Symmetric Sums of Squares

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Goal

Certify the nonnegativity of a symmetric polynomial over the hypercube.

Our key result: the runtime does not depend on the number of variables of the polynomial

1. Background
2. Our setting
3. Results
4. Flag algebras
5. Future work

Sums of squares modulo an ideal

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Certify $p \geq 0$ over the solutions of a system of polynomial equations.

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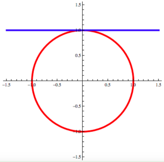
Example

Show that $1 - y \geq 0$ whenever $x^2 + y^2 = 1$



$$1 - y = \left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y-1}{\sqrt{2}}\right)^2 - \frac{1}{2}(x^2 + y^2 - 1)$$
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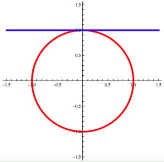

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- Ideal $\mathcal{I} \subseteq \mathbb{R}[\mathbf{x}]$
- $V_{\mathbb{R}}(\mathcal{I})$=its real variety
- p is sos modulo $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$
  (i.e., if $\exists\ h \in \mathcal{I}$ such that $p = \sum_{i=1}^{l} f_i^2 + h$)
- p is d-sos mod $\mathcal{I}$ if $p \equiv \sum_{i=1}^{l} f_i^2 \mod \mathcal{I}$ where $\deg(f_i) \leq d\ \forall\ i$

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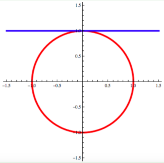

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Our problem

Let $\mathcal{V}_{n,k} = \{0,1\}^{\binom{n}{k}}$ be the k-subset discrete hypercube
$\rightarrow$ coordinates indexed by k-element subsets of $[n]$

Goal

Minimize a symmetric* polynomial over $\mathcal{V}_{n,k}$

*symmetric = $\mathfrak{S}_n$-invariant

$$\mathfrak{s} \cdot x_{i_1 i_2 \ldots i_k} = x_{\mathfrak{s}(i_1)\mathfrak{s}(i_2)\ldots \mathrm{s}(i_k)} \ \forall \mathfrak{s} \in \mathfrak{S}_n$$

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How?
By finding sos certificates over $\mathcal{V}_{n,k}$ that exploit symmetry, i.e., that we can find in a runtime independent of n.

$k = 1$: see Blekherman, Gouveia, Pfeiffer (2014)
$k \geq 2$: ?

Examples of such problems

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  Given a fixed graph H, determine the limiting edge density of a H-free graph on n vertices as $n \to \infty$

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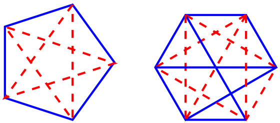
  Color the edges of K_n ruby or sapphire. Find the smallest n for which you are guaranteed a ruby clique of size r or a sapphire clique of size s



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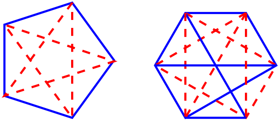

Focus on $\mathcal{V}_n := \mathcal{V}_{n,2} = \{0,1\}^{\binom{n}{2}}$
$\to$ coordinates are indexed by pairs ij, $1 \leq i < j \leq n$

Passing to optimization - Turán-type problem

Example

Forbidding triangles in a graph on n vertices, find

$$\boxed{\begin{aligned} &\max \frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq n} x_{ij} \\ &\text{s.t. } x_{ij}^2 = x_{ij} && \forall 1 \leq i < j \leq n \\ &\quad\ x_{ij}x_{jk}x_{ik} = 0 && \forall 1 \leq i < j < k \leq n \end{aligned}}$$

In particular, show that this is at most $\frac{1}{2} + O(\frac{1}{n})$

$\rightarrow$ show that $\frac{1}{2} + O(\frac{1}{n}) - \frac{1}{\binom{n}{2}} \sum_{1\leq i<j\leq n} x_{ij} \geq 0$

Issue with passing to optimization - Turán-type problem

Example (continued)

