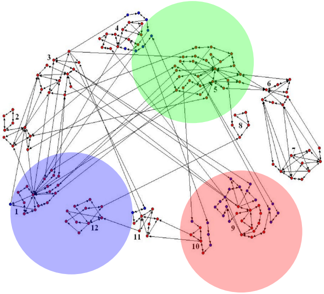
From Weak to Strong LP Gaps for all CSPs



Mrinalkanti Ghosh

Madhur Tulsiani

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Max-3-SAT

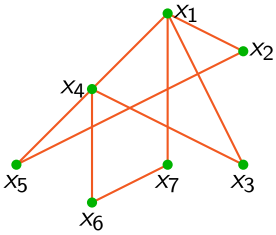
$$x_1 \vee x_{22} \vee \overline{x}_{19}$$

$$x_3 \vee \overline{x}_9 \vee x_{23}$$

$$x_5 \vee \overline{x}_7 \vee \overline{x}_9$$

$$\vdots$$

Max-Cut



$$x_1 \neq x_2$$

$$x_2 \neq x_5$$

$$x_3 \neq x_4$$

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Max-k-CSP$_q$

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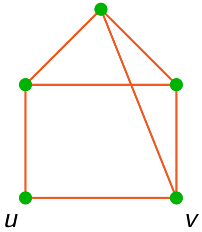
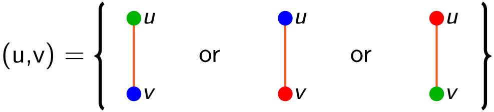


- For a graph, given:
  - Set of colors: $[q]$
  - Constraints: one for each edge $(u, v) \in E$



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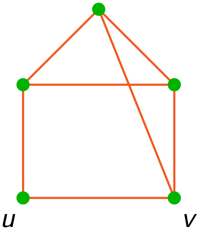

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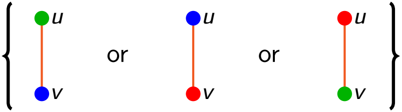

- Each constraint is a bijection from $[q]$ to $[q]$. Can in fact consider difference equations

$$x_u - x_v = c_{uv} \pmod{q}$$

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- Unique Games: $f \equiv$ EQUAL. For i^{th} constraint (u, v), let $i_1 = u$, $i_2 = v$ and let $b_{i,2} - b_{i,1} = c_{uv}$

$$x_u - x_u = c_{uv} \quad \Leftrightarrow \quad x_{i_1} + b_{i,1} = x_{i_2} + b_{i,2}.$$

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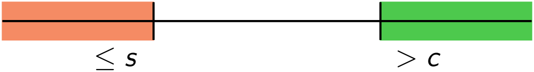

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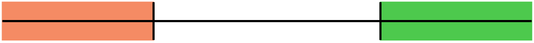

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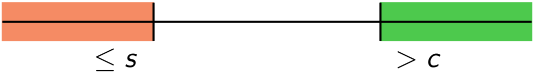
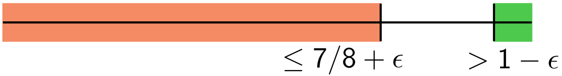


- Goal: Distinguish the cases $\mathrm{OPT}(\Phi) \leq s$ and $\mathrm{OPT}(\Phi) > c$.
- If for some $\gamma \leq 1$, all pairs $(\gamma \cdot c, c)$ can be solved, then can approximate within factor γ.

Characterizing approximability

- Max-3-SAT [Håstad 97]: For all $\epsilon > 0$, distinguishing $(7/8 + \epsilon, 1 - \epsilon)$ is NP-hard ($s < 7/8$ is trivial).



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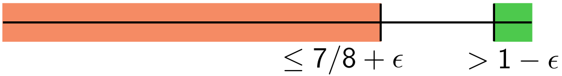
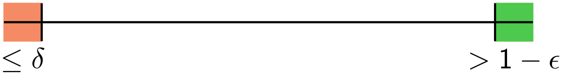


- Unique Games Conjecture [Khot 02]: For all $\delta, \epsilon > 0$, there exists q such that it is NP-hard to distinguish $(\delta, 1 - \epsilon)$ for UG with domain $[q]$.



A dichotomy assuming the UGC

- [Raghavendra 08]: For all q, for all f, if a basic SDP cannot distinguish (s, c) for Max-k-CSP$_q$(f), then for all $\epsilon > 0$, it is NP-hard to distinguish $(s + \epsilon, c - \epsilon)$ assuming the UGC.

