From Minimum Cut to Submodular Minimization
Leveraging the Decomposable Structure

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Minimum s-t Cut

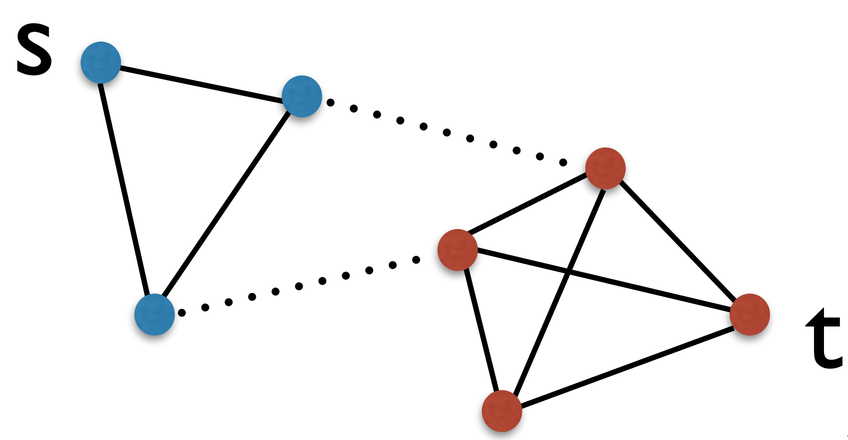
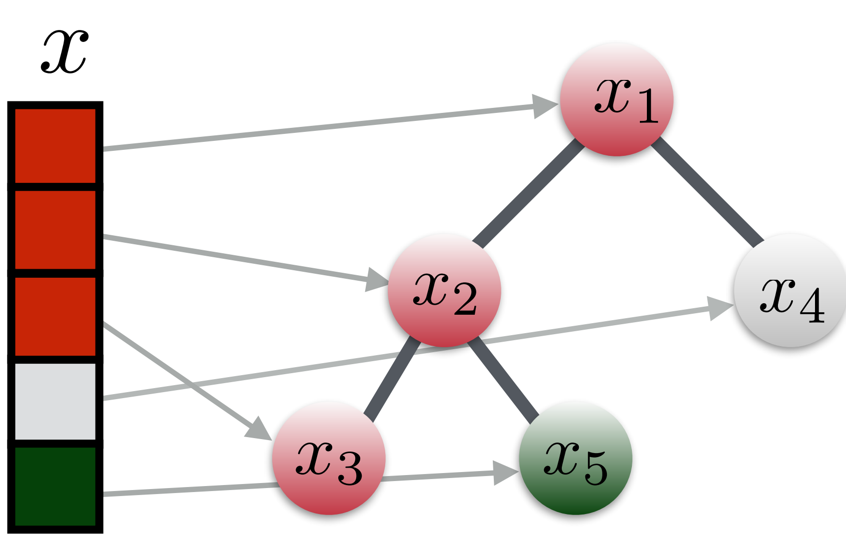


Image Segmentation

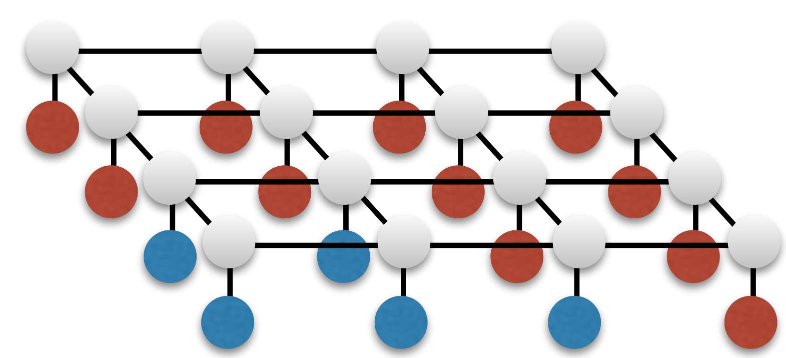
Submodular Minimization

$$\min_{S \subseteq V} f(S)$$



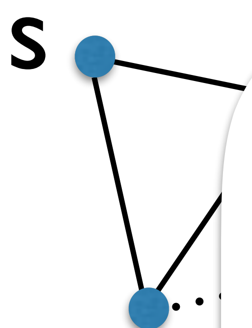
Structured Sparsity

[Bach '10]