Find $Q \succeq 0$ and $d \in \mathbb{Z}^+$ such that

$$\frac{1}{2} + O\left(\frac{1}{n}\right) - \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} x_{ij} \equiv v^\top Q v \mod \mathcal{I}$$

where v =vector of basis elements of $(\mathbb{R}[x]/\mathcal{I})_d$ and

$$\mathcal{I} = \langle x_{ij}^2 - x_{ij} \ \forall 1 \le i < j \le n, \\ x_{ij}x_{jk}x_{ik} \ \forall 1 \le i < j < k \le n \rangle$$

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The runtime would be $\binom{n}{2}^{O(d)}$

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Foreshadowing

Example

The following is a sos proof of Mantel's theorem

$$\begin{pmatrix} 1 & q_1 \end{pmatrix} \begin{pmatrix} \frac{(n-1)^2}{2} & -\frac{2(n-1)}{n} \\ -\frac{2(n-1)}{n} & \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \end{pmatrix} + \mathrm{sym}\left(\begin{pmatrix} q_2 \end{pmatrix} \begin{pmatrix} \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} q_2 \end{pmatrix} \right)$$

where $q_1 = \sum_{i<j} x_{ij}$ and $q_2 = \sum_{i<j} x_{ij} - \frac{n-2}{2} \sum_{i=1}^{n-1} x_{in}$

Key features of desired sos certificates:

- exploits symmetry
- constant size
- entries are functions of n

Representation theory needed for exploiting symmetry

- $(\mathbb{R}[x]/\mathcal{I})_d =: V = \bigoplus_{\lambda \vdash n} V_\lambda$ isotypic decomposition
  - partition $\lambda = (5, 3, 3, 1)$ for $n = 12$

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- $V_\lambda = \bigoplus_{\tau_\lambda} W_{\tau_\lambda}$
  - shape of λ:

  standard tableau τ_λ:

1	4	5	6	9
2	7	10		
3	8	12		
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  - $\mathfrak{R}_{\tau_\lambda}$:=row group of τ_λ (fixes the rows of τ_λ)
  - $W_{\tau_\lambda} := (V_\lambda)^{\mathfrak{R}_{\tau_\lambda}}$ = subspace of V_λ fixed by $\mathfrak{R}_{\tau_\lambda}$
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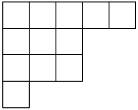

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$$\boxed{V = \bigoplus_{\lambda \vdash n} \bigoplus_{\tau_\lambda} W_{\tau_\lambda}}$$

Note: $\dim(V) = \sum\limits_{\lambda \vdash n} m_\lambda n_\lambda$

Gatermann-Parrilo symmetry-reduction technique

Recall: p d-sos mod $\mathcal{I} \Leftrightarrow \exists\ Q \succeq 0$ s.t. $p \equiv v^\top Q v$ mod $\mathcal{I}$
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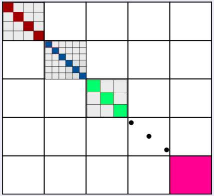
Theorem (Gatermann-Parrilo, 2004)

For each λ, fix τ_λ and find a symmetry-adapted basis $\{b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda}\}$ for W_{τ_λ}.



If p is symmetric and d-sos mod $\mathcal{I}$, then

$$p \equiv \sum_{\lambda \vdash n} \mathrm{sym}(b^\top Q_\lambda b),$$

where $b = (b_1^{\tau_\lambda}, \ldots, b_{m_\lambda}^{\tau_\lambda})^\top$ and $Q_\lambda \succeq 0$ has size $m_\lambda \times m_\lambda$.

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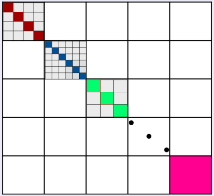

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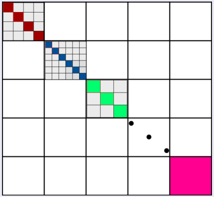

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Succinct SOS

Theorem (RSST, 2016)

If p is symmetric and d-sos, then it has a symmetry-reduced sos certificate that can be obtained by solving a SDP of size independent of n by keeping only a few partitions in Gatermann-Parrilo.

