Volker Kaibel

Constructing Extended Formulations

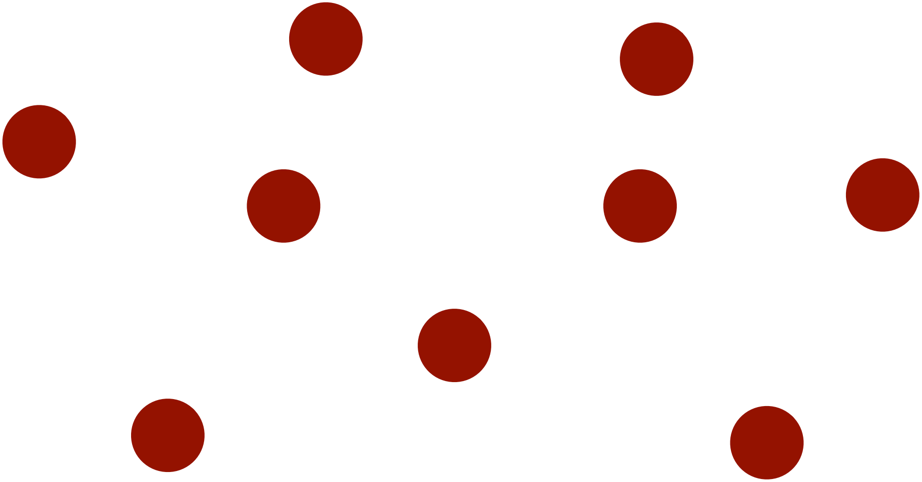
Nov 6, 2017

Simons Institute, Berkeley

OTTO VON GUERICKE
UNIVERSITÄT
MAGDEBURG

Outline

Convex Hulls and Linear Programming



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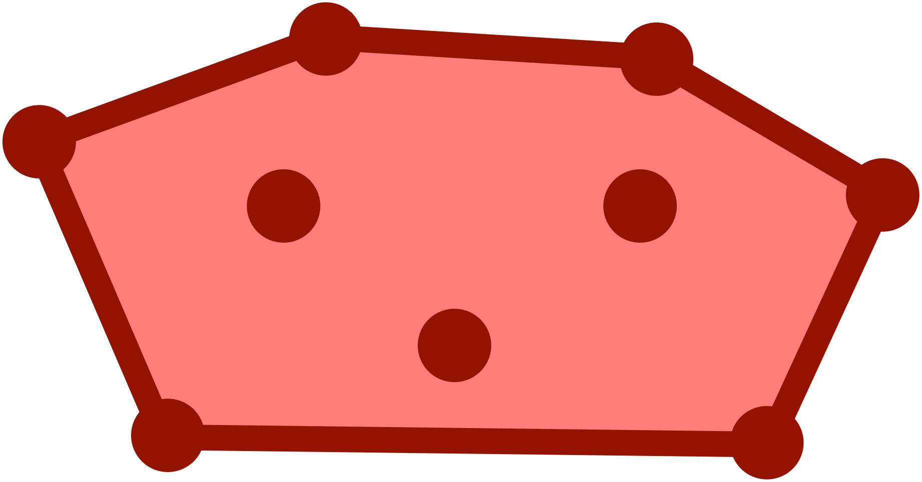

Convex Hulls and Linear Programming

From points to polytopes

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For $|X| < \infty$ there is $A, b \;:\; \mathrm{conv}(X) = \{x \in \mathbb{R}^n \;:\; Ax \leq b\}$

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LP-algorithms

Efficient both in theory and praxis.

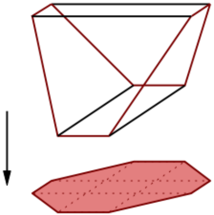
Representations as Projections

Extension of a Polytope $P \subseteq \mathbb{R}^n$:

A polytope $Q \subseteq \mathbb{R}^d$ and a linear projection $p : \mathbb{R}^d \to \mathbb{R}^n$ with

$$P = p(Q).$$

Size: Number of facets of Q



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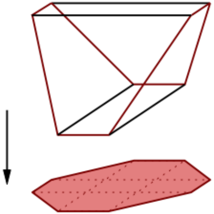

Extended Formulation of P:

Linear description of some extension of P

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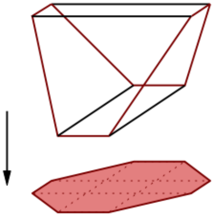

Extension Complexity of P:

$$\mathrm{xc}(P) = \text{smallest size of any extension of } P$$

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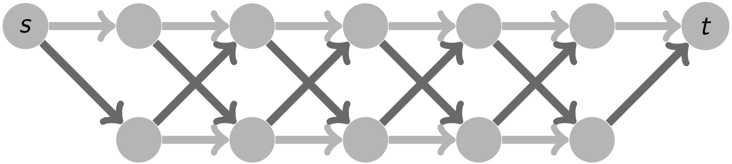
$$\text{xc}(\text{conv}\{v \in \{0,1\}^n \ : \ v \text{ has even } \# \text{ of 1's}\}) \leq 4n - 4$$



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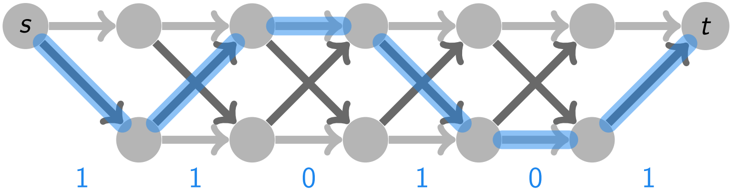

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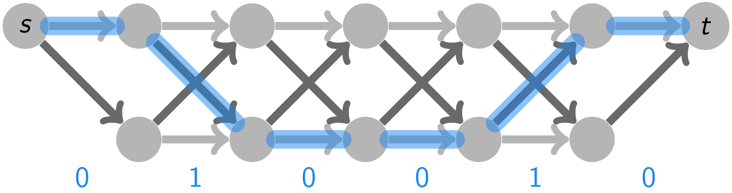

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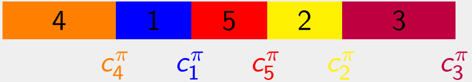

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Queyranne 1993

For $0 < p_1 \leq \cdots \leq p_n$: Description by one equation and

$$\sum_{i \in I} p_i x_i \geq \sum_{i=1}^{|I|} p_i \sum_{j=1}^{i} p_j \quad \text{for all } \varnothing \neq I \subseteq [n]$$

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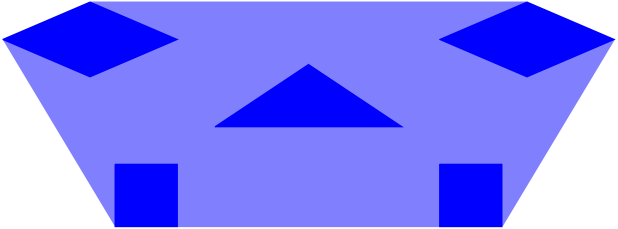
Wolsey 1986

The cube $Q = [0,1]^{\binom{[n]}{2}}$ projects to $\mathrm{P}_{\mathrm{ct}}(p_1, \dots, p_n)$ via

$$x_i = \sum_{j=1}^{i-1} p_j y_{\{i,j\}} + \sum_{j=i+1}^{n} p_j (1 - y_{\{i,j\}}) \quad \text{for all } i \in [n]\,.$$

Outline

Unions of Polytopes



Balas 1975

For polytopes $P_1, \ldots, P_q \subseteq \mathbb{R}^m$ (with $\dim(P_i) > 0$)

$$\mathrm{xc}(\mathrm{conv}(\bigcup_{i=1}^{q} P_i)) \leq \sum_{i=1}^{q} \mathrm{xc}(P_i)$$

holds.

Matching Polytopes

Matchings with ℓ edges

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$\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ is described by $x \geq \mathbf{0}$, $x(E) = \ell$, and:

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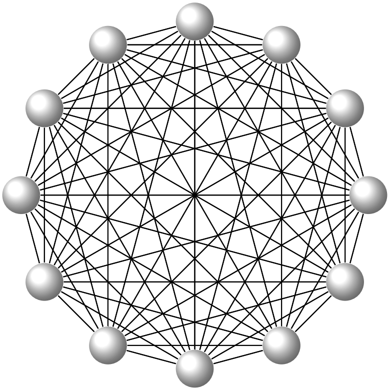
The Strategy

1. Cover by few subproblems.
2. Find small (extended) formulations for subproblems.
3. Take (convex hull of) union.

Special Matchings



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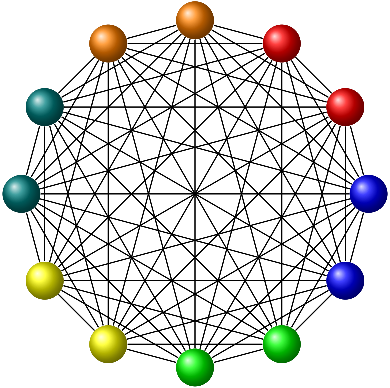

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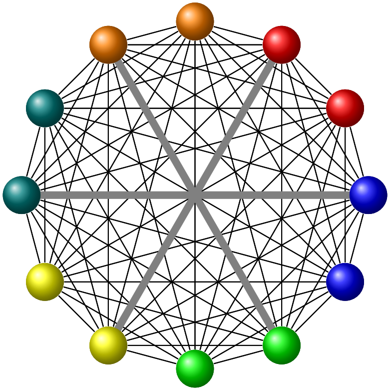

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For $V = W_1 \uplus \cdots \uplus W_{2\ell}$, a matching $M \subseteq E$ is colorful if it matches exactly one node from each set W_i.

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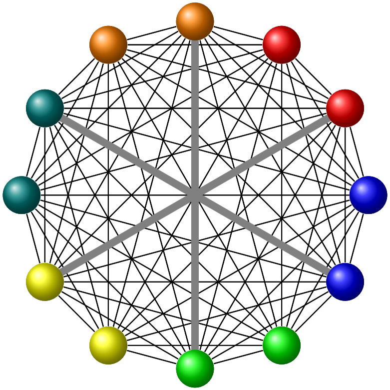

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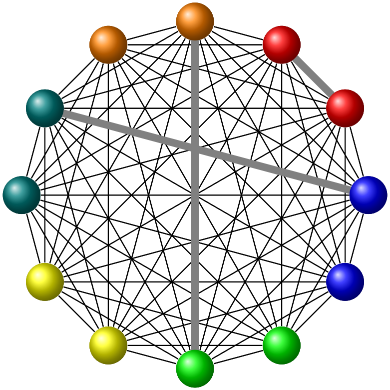

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$$x(E(W_i)) = 0 \quad \text{for all } i \in \{1,\ldots,2\ell\}$$
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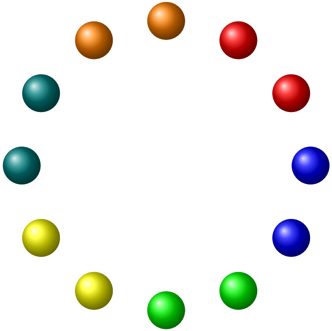

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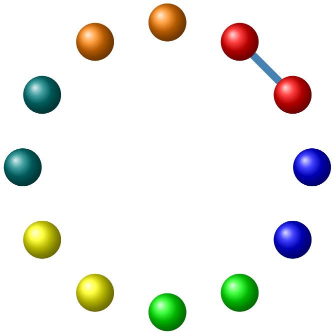

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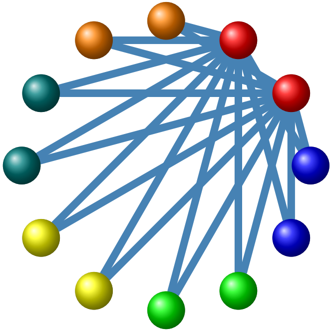

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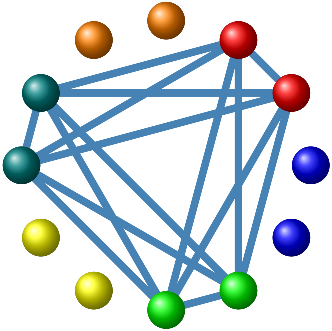
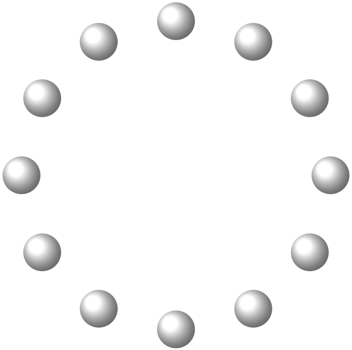


Making all Matchings Colorful

(n, k)-perfect hash function family of size q

$\phi_1, \dots, \phi_q : [n] \to [k]$ such that for all $W \subseteq [n]$ with $|W| = k$ there is some $i \in [q]$ with ϕ_i bijective on W



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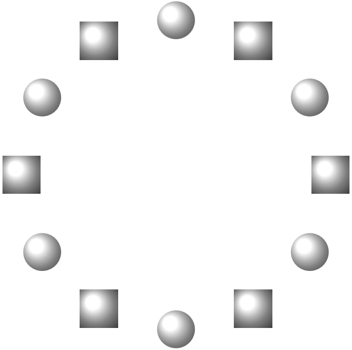

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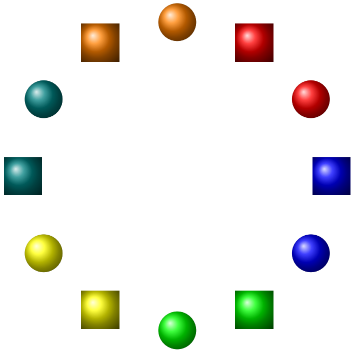

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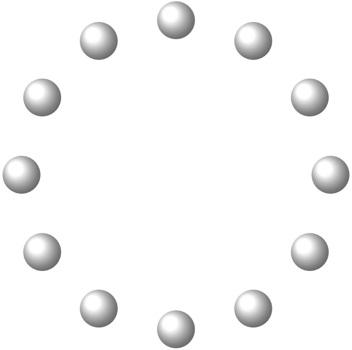

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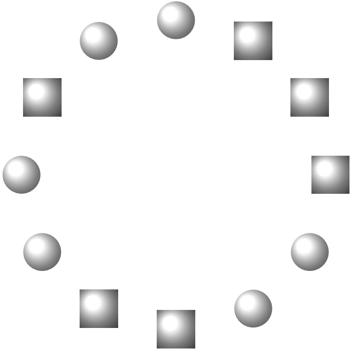