MAP Inference

Minimum s-t Cut



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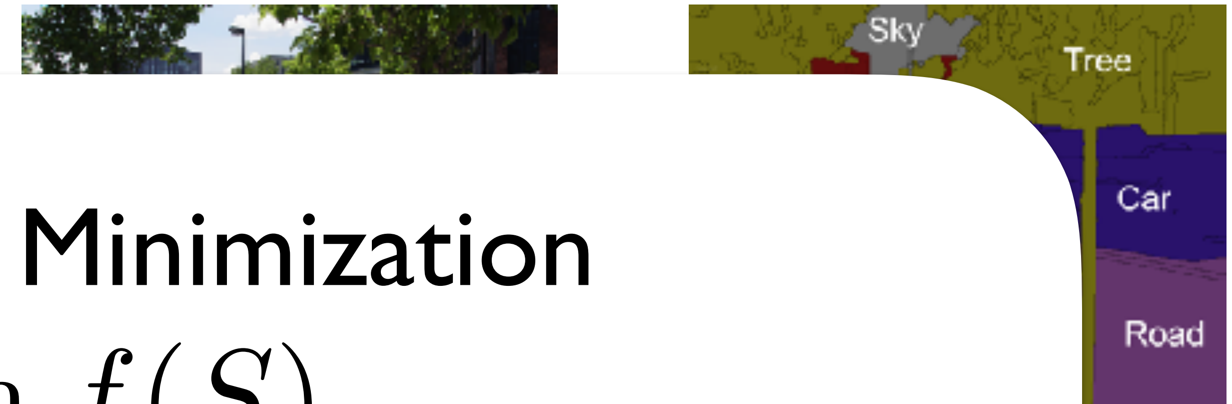

Submodular Minimization

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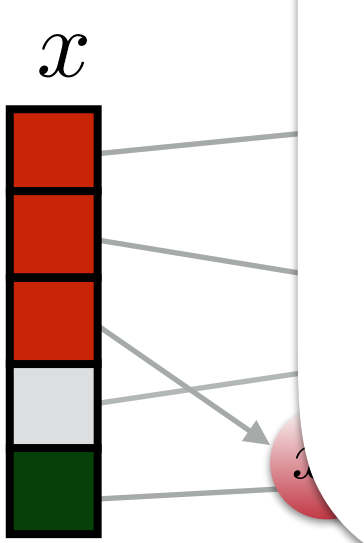
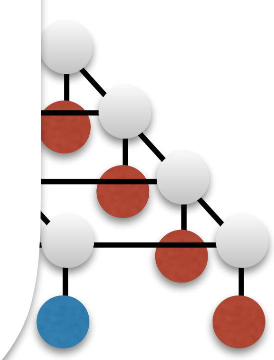


Structured Sparsity
[Bach '10]