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Example

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$$\begin{pmatrix} 1 & q_1 \end{pmatrix} \begin{pmatrix} \frac{(n-1)^2}{2} & -\frac{2(n-1)}{n} \\ -\frac{2(n-1)}{n} & \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \end{pmatrix} + \mathrm{sym}\left(\begin{pmatrix} q_2 \end{pmatrix} \begin{pmatrix} \frac{8}{n^2} \end{pmatrix} \begin{pmatrix} q_2 \end{pmatrix} \right)$$

$\rightarrow$ kept partitions $(n) = \overbrace{\square\square\square\square\square\square}^{n}$ and $(n-1, 1) = \begin{array}{l} \overbrace{\square\square\square\square\square}^{n-1} \\ \square \end{array}$

Bypassing symmetry-adapted basis

Theorem (RSST, 2016)

In Gatermann-Parrilo, instead of a symmetry-adapted basis, one can use

- *a spanning set for W_{τ_λ} for $\lambda \geq_{\text{lex}} \overbrace{\ydiagram{6,1,1,1}}^{n-2d}$.*
- *of size independent of n*
- *that is easy to generate*

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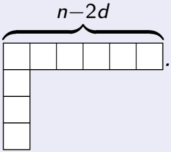
- *a spanning set for W_{τ_λ} for $\lambda \geq_{\text{lex}}$* 

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Examples of spanning sets containing W_{τ_λ}

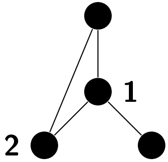
- $\text{sym}_{\tau_\lambda}(x^m) := \frac{1}{|\mathfrak{R}_{\tau_\lambda}|} \sum_{\mathfrak{s} \in \mathfrak{R}_{\tau_\lambda}} \mathfrak{s} \cdot x^m$
- an appropriate Möbius transformation

Razborov's flag algebras for Turán-type problems

Use flags (=partially labelled graphs) to certify a symmetric inequality that gives a good upper bound for Turán-type problems

Key features:

- sums of squares of graph densities
- n disappears
- asymptotic results for dense graphs



Theorem (Razborov, 2010)

If $\mathcal{A} = \{K_4^3\}$, *then* $\max_{G:|V(G)|\to\infty} d(G) \leq 0.561666.$
If $\mathcal{A} = \{K_4^3, H_1\}$, *then* $\max_{G:|V(G)|\to\infty} d(G) = 5/9.$

Connection of spanning sets to flag algebras

Theorem (RSST, 2016)

Flags provide spanning sets for W_{τ_λ} of size independent of n.

If p is symmetric and d-sos, then its nonnegativity can be established through flags on kd vertices (even in restricted cases).

Theorem (R., Singh, Thomas, 2015)

Every flag sos polynomial of degree kd can be written as a succinct d-sos.

Theorem (RSST, 2016)

Flag methods are equivalent to standard symmetry-reduction methods for finding sos certificates over discrete hypercubes.

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There exists a family of symmetric nonnegative polynomials of fixed degree that cannot be certified exactly with any fixed set of flags, namely

$$\frac{1}{\binom{n}{2}^2}\left(\sum_{e\in E(K_n)} x_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor\right)\left(\sum_{e\in E(K_n)} x_e - \left\lfloor \frac{\binom{n}{2}}{2} \right\rfloor - 1\right) + O(\frac{1}{n^2})$$

Open problems

- Find a concrete family of nonnegative polynomials on $\binom{n}{k}$ variables that one cannot approximate up to an error of order $O(\frac{1}{n})$ with finitely many flags or with sums of squares of fixed degree.
- Provide certificates for open problems over $\mathcal{V}_{n,k}$ using symmetric sums of squares.

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Thank you!