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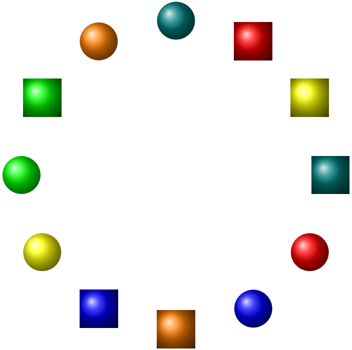

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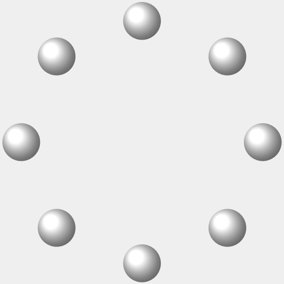

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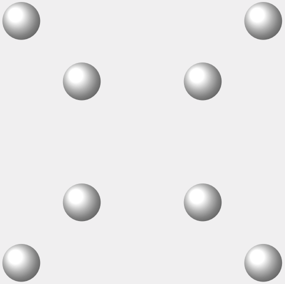

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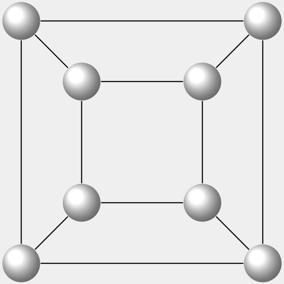

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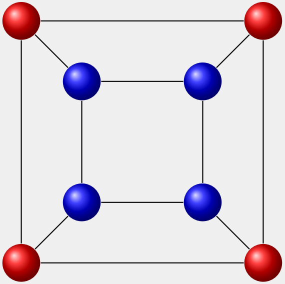

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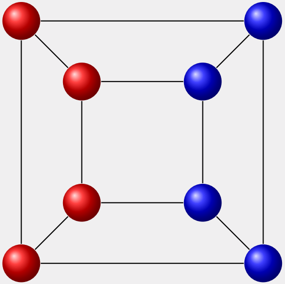

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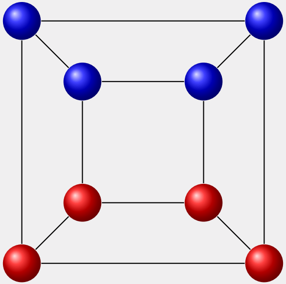

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K, Pashkovich, Theis 2010

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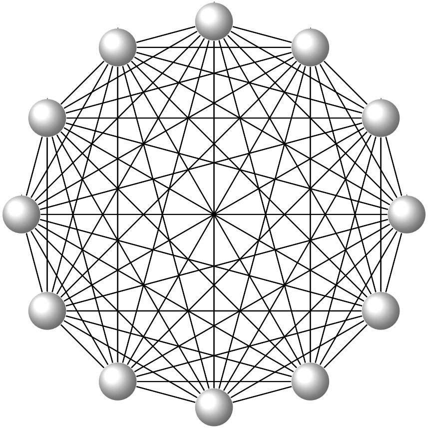
Rothvoss 2014

$$\mathrm{xc}(P_{\text{match}}(n)) \geq 2^{\Omega(n)}$$

Outline

Special Cycles



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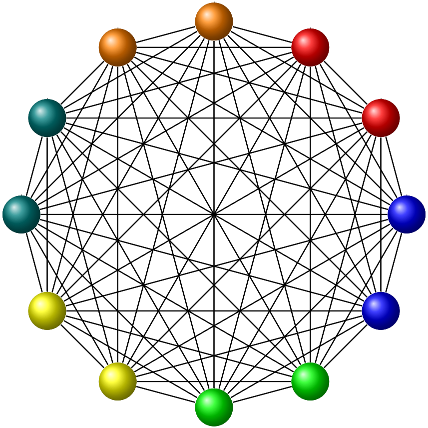

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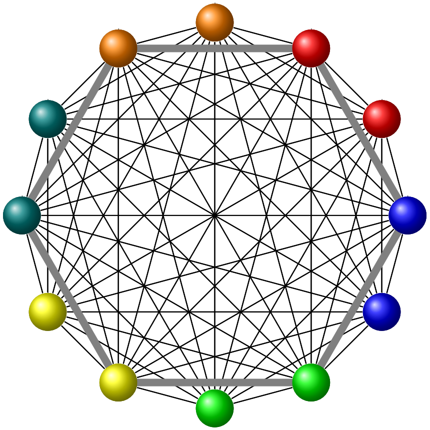

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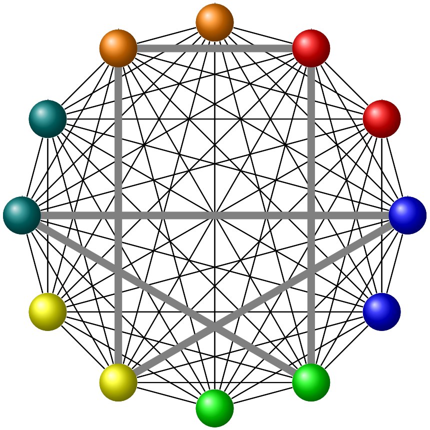

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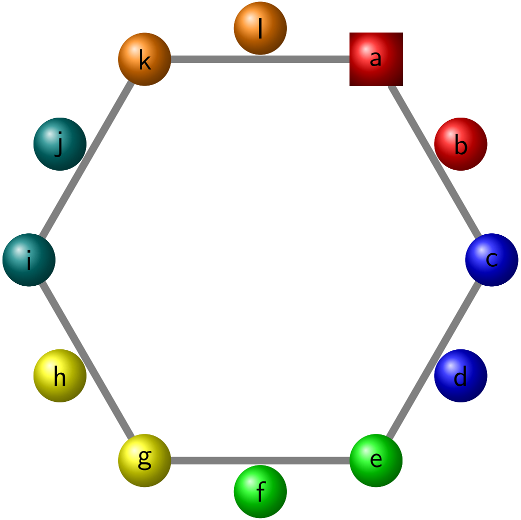

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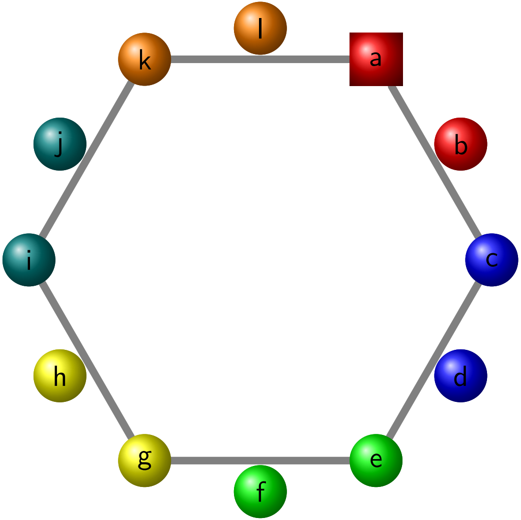

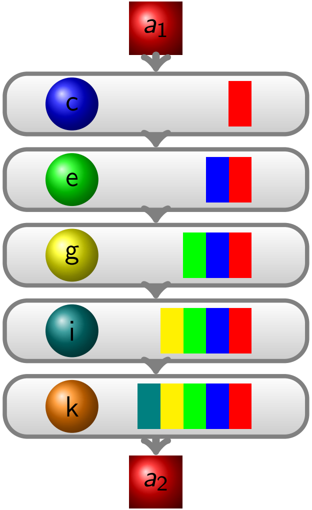

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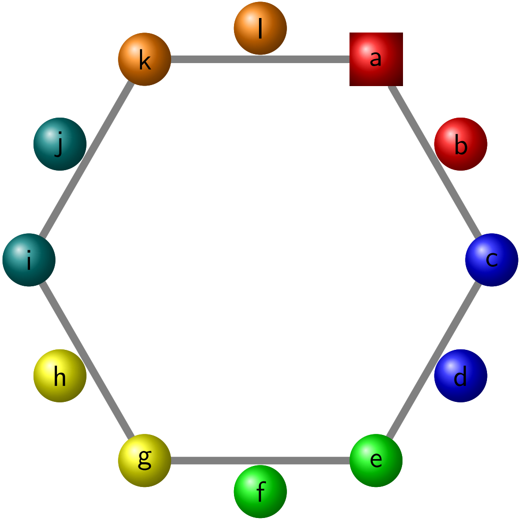

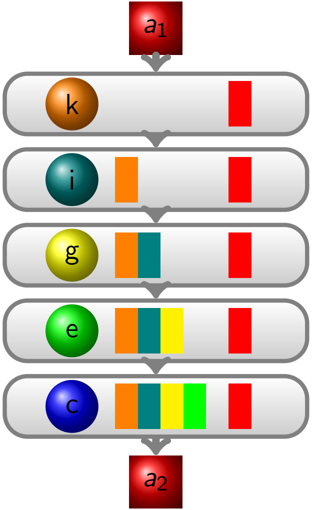

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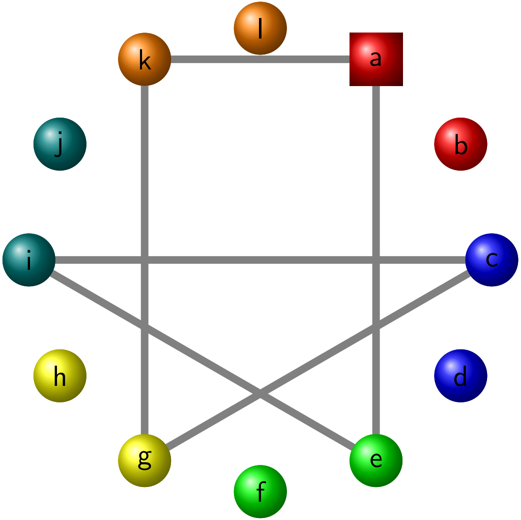

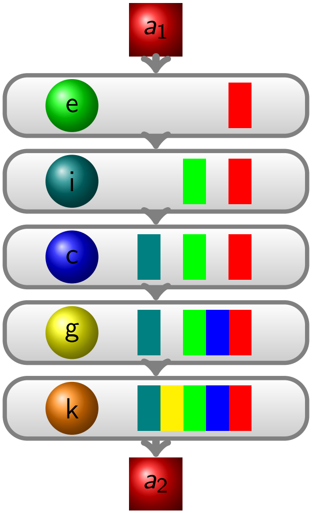

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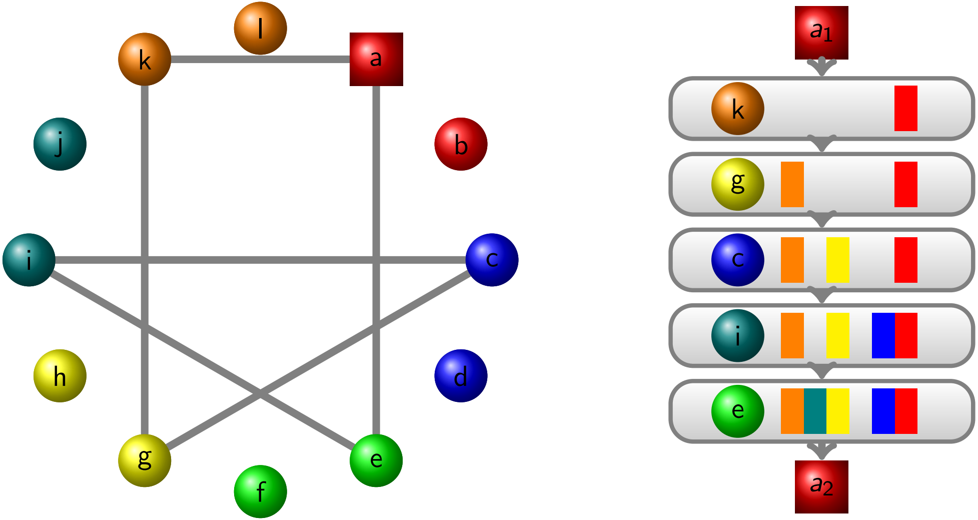
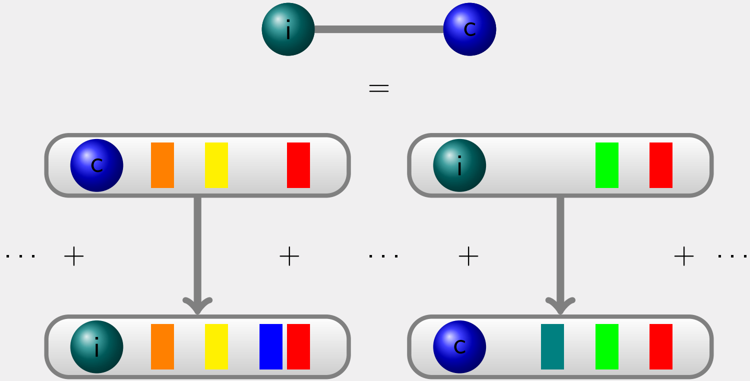


Colorful Cycle Polytopes (Prescribed Node)

Extended Formulation via

- a_1-a_2 flows of value one
- and projection



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Fiorini, Massar, de Wolff, Tiwary, Pokutta 2012

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Hyperpath Polytopes

Martin, Rardin, Campbell 1990

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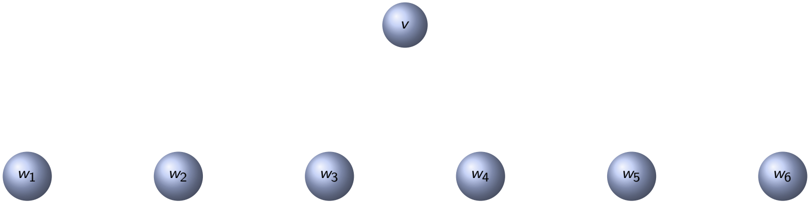

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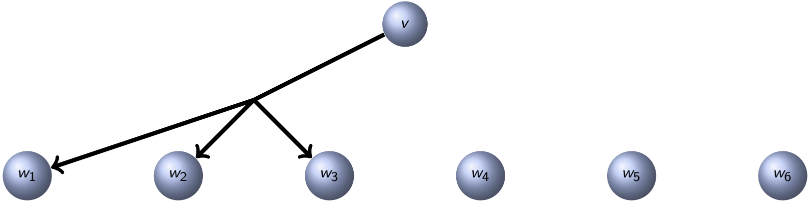

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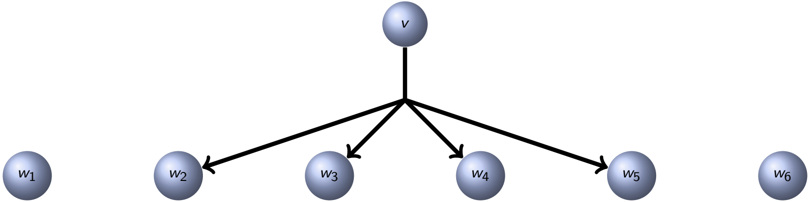