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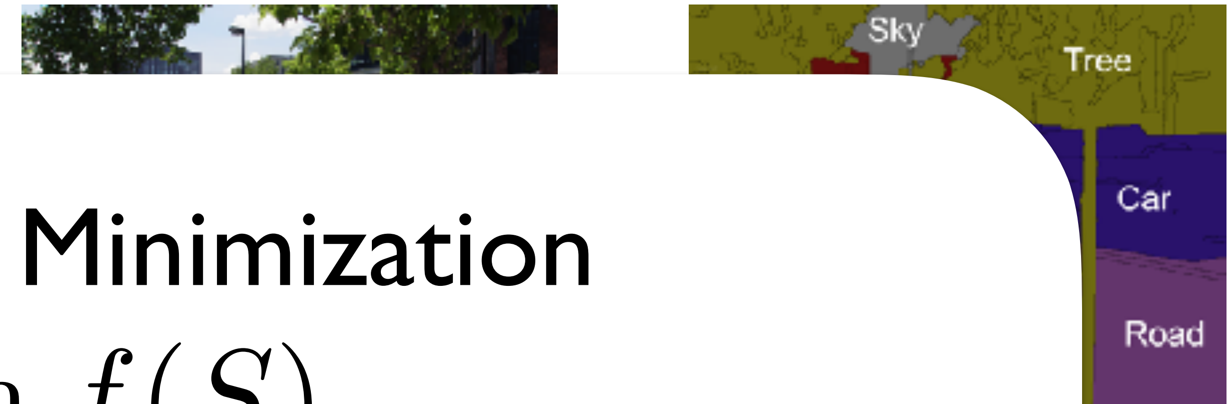

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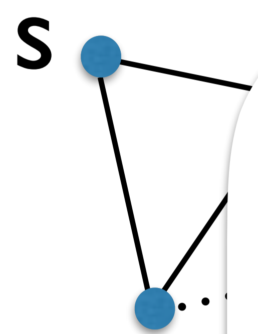
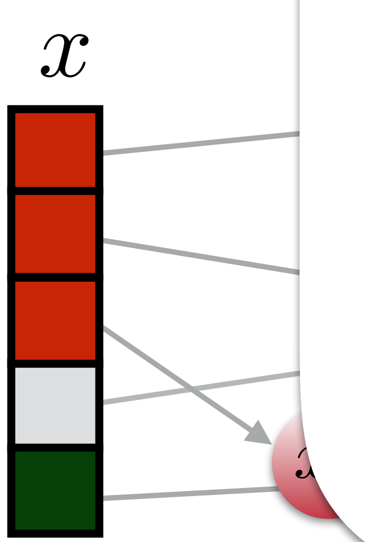


Submodular Minimization

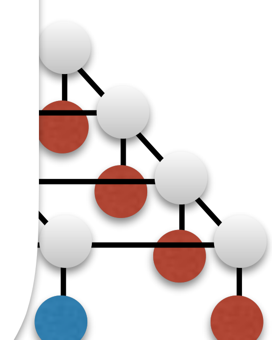
$$\min_{S \subseteq V} f(S)$$

Solvable in polynomial time

Combinatorial, ellipsoid, cutting plane,…

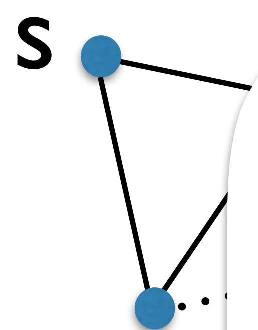


Structured Sparsity
[Bach '10]



MAP Inference

Minimum s-t Cut



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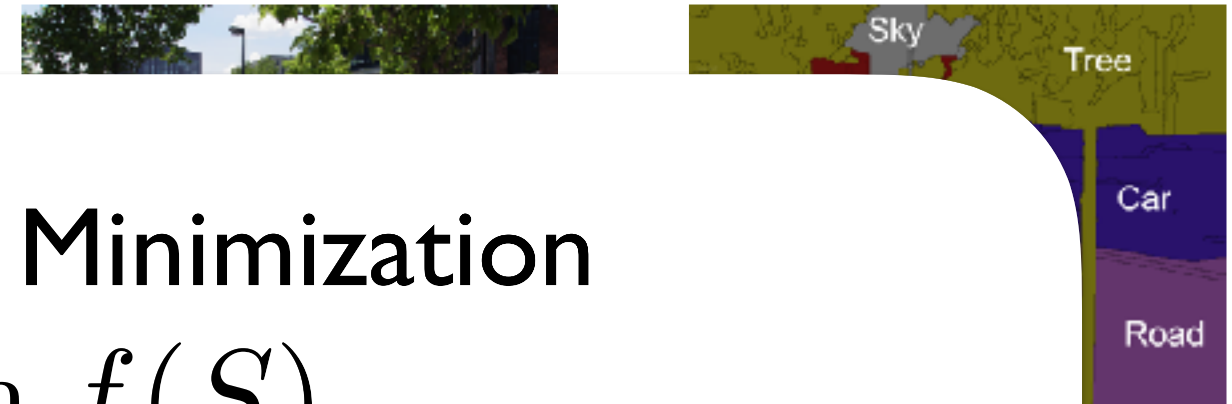