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- "All-or-nothing": Either a simple algorithm (approximately solvable in almost linear time) can distinguish (s, c) or it is NP-hard to do so.
- Equivalent to UGC (because UG is a 2-CSP).

An unconditional version for LPs [Ghosh T 17]

- For all q, for all f, if a basic LP cannot distinguish (s, c) for Max-k-CSP$_q$(f), then for all $\epsilon > 0$, no LP of any polynomial size in the Sherali-Adams hierarchy can distinguish $(s + \epsilon, c - \epsilon)$.

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- "All-or-not-much" for LPs: If a simple (linear size) LP cannot do it, neither can any polysize LP extended formulation.

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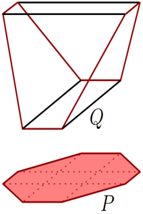

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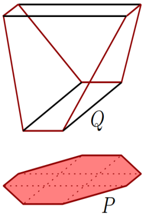

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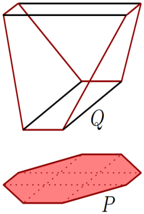

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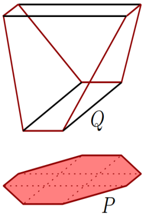

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- Optimize objective objective $\langle w_\Phi, x \rangle$ (depending on Φ) over P.

Integer Program for CSPs

Variables: $Z_{(i,b)}$ for $i \in [n]$ and $b \in [q]$

Constraints: $(Z_{(i,b)})^2 = Z_{(i,b)} \quad \forall\ i \in [n], b \in [q]$

$$\sum_{b \in [q]} Z_{(i,b)} \ = \ 1 \quad \forall\ i \in [n]$$

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Maximize: $$\frac{1}{m} \cdot \sum_{C} \sum_{\alpha \in [q]^{S_C}} \left(\prod_{i \in S_C} Z_{(i,\alpha_i)} \right) \cdot f\left(\alpha + (b_{i,1}, \ldots, b_{i,k})\right)$$

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Linear Program: For variables $X_{(S,\alpha)} \in [0,1]$ satisfying consistency

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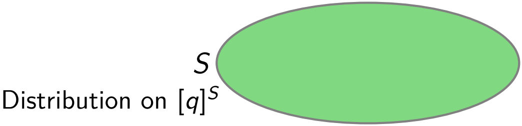

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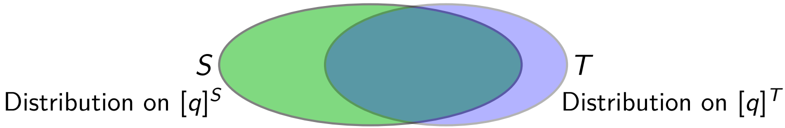

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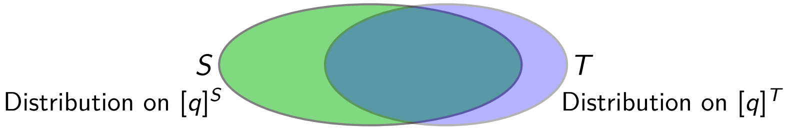

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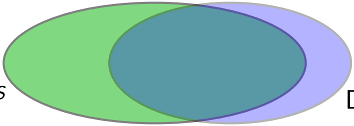

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- $n^{O(t)} \cdot q^t$ variables.

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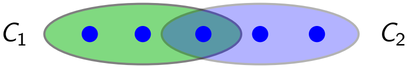

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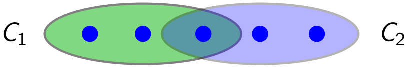
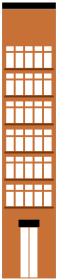
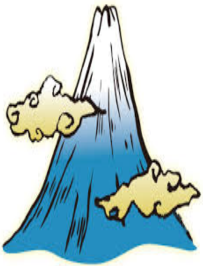


- $O(q^k \cdot m + q \cdot n)$ variables.