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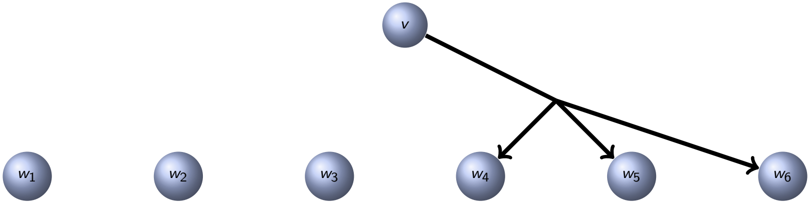
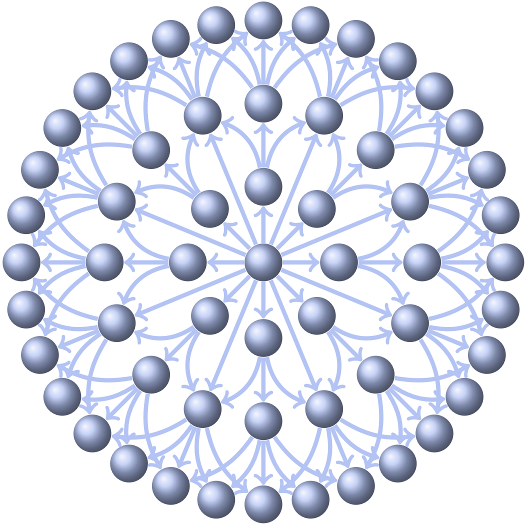


Outline

1. The Concept
2. Disjunctive Programming
3. Dynamic Programming
4. Branched Polyhedral Systems
5. Dualization
6. Redundant Information
7. Reflections

Branched Combinatorial/Polyhedral Systems



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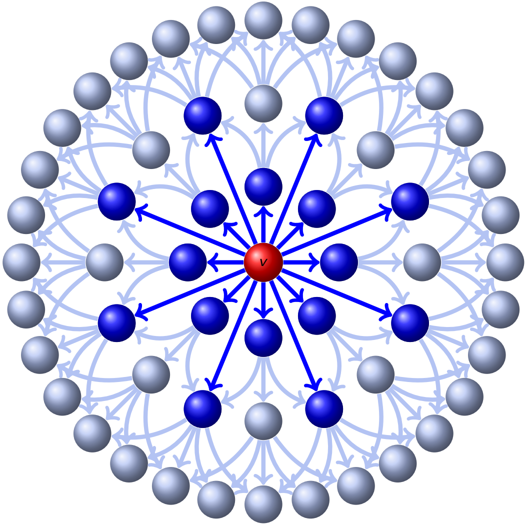

For each non-sink v

$$\mathcal{S}^{(v)} \subseteq 2^{\mathrm{N}^{\mathrm{out}}(v)}$$

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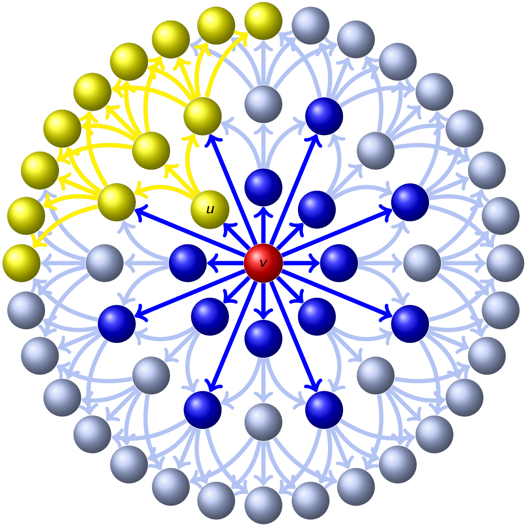

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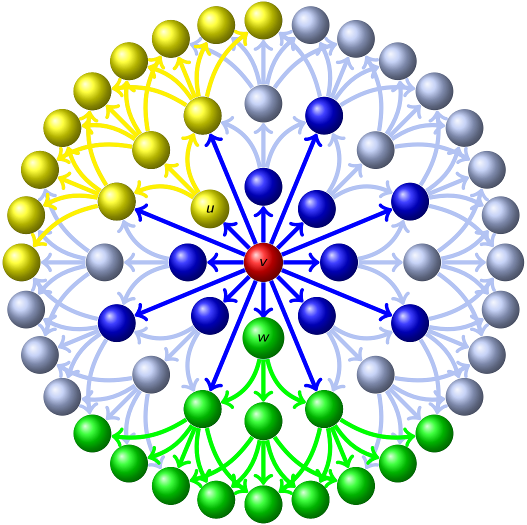

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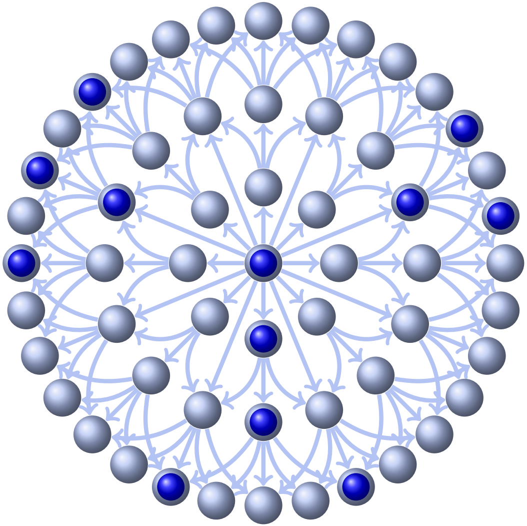

Feasible set $F \subseteq V$

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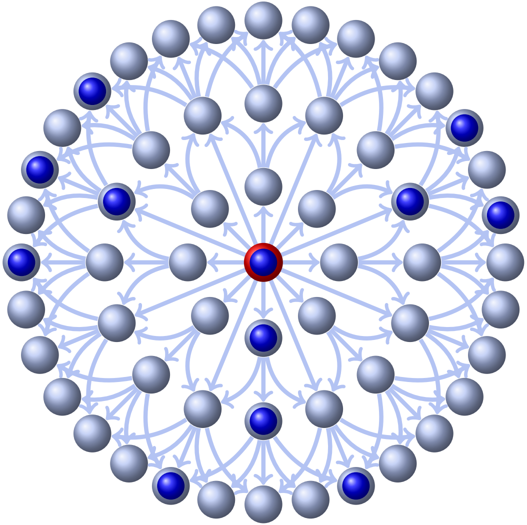

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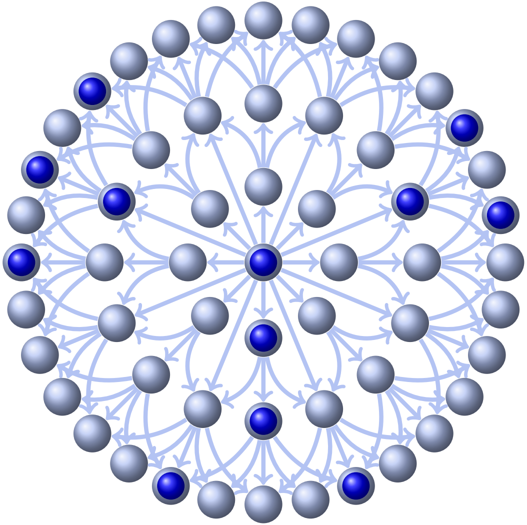

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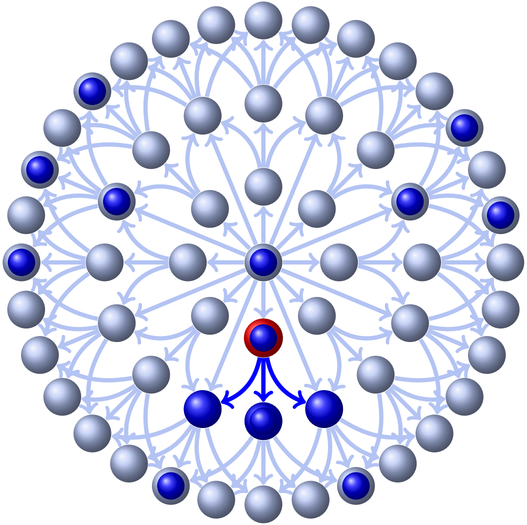

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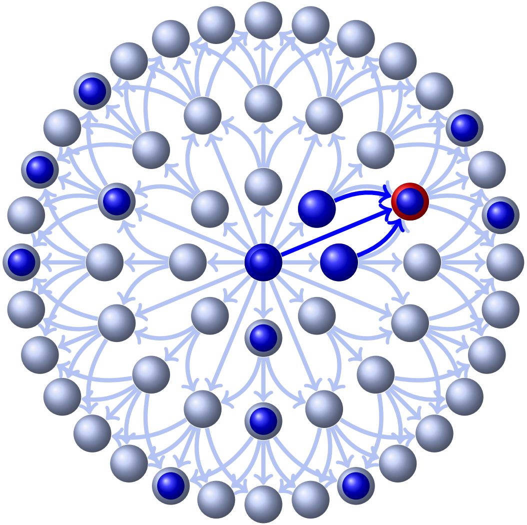

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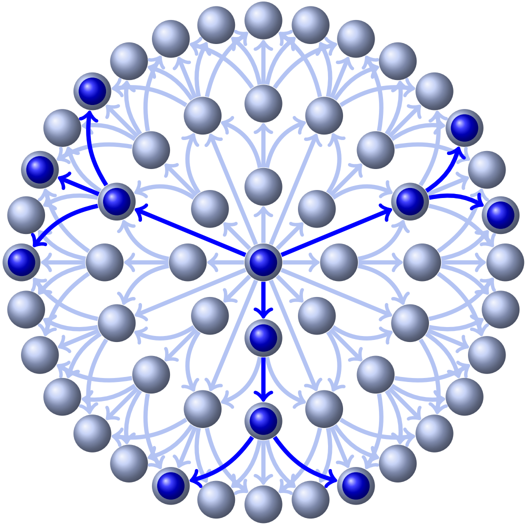

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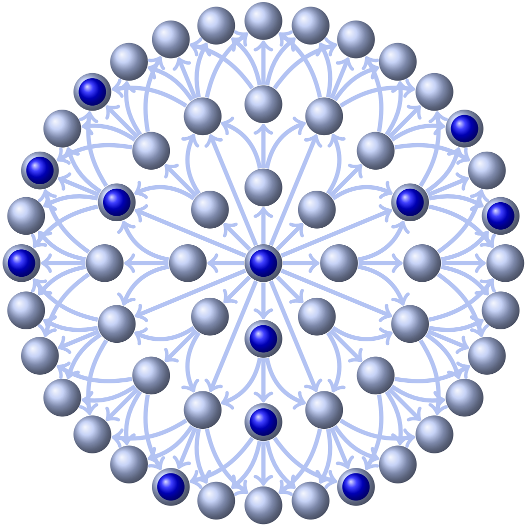


0/1-Polytope $P(\mathcal{B})$

$$\mathrm{conv}(\{\chi(F) : F \text{ feasible}\})$$

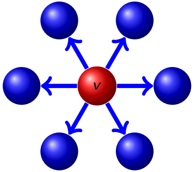
Extended Formulation

K & Loos 2010

$P(\mathcal{B})$ is described by the following extended formulation:

$$
\begin{aligned}
x_s &= 1 && \\
x_v &= y(\delta^{\text{in}}(v)) && \text{for all } v \neq s \\
A^{(v)} y_{\delta^{\text{out}}(v)} - x_v b^{(v)} &\leq 0 && \text{for all non-sinks } v \\
x_v &\geq 0 && \text{for all non-sinks } v
\end{aligned}
$$

(if $A^{(v)} z \leq b^{(v)}$ describe $P^{(v)} = \text{conv}(\{\chi(S) : S \in \mathcal{S}^{(v)}\})$)



Outline

Extended Formulatations via Duality

Observation due to Martin (1991)

$$\mathrm{xc}\{x : Ax \le \beta \cdot \mathbf{1}\} \le \mathrm{xc}(\mathrm{conv}\{\text{rows of } A\}) + 1$$

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Background: LP-Duality

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Application to Spanning Tree Polytopes

$$\mathrm{P}_{\mathrm{spt}}(G) = \{x \in \mathbb{R}^E_+ \, : \, x(E) = |V| - 1, \, x(E(U)) \le |U| - 1 \text{ for all } \varnothing \ne U \subsetneq V\}$$

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$$x(E(U)) \le |U| - 1 \iff (\chi(U)^{\mathrm{t}}, \chi(F)^{\mathrm{t}}) \begin{pmatrix} -\mathbf{1}_V \\ x \end{pmatrix} \le -1 \;\forall F \subseteq E(U)$$

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Extended Formulations for Non-Empty-Subgraphs Polytopes

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Martin 1991

$$\operatorname{xc}(\mathrm{P}_{\text{spt}}(G)) \le |V| \cdot (2|V| + 2|E|) + |E| + 1 = 2|V||E| + 2|V|^2 + |E| + 1$$

Non-Extended Formulations of Nonempty-Subgraphs Polytopes

Conforti, K, Walter, Weltge (2015)

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Corollary

$$|\,\mathrm{xc}(\mathrm{P}_{\text{spt}}(G)) - \mathrm{xc}(\mathrm{P}_{\text{ne-sub}}(G))| \le 2|V| + |E|$$

Graphs of Bounded Genus

Djidjev & Venkatesan (1995), Hutchinson & Miller (1987)

Each graph G of genus g has a subset of

$$O(\sqrt{g|V|})$$

nodes whose removal leaves a planar graph.