Submodular Minimization

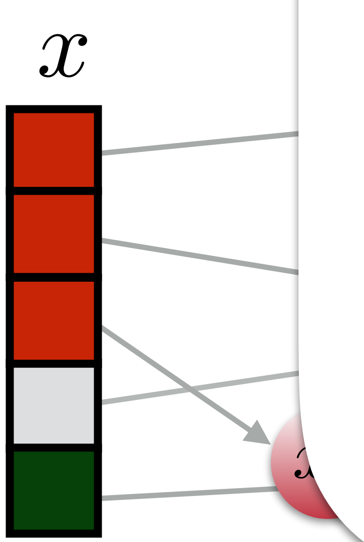
$$\min_{S \subseteq V} f(S)$$

Solvable in polynomial time

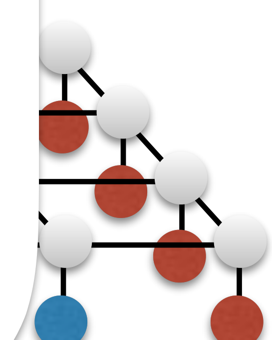
Combinatorial, ellipsoid, cutting plane,…

Very high running times $\Omega(|V|^3)$

Leverage structure to obtain faster algos



Structured Sparsity
[Bach '10]



MAP Inference

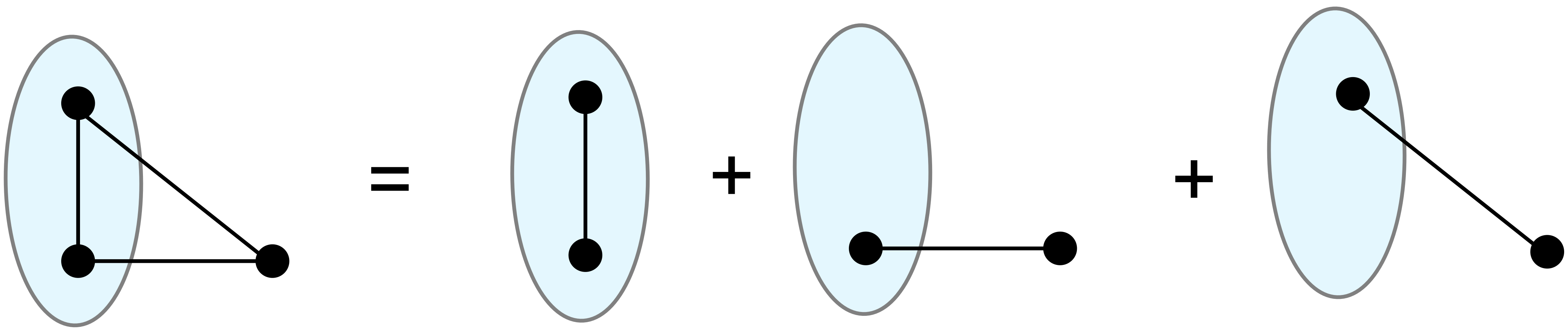
Decomposable Functions

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

Simple f_i : fast subroutine for minimizing $f_i(S) + w(S)$ for any linear function w

[Stobbe, Krause '10; Kolmogorov '12; Jegelka, