes certain form, one can obtain the optimal policy explicitly.
- In other cases, approximate dynamic programming method can be applied to obtain approximate solution. And usually the results are very close to optimal.
- Mostly in Airline Revenue Management, see Gallego and van Ryzin (1994, 1997), Talluri and van Ryzin (1998, 1999), etc.

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- Non-parametric approach only poses few requirements on the demand function thus is very robust to model uncertainty.
- In a non-parametric learning algorithm, more price experimentations have to be made and the question is how to reduce the learning cost.

Evaluation of the Learning Algorithm: Asymptotic Regret I

For any pricing policy/algorithm π, denote its expected revenue by $J^\pi(B, T; \lambda)$. Also denote the optimal expected revenue as $J^*(B, T; \lambda)$. Then, we consider the regret

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Evaluation of the Learning Algorithm: Asymptotic Regret II

- However it is still very hard to evaluate the regret for a low volume.
- Therefore we consider a high-volume regime where the inventory B, together with the demand rate λ, grows proportionally (multiplied in an positive integer n) and consider the asymptotic behavior of $R^{\pi}(n \cdot B, T; n \cdot \lambda)$.

Evaluation of the Learning Algorithm: Asymptotic Regret II

- However it is still very hard to evaluate the regret for a low volume.
- Therefore we consider a high-volume regime where the inventory B, together with the demand rate λ, grows proportionally (multiplied in an positive integer n) and consider the asymptotic behavior of $R^{\pi}(n \cdot B, T; n \cdot \lambda)$.
- This type of evaluation criterion is widely adopted.

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- The algorithms for both cases use one-time learning, that is, learning first and doing second. In the learning period, a number of prices are tested and the best one is selected to be implemented in the doing period.
- As presented earlier, under the auction model, the best learning algorithm can achieve an asymptotic regret of $O(n^{-1/2})$.

Could we close the gaps?

Assumptions on the Demand Function

- $\lambda(p)$ is bounded
- $\lambda(p)$ (and $r(p) = p\lambda(p)$) is Lipschitz continuous. Also there exists an inverse demand function $\gamma(\lambda)$ that is also Lipschitz continuous.
- $r(\lambda) = \lambda\gamma(\lambda)$ is (strictly) concave
- $r''(\lambda)$ exists and $r''(\lambda) \leq -\alpha < 0$ for a fixed positive number α.

Positive and Negative Results

Theorem

Let the above assumptions hold. Then, there exists an admissible pricing policy π*, such that for all* $n \geq 1$*,*

$$\sup_{\lambda \in \Gamma} R_n^{\pi_\delta}(n \cdot B, T; n \cdot \lambda) \leq C(\log n)^{4.5} \cdot n^{-1/2}.$$

On the other hand, there exists a set of demand function Γ *parameterized by a single parameter satisfying the above assumption, such that for any admissible pricing policy* π*, for all* $n \geq 1$

$$\sup_{\lambda \in \Gamma} R_n^{\pi}(n \cdot B, T; n \cdot \lambda) \geq \frac{C}{\sqrt{n}}$$

for some constant C *that only depends on* Γ*,* B *and* T*.*

Wang, Deng and Y [2014]

Description of the Algorithm

The algorithm is a dynamic pricing algorithm, where we integrate the "learning" and "doing" periods. Specifically, we

- Divide the time into geometric intervals
- Keep a shrinking admissible price range
- Perform and apply price experimentation in each time interval within the current price range
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The key is to balance and interplay demand learning (exploration) and near-optimal pricing (exploitation).

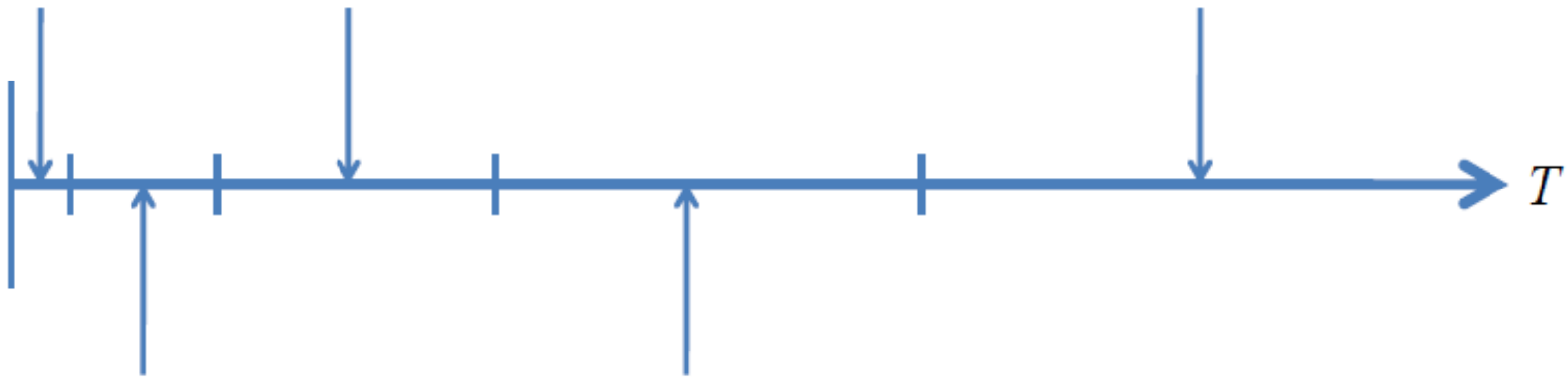
Geometric Pace of Price Testing

τ: Time spent in each period
κ: Number of price tested

Iter 1
τ_1=0.0035;
κ_1=12;
p range: 0.1-10
p^u=5.05
p^c = 3.4

Iter 3
τ_3=0.15
κ_3=5;
p range: 4.90-5.35
p^u=5.07
p^c = 4.90

Apply price *5.01* until the end



Iter 2
τ_2=0.037;
κ_2=7;
p range: 3.99-5.69
p^u=5.21
p^c = 3.99

Iter 4
τ_4=0.35;
κ_4=4;
p range: 4.96-5.18
p^u=5.01
p^c = 4.96

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- Buy-and-sell or double market?
- Multi-item price-posting market?
- More general online optimization?