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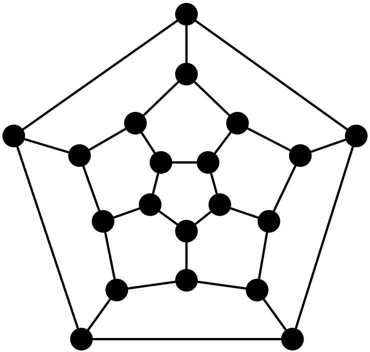
Fiorini, Huynh, Joret, Pashkovich (2016)

Each graph G of genus g has

$$\mathrm{xc}(\mathrm{P}_{\mathrm{spt}}(G)) \leq O(\sqrt{g} \cdot |E|\sqrt{|V|})\,.$$

Outline

Spanning Trees in Planar Graphs



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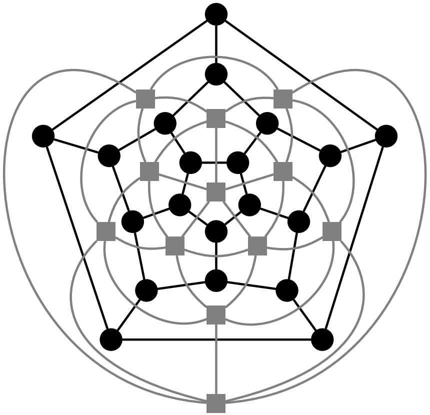

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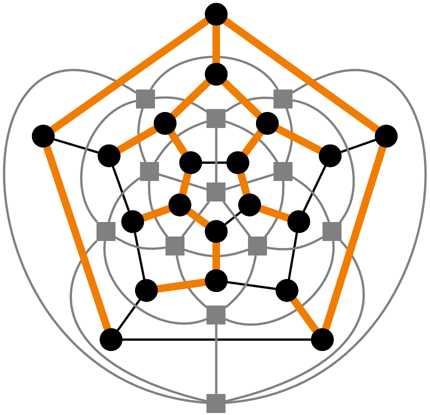

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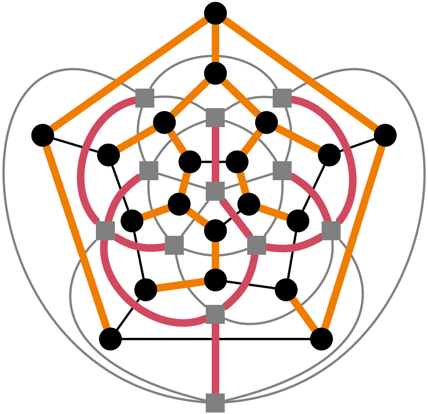

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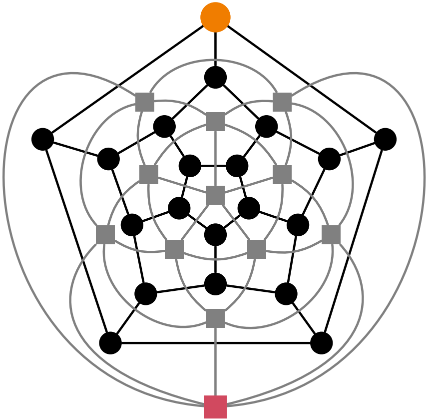

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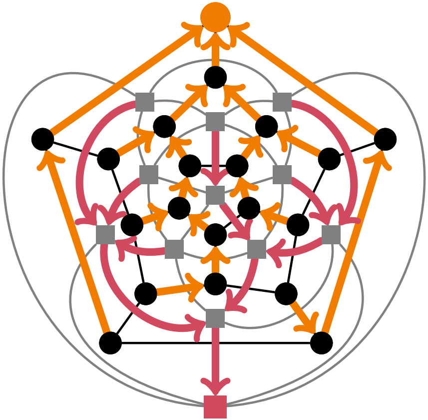

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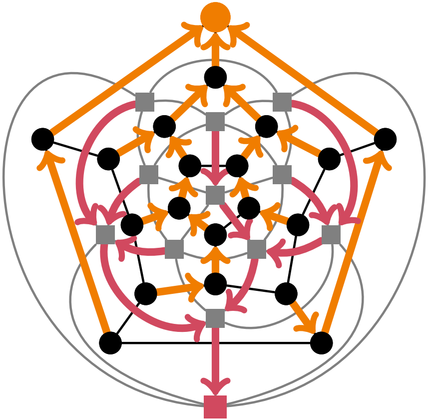

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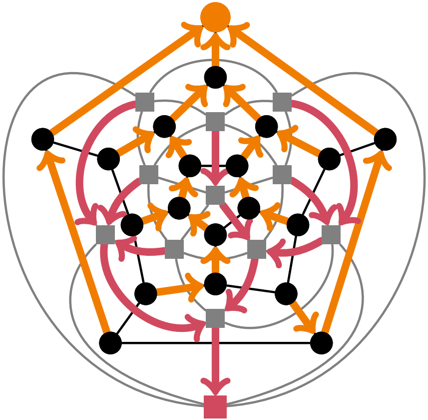
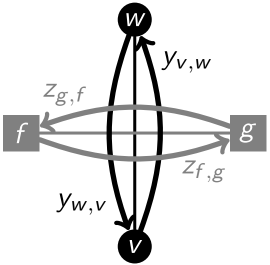


$$\mathrm{P}_{\mathrm{arb}}(G) = \mathrm{conv}\{(\chi(\vec{T}), \chi(\vec{T}^{\star})) \,:\, T \text{ spanning tree of } G\}$$
$$\mathrm{P}_{\mathrm{spt}}(G) = p(\mathrm{P}_{\mathrm{arb}}(G)) \text{ with linear projection } p$$

Linear Description of $\mathrm{P}_{\mathrm{arb}}(\cdot)$



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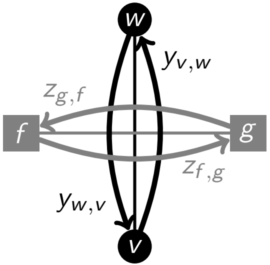

Williams 2001

$$
\begin{aligned}
y_{v,w} + y_{w,v} + z_{f,g} + z_{g,f} &= 1 && \forall \{v, w\} \in E \\
\textstyle\sum_w y_{v,w} &= 1 && \forall v \neq \text{root} \\
\textstyle\sum_g z_{f,g} &= 1 && \forall w \neq \text{root} \\
y_{v,w}, y_{w,v}, z_{f,g}, z_{g,f} &\geq 0 && \forall \{v, w\} \in E
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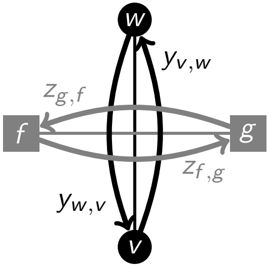

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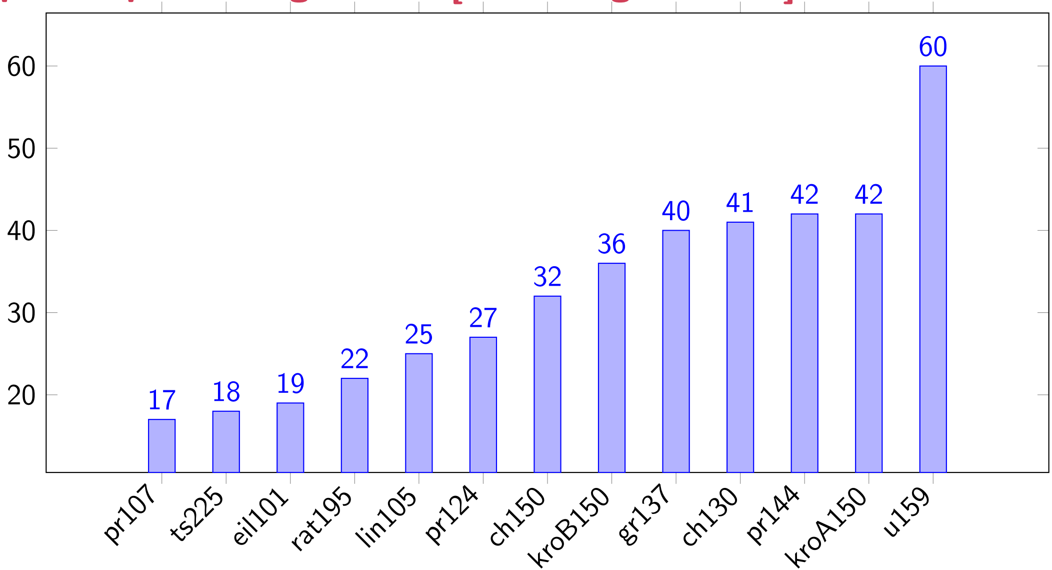
Thus $\mathrm{xc}(\mathrm{P}_{\mathrm{spt}}(G)) \leq \mathrm{O}(n)$ for planar G on n nodes.

Speed-Ups for Degree ≤ 3 [K & Sorgatz 2012]



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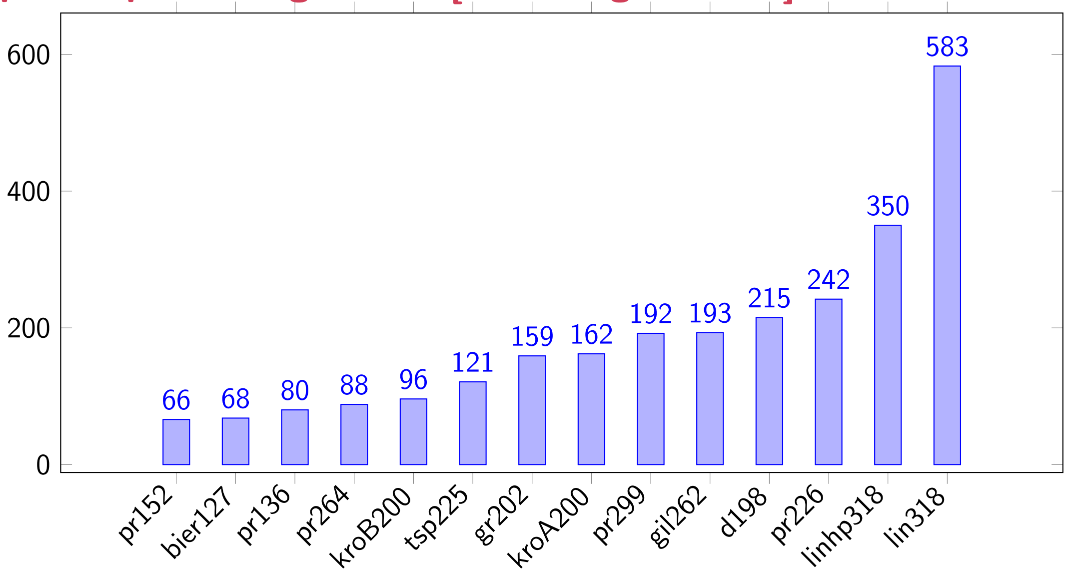

Polynomial Spanning Tree Optimization

Setup for $G = (V, E)$, $\mathcal{M} \subseteq 2^E$ (acyclic subsets)

- $\Omega(\mathcal{M}) := \{(x, (y_M)_{M \in \mathcal{M}}) \,:\, x$ incidence vector of a spanning tree in G, $y_M = \prod_{e \in M} x_e$ for all $M \in \mathcal{M}\}$
- Linear optimization over $\Omega(\mathcal{M}) \iff$ Optimization of polynomials with support in $\mathcal{M}$
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Relaxations

- $\mathcal{M}' \subseteq \mathcal{M}$: $\Omega(\mathcal{M}', \mathcal{M}) := \{(x, (y_M)_{M \in \mathcal{M}}) : x\ (\dots), y_M = \prod_{e \in M} x_e$ for all $M \in \mathcal{M}'\}$
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- For $\mathcal{M} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_t$: $P(\mathcal{M}) \subseteq P(\mathcal{M}_1, \mathcal{M}) \cap \dots \cap P(\mathcal{M}_t, \mathcal{M})$

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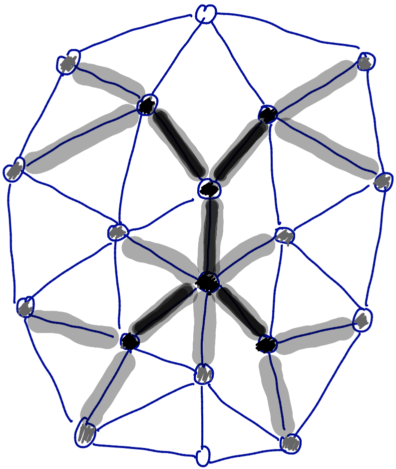
Disjunctive extended formulation

$$\text{xc}(P(\mathcal{M})) \leq |V|^{\text{width of } \mathcal{M}} \cdot \text{xc}(\text{P}_{\text{spt}}(G))$$

A Single Chain $\mathcal{M}$ of Trees



$[M_1 \subsetneq M_2 =: \overline{M}]$

Fischer, Fischer, McCormick 2016

$P(\mathcal{M})$ is described by

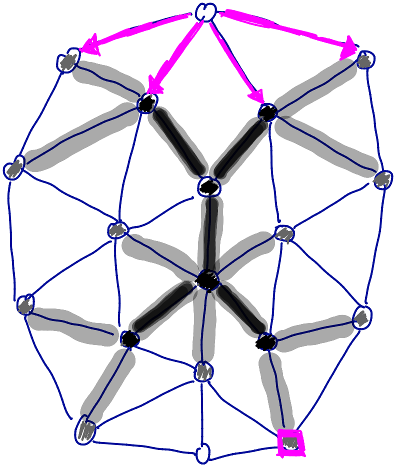
- $x \in \mathrm{P_{spt}}(G)$, $y \geq \mathbf{0}$
- $y_i \leq x_e$ for all $i, e \in M_i \setminus M_{i-1}$
- $y_i - y_{i-1} \geq \sum_{e \in M_i \setminus M_{i-1}} -|M_i \setminus M_{i-1}|$ for all i
- $y_i \leq y_{i-1}$ for all i
- $x(\cup_j E[S_j]) + \sum_i \beta_i y_i \leq \sum_j ([S_j| - 1)$ for all pairwise disjoint S_j

(Similarly even for general matroids.)