Inaccurate pictorial representations



SA hierarchy



Extended Formulations

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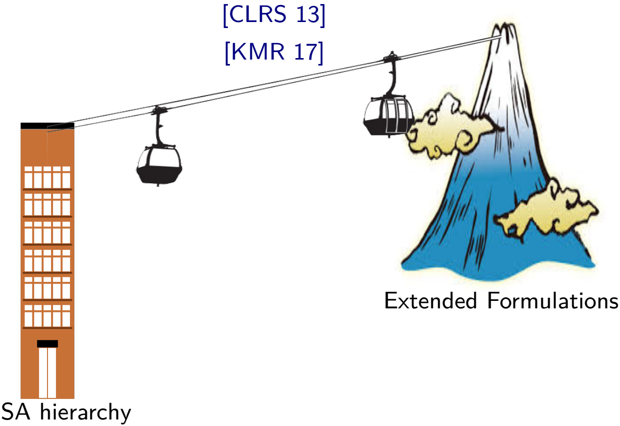


Inaccurate pictorial representations

Max-Cut [CMM 09]

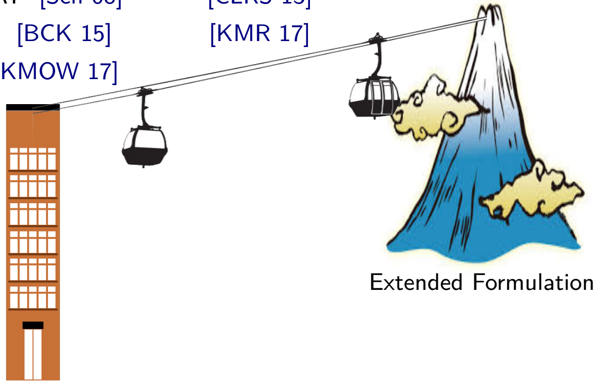
Max-3-SAT [Sch 08]

Pairwise [BCK 15]

[KMOW 17]

[CLRS 13]

[KMR 17]



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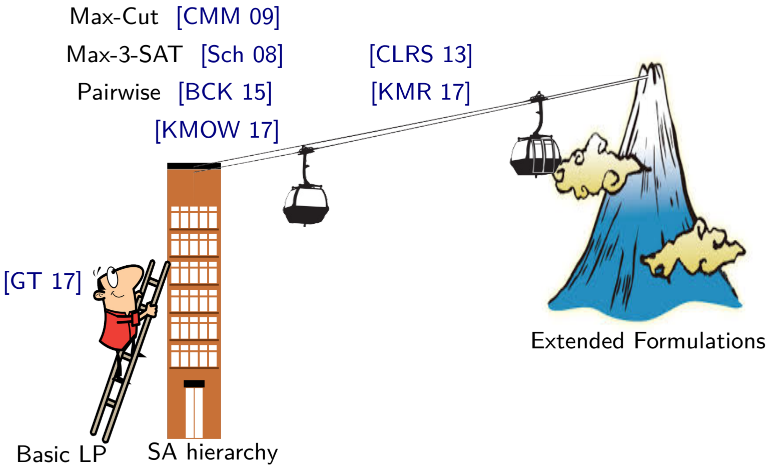

A more precise version

- [Ghosh T 17]: For all q, for all f, if basic LP cannot distinguish (s, c) for Max-k-CSP$_q$(f), then for all $\epsilon > 0$, no LP given by $t = O_\epsilon\left(\frac{\log n}{\log \log n}\right)$ levels of the Sherali-Adams hierarchy can distinguish $(s + \epsilon, c - \epsilon)$.

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- "Escalate" a hard instance for basic LP to a hard instance for Sherali-Adams.

What is a hard instance ($c = 1$)

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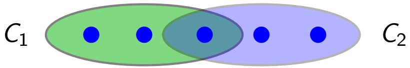

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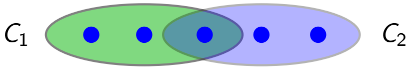

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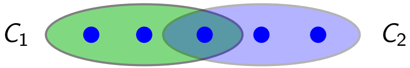

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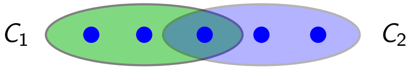

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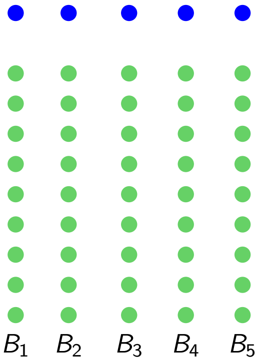

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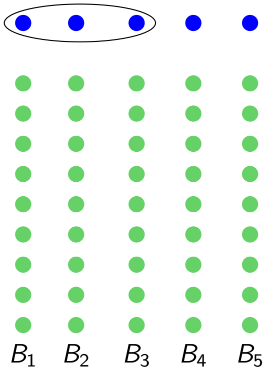

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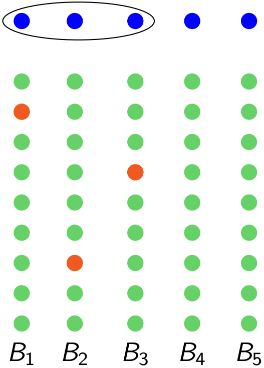

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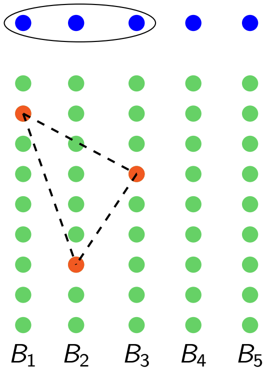

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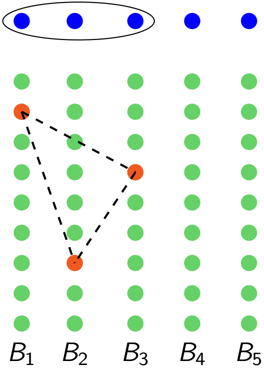

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- Similar constructions used by [GL 15], [KTW 14]

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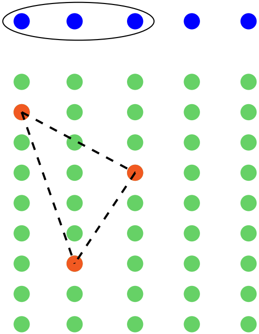

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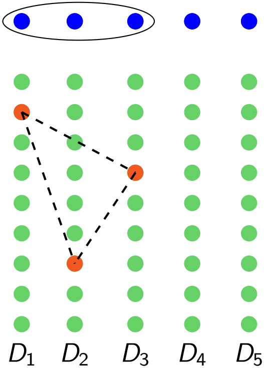

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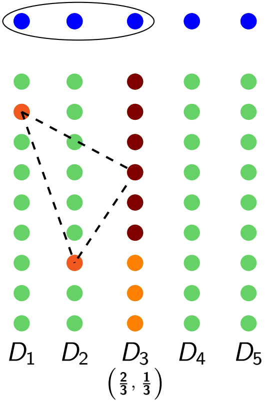

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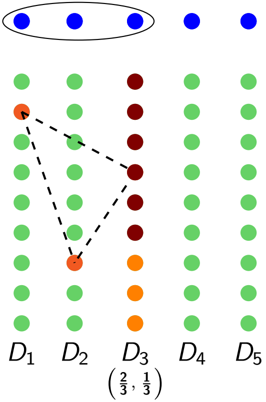

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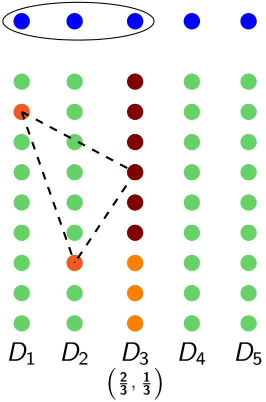

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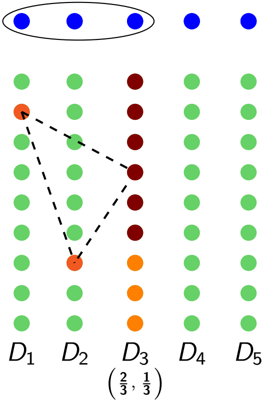

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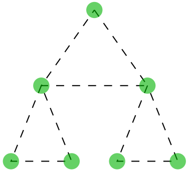
Propagation on trees



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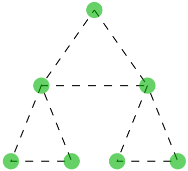

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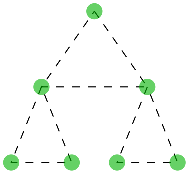

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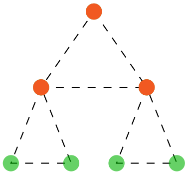

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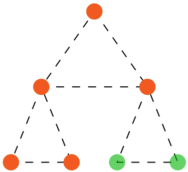

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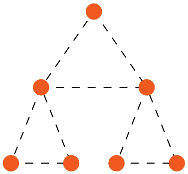

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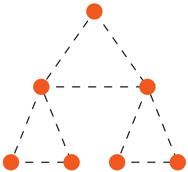

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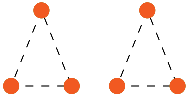

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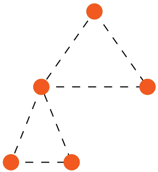

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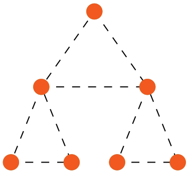

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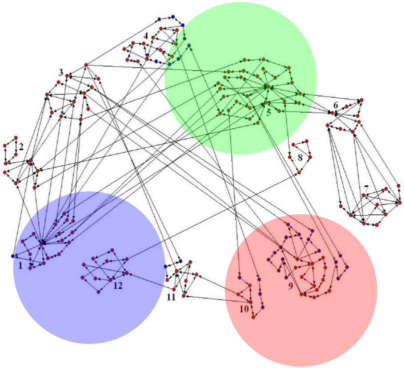

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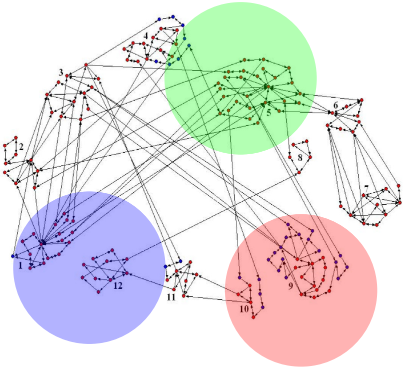

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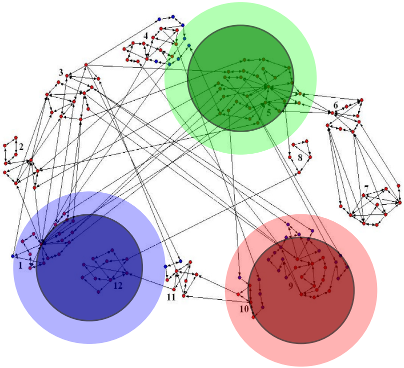

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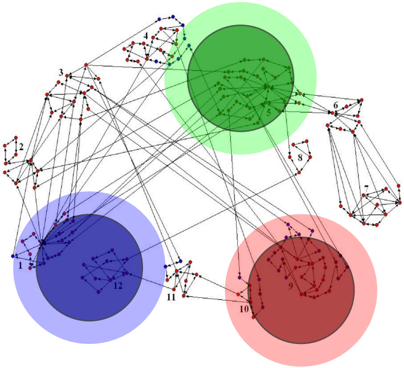

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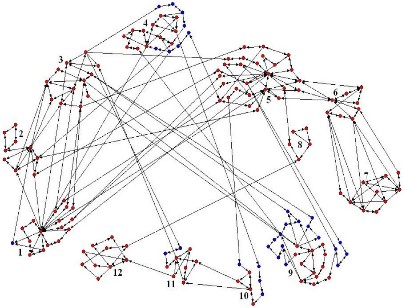

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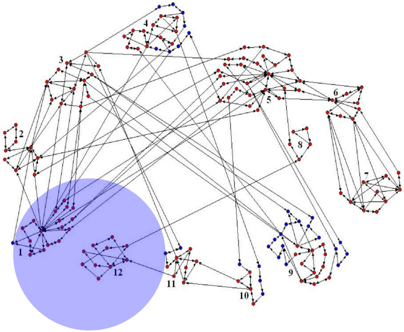

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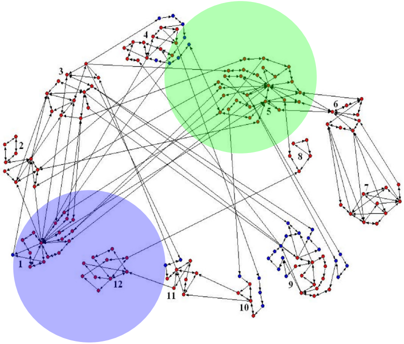

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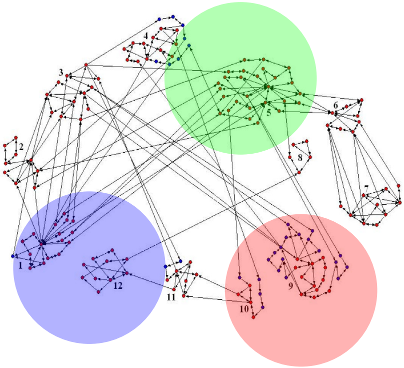

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- For sets S and T, can one consistently discard bad Gaussian projections?

Open Problems

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- Can one avoid loss of ϵ in c when $c = 1$ (relevant for refutation)? Exact refutation addressed by [TZ 16].

Thank You

Questions?