For a single pair of edges see also:

- Buchheim & Klein 2014
- Fischer & Fischer 2013

The Extended Formulation $Q(\mathcal{M})$



$[M_1 \subseteq M_2 =: \overline{M}]$

Friesen & K 2017

$$\mathrm{xc}(P(\mathcal{M})) \leq \mathrm{xc}(\mathrm{P}_{\mathrm{spt}}(G)) + \mathrm{O}(|\overline{M}| \cdot |E|)$$

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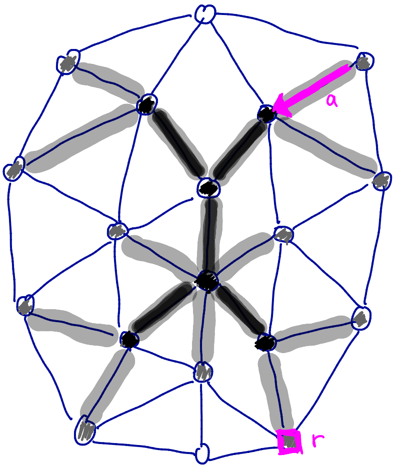

$[M_1 \subseteq M_2 =: \overline{M}]$ $\quad y_2 \leq z_a^r$

Friesen & K 2017

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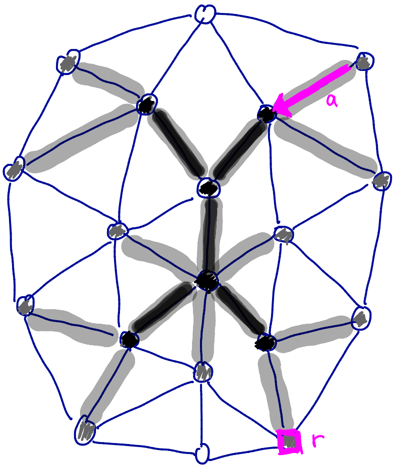


$[M_1 \subseteq M_2 =: \overline{M}]$ $y/2 \leq z_a^r$

Friesen & K 2017

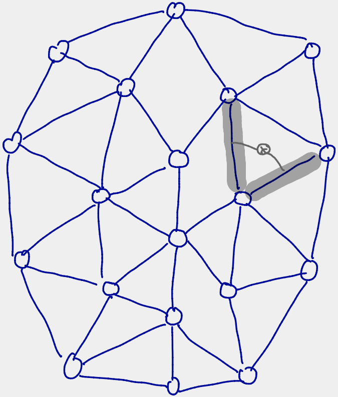
$$\mathrm{xc}(P(\mathcal{M})) \leq \mathrm{xc}(\mathrm{P_{spt}}(G)) + O(|\overline{M}| \cdot |E|)$$

Disjunction yields:

$$\mathrm{xc}(P(\mathcal{M})) \leq O(|\overline{M}|) \cdot \mathrm{xc}(\mathrm{P_{spt}}(G))$$

A Single Product

One adjacent pair



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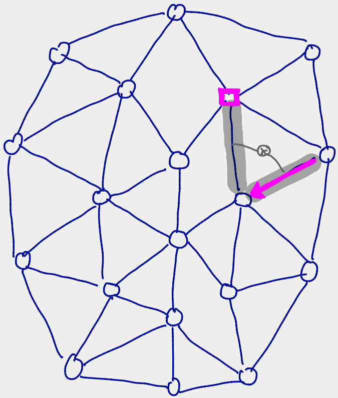

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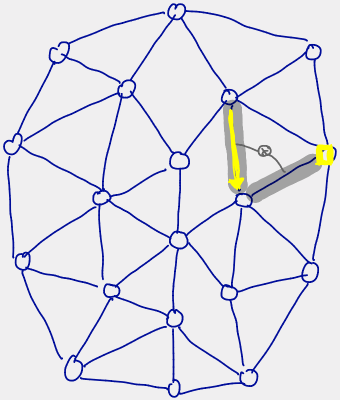

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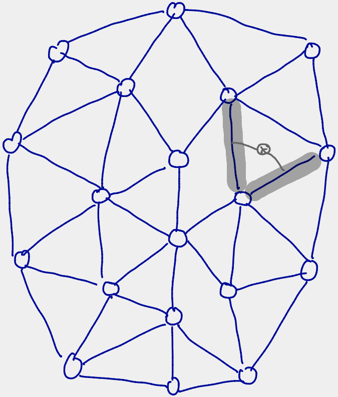
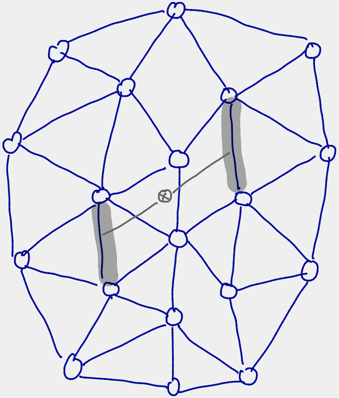


One non-adjacent pair



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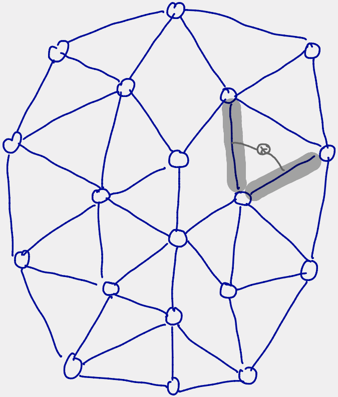

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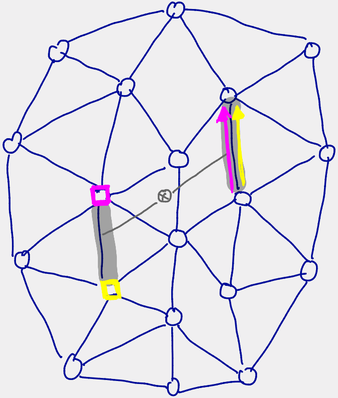

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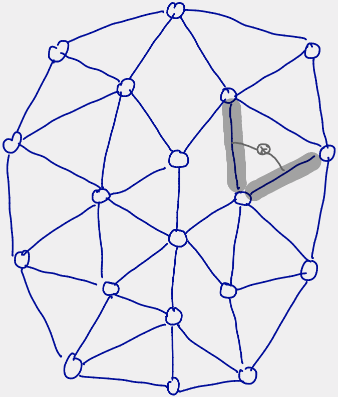

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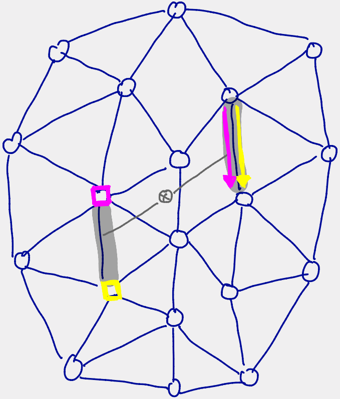

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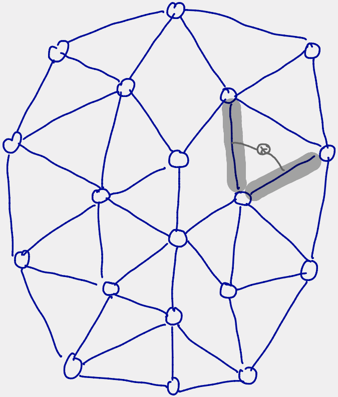

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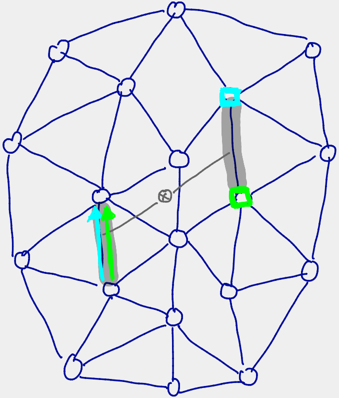

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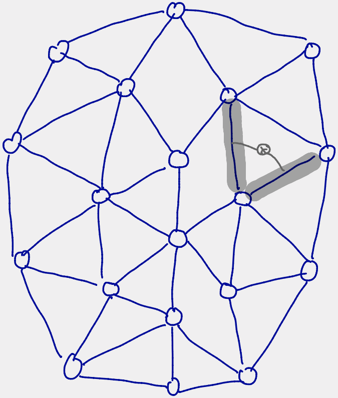

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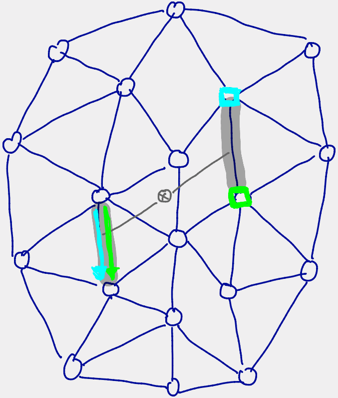
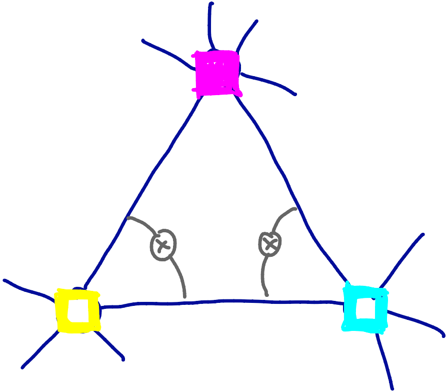


A Strengthened Relaxation



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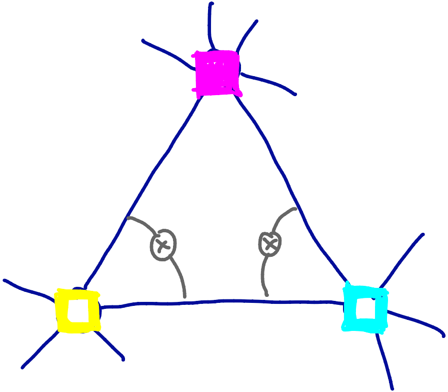


Friesen & K 2017

The relaxation provided by

$$Q(\mathcal{M}_1, \mathcal{M}) \cap \cdots \cap Q(\mathcal{M}_t, \mathcal{M})$$

with re-used z-variables is in general stronger than $P(\mathcal{M}_1, \mathcal{M}) \cap \cdots \cap P(\mathcal{M}_t, \mathcal{M})$.

An Approach to Obtain Relaxations

For polytope $P \subseteq \mathbb{R}^d$

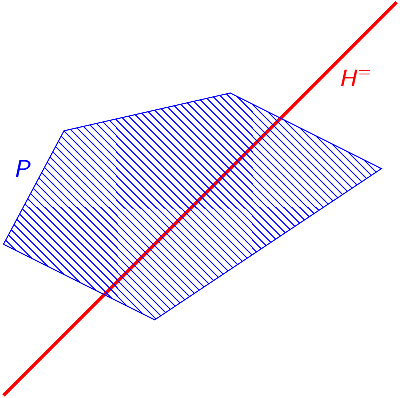
1. Choose $P_1, \dots, P_r \supseteq P$.
2. Construct extensions Q_i of P_i with preimages $z_i(v)$ of the vertices v of P.
3. Identify valid linear inequalities for the $(v, z_1(v), \dots, z_r(v))$'s defining a polyhedron S.

Then $\{(x, z_1, \dots, z_r) \in S \,:\, z_i \in Q_i\}$ is an extension of a polyhedron R with

$$P \subseteq R \subseteq P_1 \cap \dots \cap P_r \,.$$

Outline

The Reflection Operation



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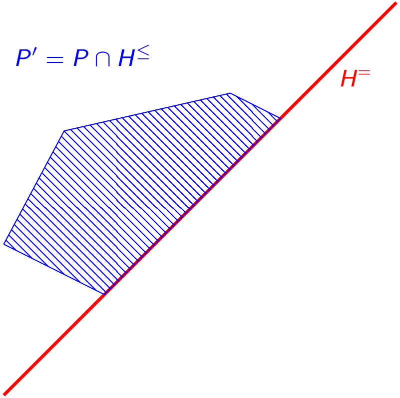

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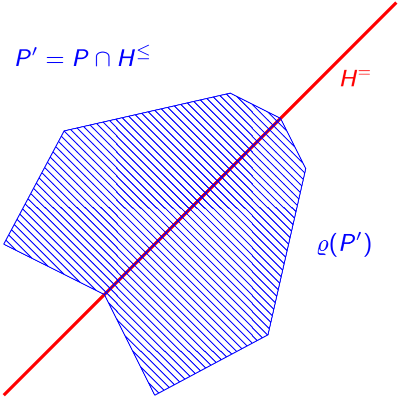

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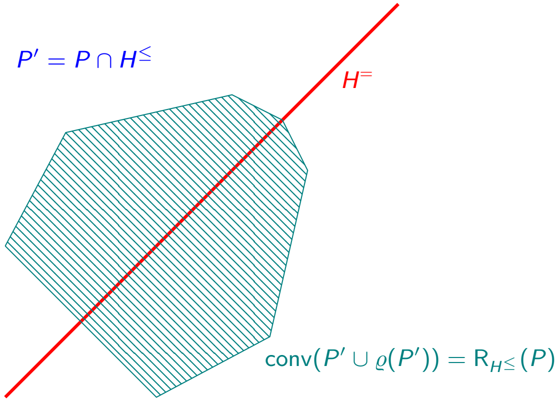

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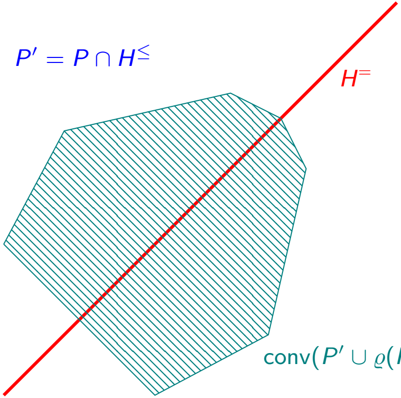

- $\mathrm{R}_{H^{\leq}}(P) = \{x + \lambda a : x \in P, \langle a, x\rangle \leq \langle a, x + \lambda a\rangle \leq 2b - \langle a, x\rangle\}$

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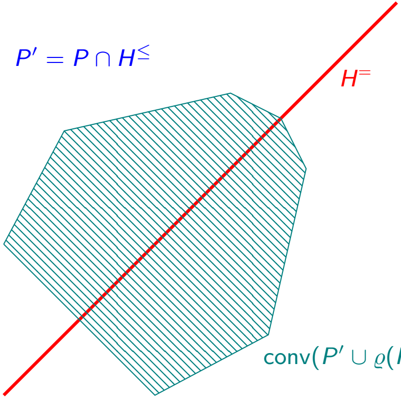

- $\mathrm{R}_{H^{\leq}}(P) = \{x + \lambda a \,:\, x \in P, \langle a, x\rangle \leq \langle a, x + \lambda a\rangle \leq 2b - \langle a, x\rangle\}$
- Thus: $\mathrm{xc}(\mathrm{R}_{H^{\leq}}(P)) \leq \mathrm{xc}(P) + 2$

Sequences of Reflection Operations

Consequence

For each sequence $H_1^{\leq}, \dots, H_r^{\leq} \subseteq \mathbb{R}^n$ of halfspaces and for each polytope $P \subseteq \mathbb{R}^n$, the polytope

$$\mathcal{R}_{H_1^{\leq},\dots,H_r^{\leq}}(P) = \mathrm{R}_{H_r^{\leq}}(\mathrm{R}_{H_{r-1}^{\leq}}(\dots \mathrm{R}_{H_1^{\leq}}(P)\dots))$$

satisfies

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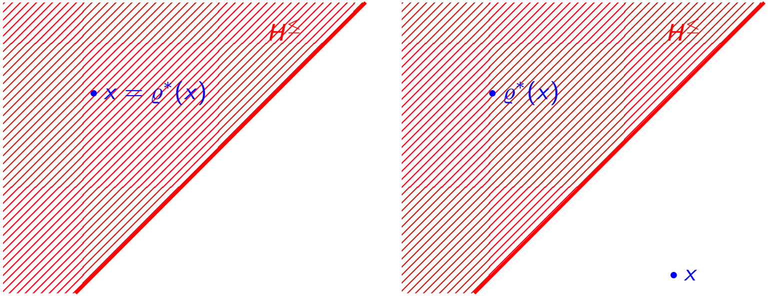
Task for target polytope Q

Find (and describe) P, design sequence $H_1^{\leq}, \ldots, H_r^{\leq}$, and prove

$$Q = \mathcal{R}_{H_r^{\leq},\ldots,H_1^{\leq}}(P)\,.$$

Conditional Reflections

Define $\varrho^\star : \mathbb{R}^n \to \mathbb{R}^n$ via $\varrho^*(x) = \begin{cases} x & \text{if } x \in H^{\leq} \\ \varrho(x) & \text{otherwise} \end{cases}$.



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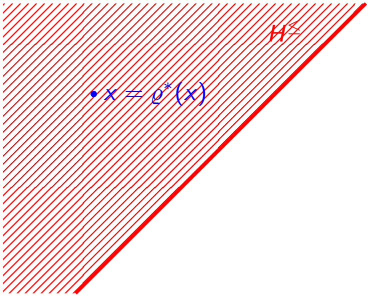

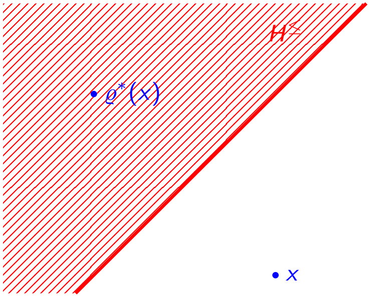

$$\varrho^*(x) \in P \quad \Rightarrow \quad x \in \mathrm{R}_{H^{\leq}}(P)$$

Generating the Target Polytope

K & Pashkovich 11

Let

- $Q = \mathrm{conv}(W)$ be some (target) polytope,
- $H_1^{\leq}, \dots, H_r^{\leq} \subseteq \mathbb{R}^n$ be a sequence of halfspaces, and
- ϱ_i (and $\varrho_i^\star$) the associated (conditional) reflections.

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whenever the following two conditions are satisfied:

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K & Pashkovich 11

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2. $\varrho_1^\star(\varrho_2^\star(\cdots(\varrho_r^\star(w)\cdots)) \in P$ for all $w \in W$.

Reflection Groups

Finite Reflection Group G

A *finite* group generated by a (finite) family $\varrho^{H_i} : \mathbb{R}^n \to \mathbb{R}^n$ $(i \in I)$ of reflections at hyperplanes $\mathbf{0} \in H_i \subseteq \mathbb{R}^n$.

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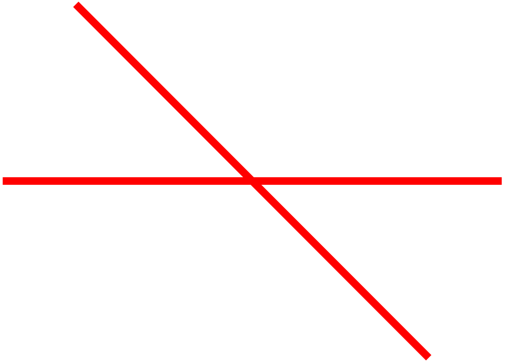
G-Permutahedron of polytope P in one region

$$\mathrm{P}^G_{\mathrm{perm}}(P) = \mathrm{conv}(\bigcup_{g \in G} g.P)$$

Example: $I_2(4)$



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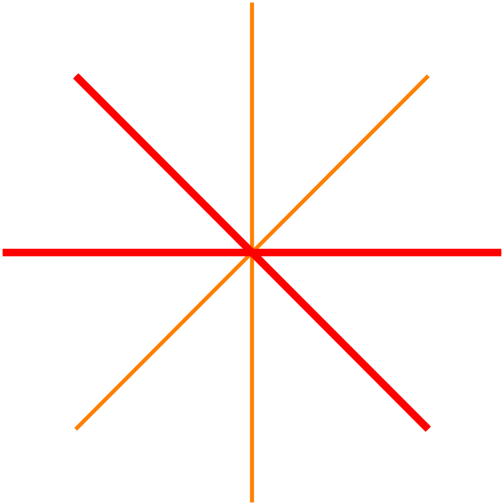

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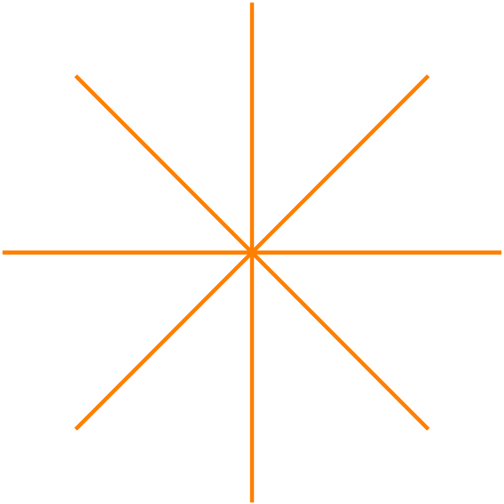

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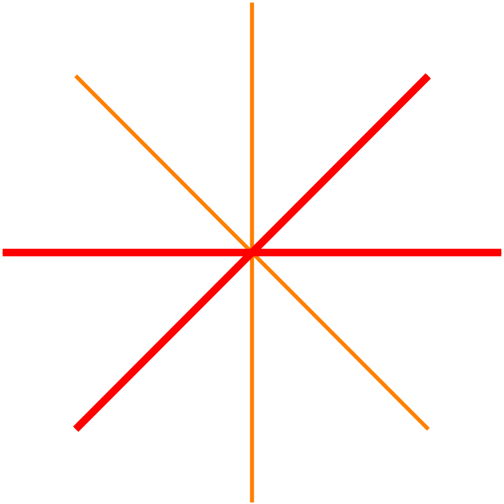

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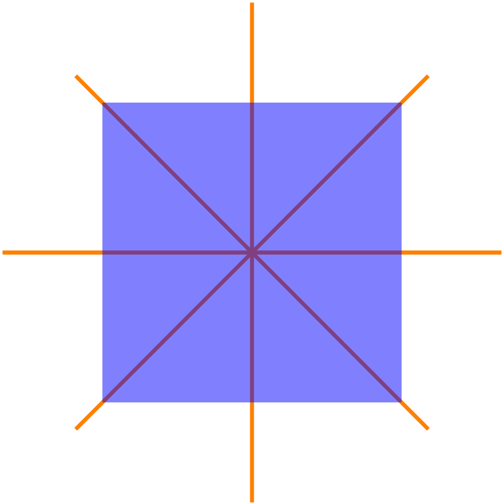

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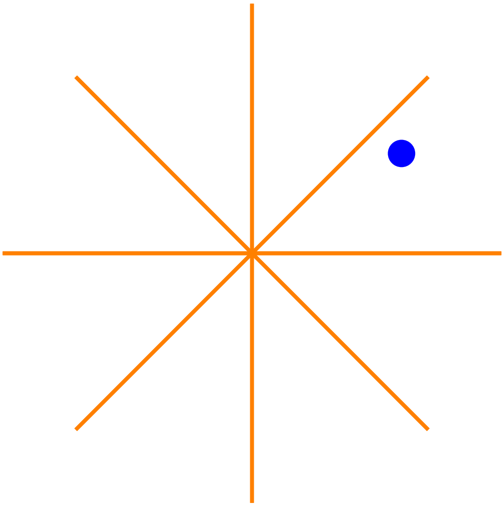

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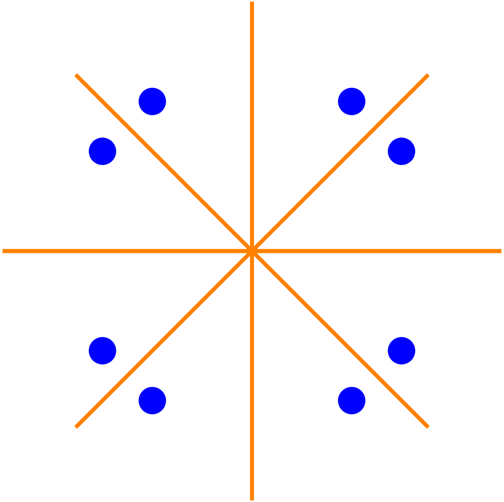

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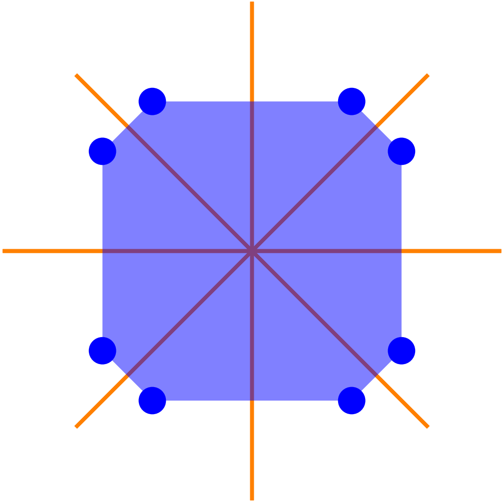
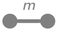
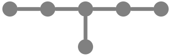
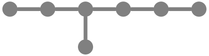
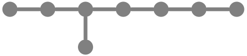


Classification of Irreducible Reflection Groups

Name	Dynkin Diagram	Regular Polytope
$I_2(m)$		m-gon
A_{n-1}		$(n-1)$-simplex
B_n		n-cube, n-cross polytope
D_n		
E_6		
E_7		
E_8		
F_4		24-cell
H_3		dodecahedron, icosahedron

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The group

- $H_\varphi = \mathrm{H}^{=}((-\sin\varphi, \cos\varphi), 0)$, $H_\varphi^{\leq} = \mathrm{H}^{\leq}((-\sin\varphi, \cos\varphi), 0)$

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- $H_\varphi = \mathrm{H}^=((-\sin\varphi, \cos\varphi), 0)$, $H_\varphi^\leq = \mathrm{H}^\leq((-\sin\varphi, \cos\varphi), 0)$
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If P lies in $\mathrm{FR}_{\mathrm{I}_2(m)}$:

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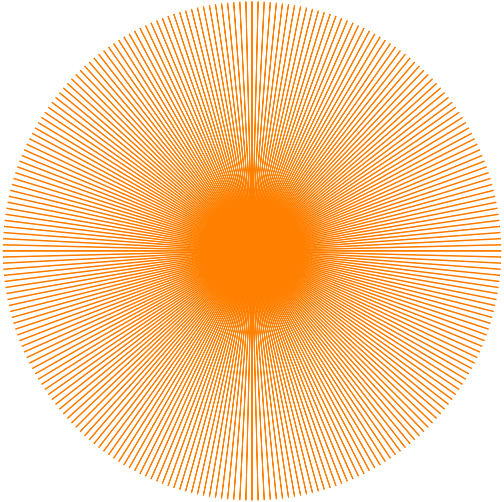
Thus we have:

$$\mathrm{xc}(\mathrm{P}_{\mathrm{perm}}^{I_2(m)}(P)) \leq \mathrm{xc}(P) + 2\lceil \log(m) \rceil + 2$$

Example: $\mathrm{I}_2(128)$



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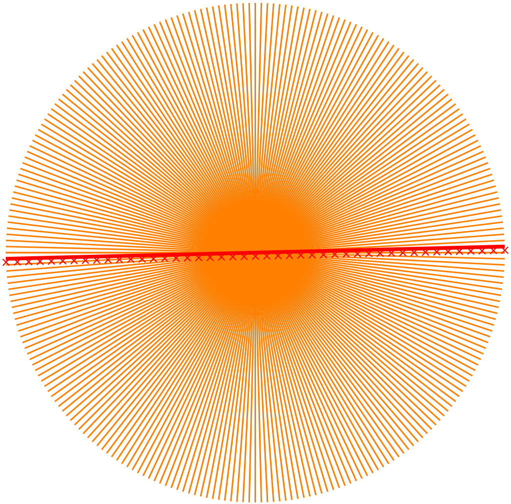

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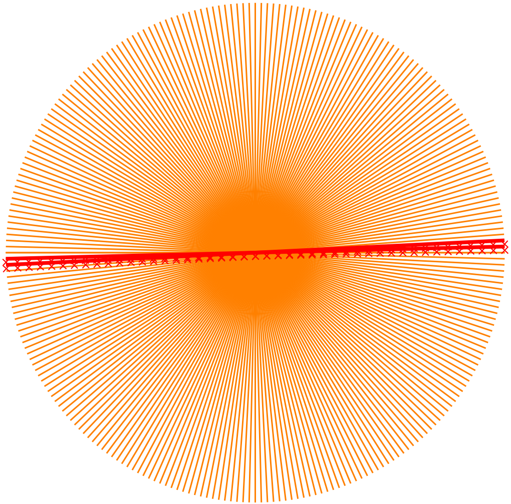

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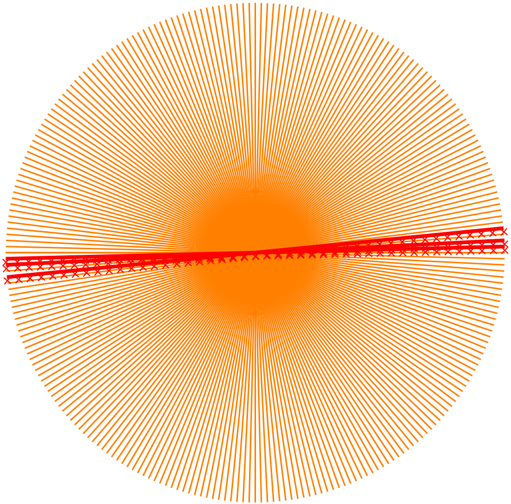

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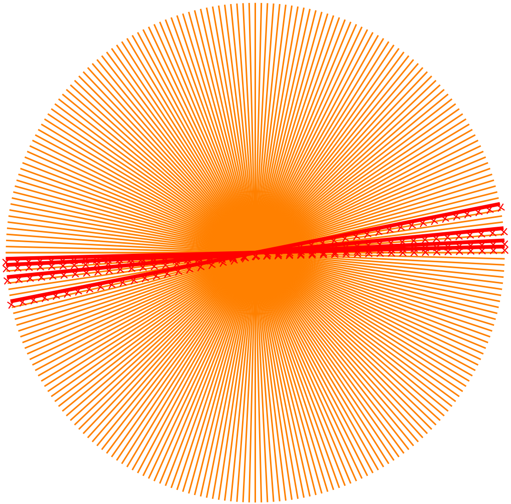

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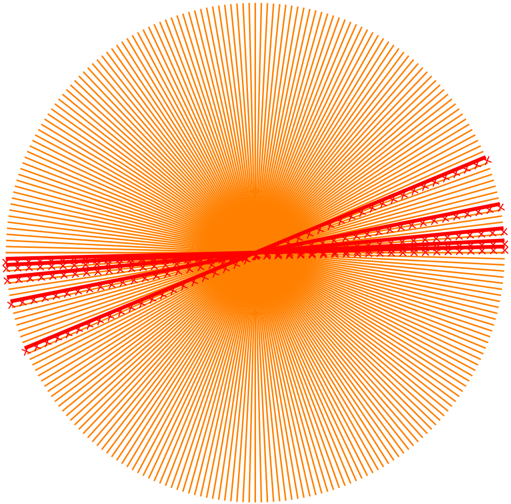

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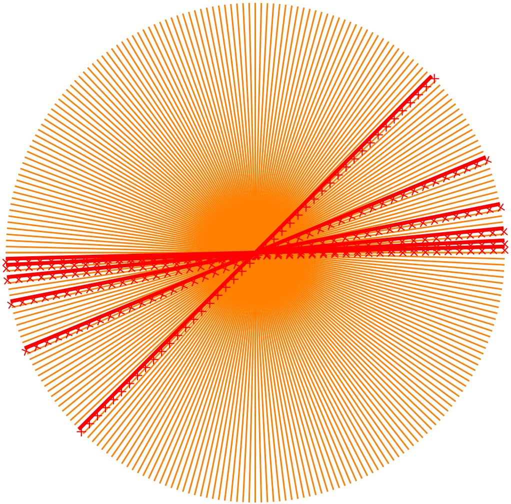

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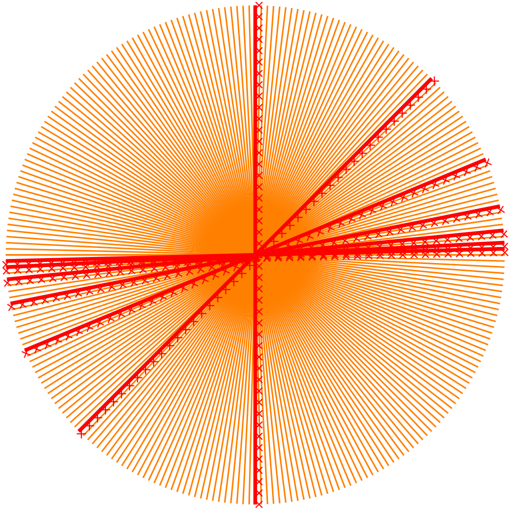

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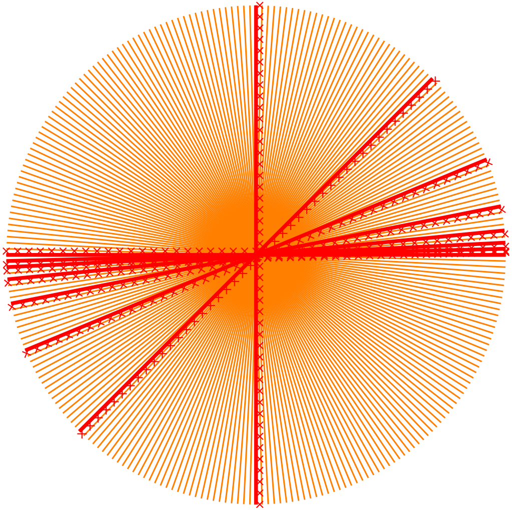

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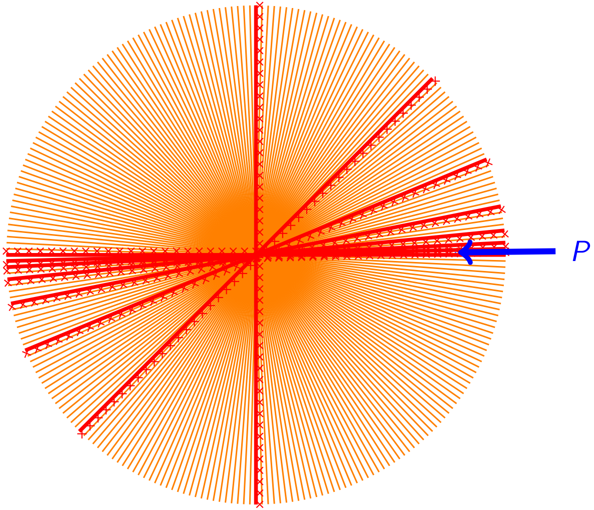

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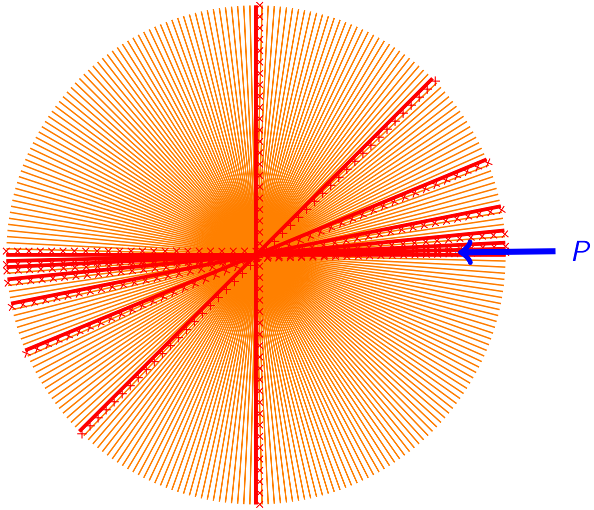

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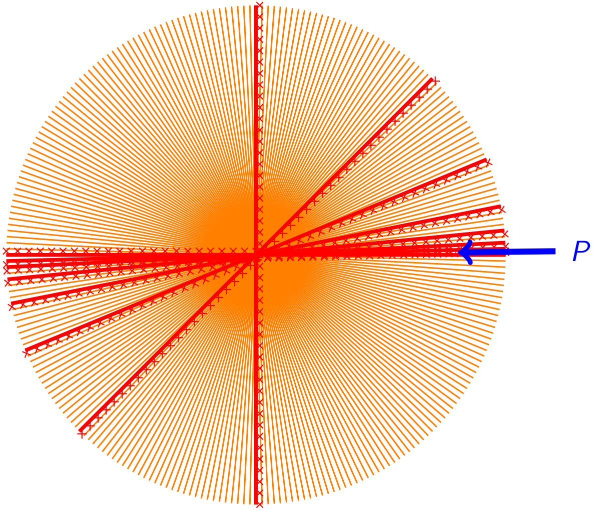

$\mathcal{R}_{H^{\leq}_{\pi/128}}(P)$

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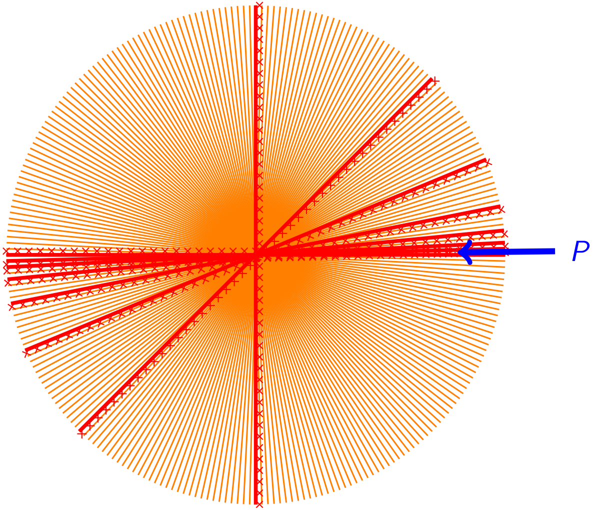

$\mathcal{R}_{H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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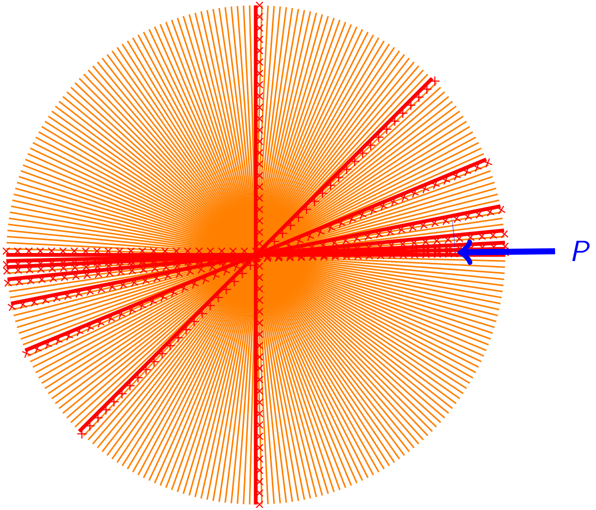

$\mathcal{R}_{H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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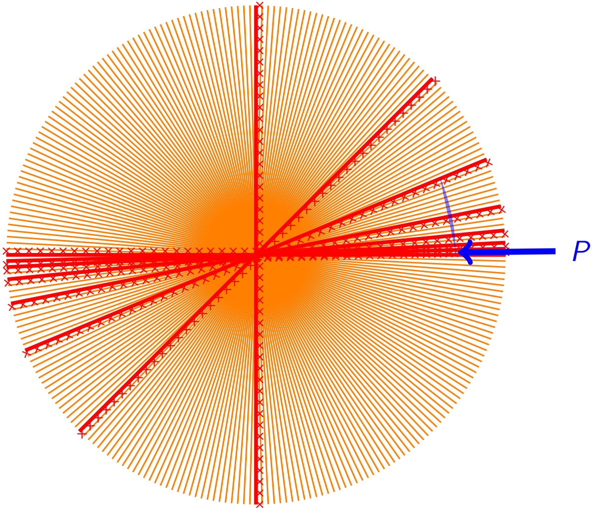

$\mathcal{R}_{H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

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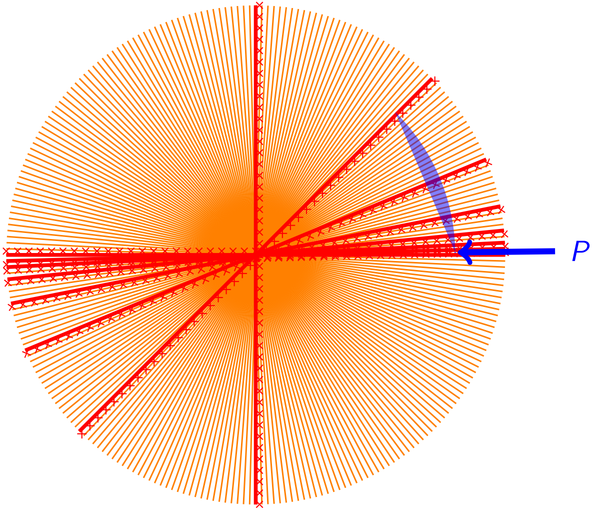
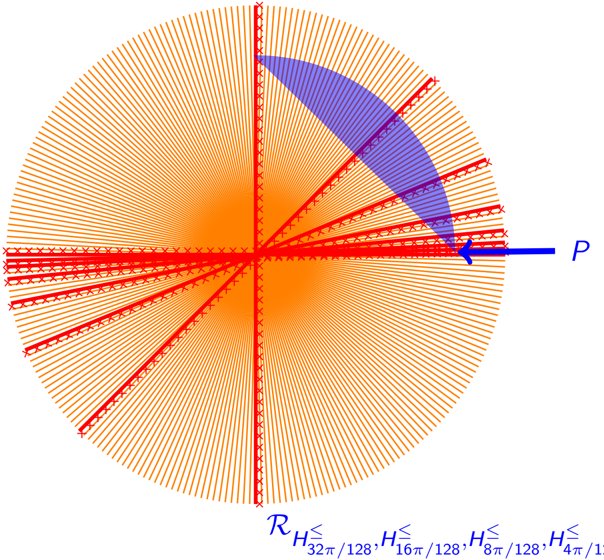


$\mathcal{R}_{H^{\leq}_{16\pi/128}, H^{\leq}_{8\pi/128}, H^{\leq}_{4\pi/128}, H^{\leq}_{2\pi/128}, H^{\leq}_{\pi/128}}(P)$

Example: $\mathrm{I}_2(128)$



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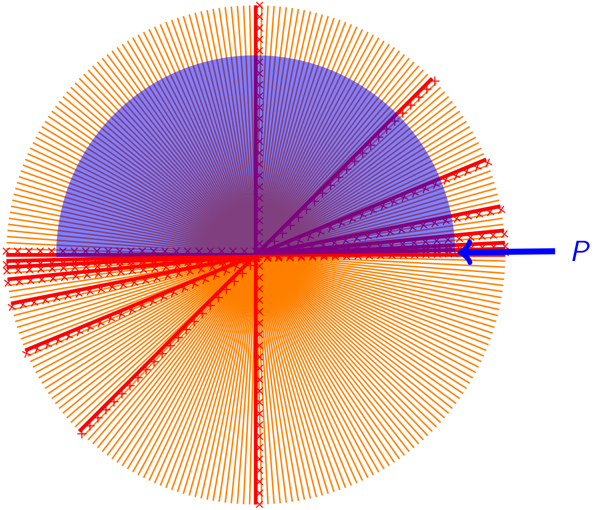

$$\mathcal{R}_{H^{\leq}_{64\pi/128},H^{\leq}_{32\pi/128},H^{\leq}_{16\pi/128},H^{\leq}_{8\pi/128},H^{\leq}_{4\pi/128},H^{\leq}_{2\pi/128},H^{\leq}_{\pi/128}}(P)$$

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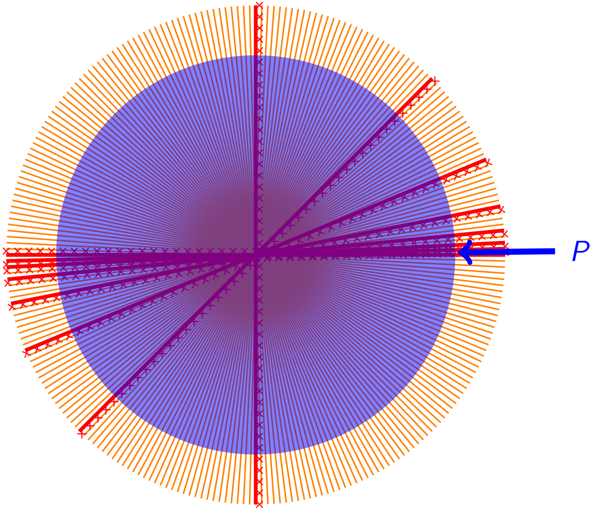

$$\mathcal{R}_{H^{\leq}_{128\pi/128},H^{\leq}_{64\pi/128},H^{\leq}_{32\pi/128},H^{\leq}_{16\pi/128},H^{\leq}_{8\pi/128},H^{\leq}_{4\pi/128},H^{\leq}_{2\pi/128},H^{\leq}_{\pi/128}}(P)$$

The Reflection Group A_{n-1}

The group

- $H_{k,\ell} = \mathrm{H}^{=}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$, $H^{\leq}_{k,\ell} = \mathrm{H}^{\leq}(\mathbb{e}_k - \mathbb{e}_\ell, 0) \quad \subseteq \mathbb{R}^n$

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The reflections

$\tau^{k,\ell} = \varrho^{H_{k,\ell}} : \mathbb{R}^n \to \mathbb{R}^n$: transposition of coordinates k and ℓ

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Conditional reflections

$$\tau^{k,\ell}_{>}(y) = \begin{cases} \tau^{k,\ell}(y) & \text{if } y_k > y_\ell \\ y & \text{otherwise} \end{cases}$$

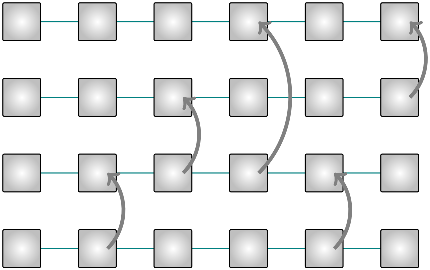
Sorting Networks

Sorting Network

Sequence $(k_1, \ell_1), \ldots, (k_r, \ell_r)$ with

$$\tau_{>}^{k_1,\ell_1} \circ \cdots \circ \tau_{>}^{k_r,\ell_r}(y) = y^{(\mathrm{sort})}$$

for all $y \in \mathbb{R}^n$.



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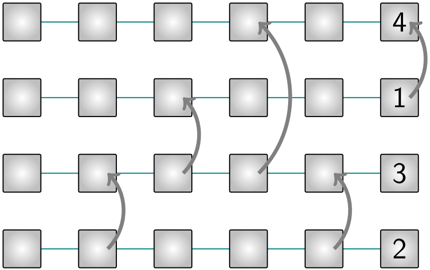

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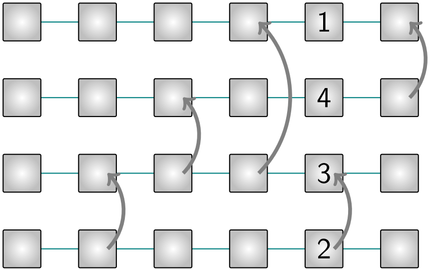

Sorting Networks

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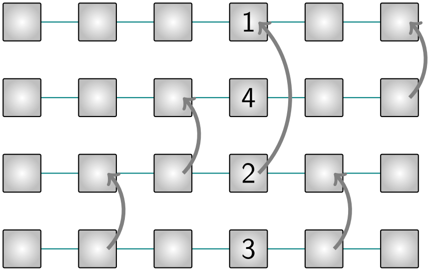

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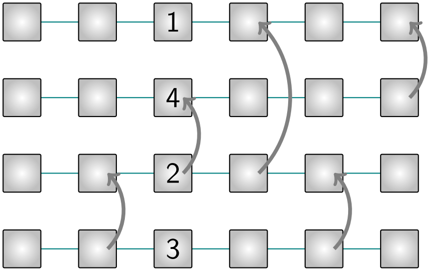

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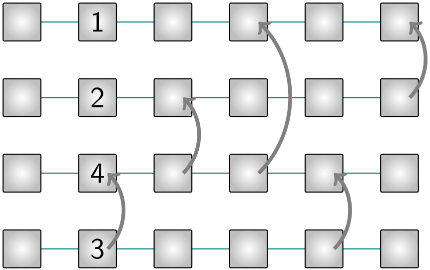

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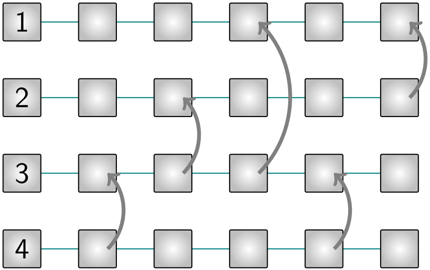

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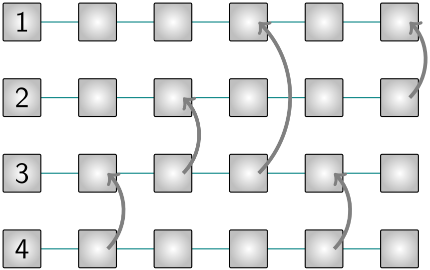

Ajtai, Komlós & Szemerédi 1983

There are sorting networks of size $r = O(n \log n)$.

Results for A_{n-1}

If P lies in $\mathrm{FR}_{\mathsf{A}_{n-1}}$:

For each sorting network $(k_1, \ell_1), \ldots, (k_r, \ell_r)$, we have

$$\mathrm{P}_{\mathrm{perm}}^{\mathsf{A}_{n-1}}(P) = \mathcal{R}_{H^{\leq}_{k_r,\ell_r}, \ldots, H^{\leq}_{k_1,\ell_1}}(P).$$

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Goemans 2009

$$xc(P_{perm}^{n}) \leq O(n \log(n))$$

General G-Permutahedra

K & Pashkovich 11

If

- G is a finite reflection group on $\mathbb{R}^n$ and
- $P \subseteq \mathbb{R}^n$ is a polytope in one region of G,

General G-Permutahedra

K & Pashkovich 11

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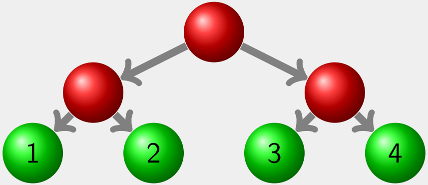
then we have:

$$\mathrm{xc}(\mathrm{P}^{G}_{\mathrm{perm}}(P)) \leq \mathrm{xc}(P) + \mathrm{O}(\log m + n \log n)$$

(where m is the largest number such that $\mathrm{I}_2(m)$ is a factor of G).

Huffman Polytopes

The set V^n_{huff} of Huffman vectors ($n = 4$)



$$(2, 2, 2, 2)$$

Huffman Polytopes

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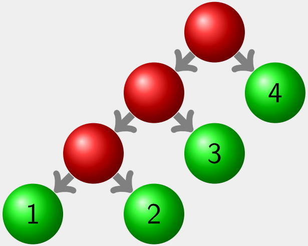
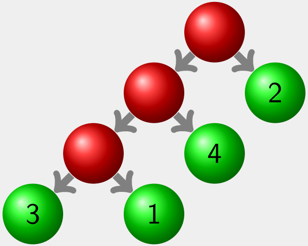


$(3, 3, 2, 1)$

Huffman Polytopes

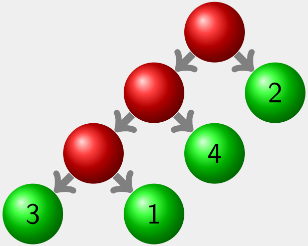
The set V^n_{huff} of Huffman vectors ($n = 4$)



$(3, 1, 3, 2)$

Huffman Polytopes

The set V^n_{huff} of Huffman vectors ($n = 4$)



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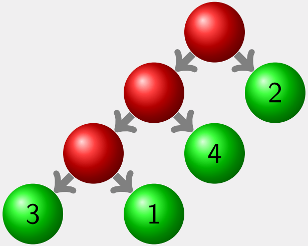

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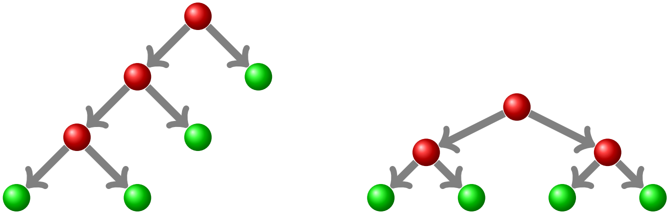
$$P^n_{\text{huff}} = \text{conv}(V^n_{\text{huff}})$$

Nguyen, Nguyen, & Maurras 10

P^n_{huff} has at least $2^{\Omega(n \log n)}$ facets.

An Extended Formulation of Size $O(n^2)$



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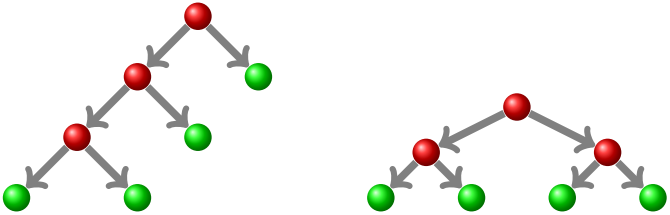

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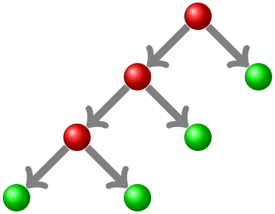

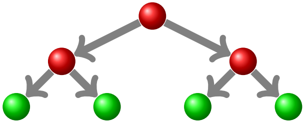

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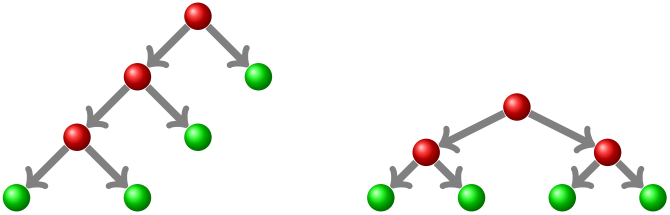

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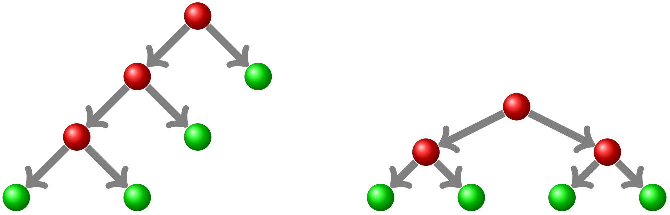

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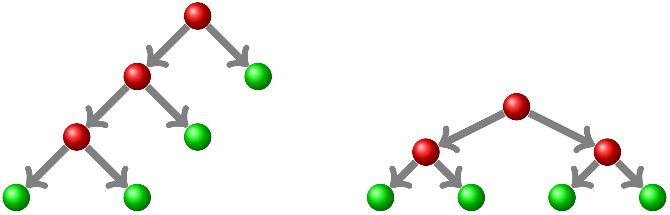

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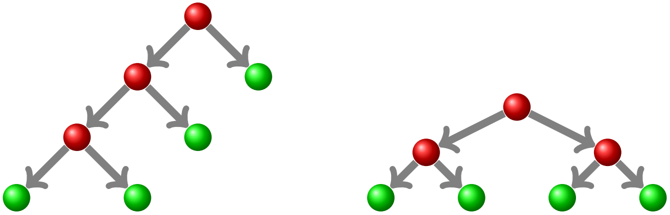

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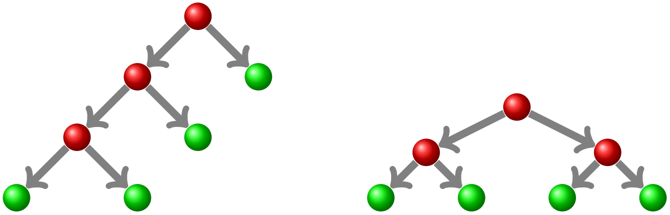

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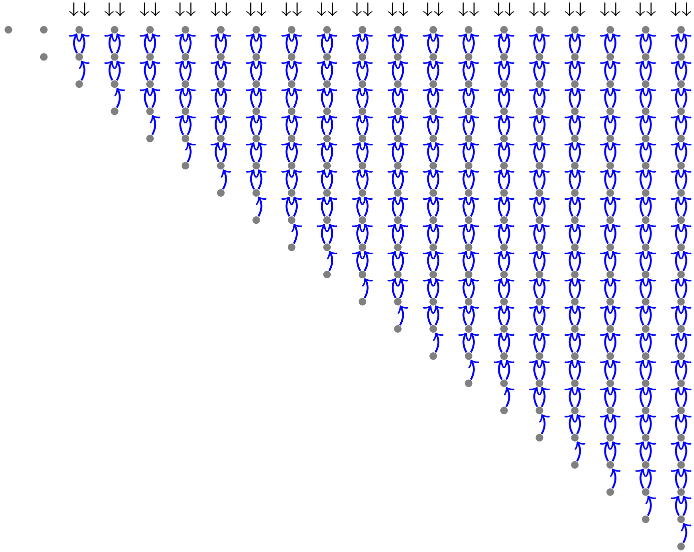
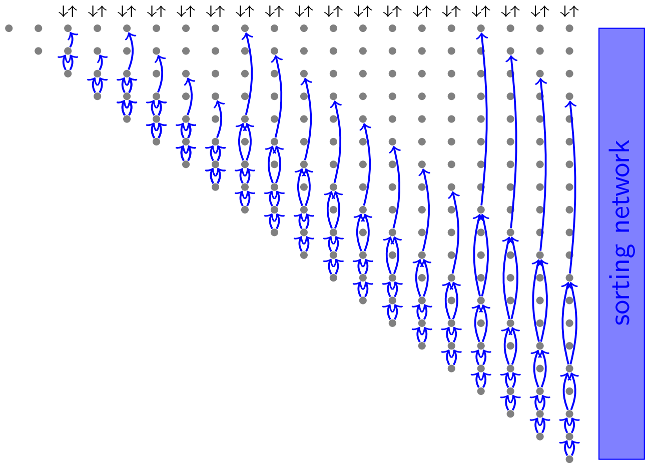
A first extended formulation

$$P^n_{\text{huff}} = \mathcal{R}_{H^{\leq}_{1,2}, \dots, H^{\leq}_{n-1,n}, H^{\leq}_{1,2}, \dots, H^{\leq}_{n-2,n-1}}(\varphi(P^{n-1}_{\text{huff}})).$$

Schematic View on the Construction



A Better Construction



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$$\text{xc}(P^n_{\text{huff}}) \leq O(n \log n)$$

Thanks for your attention.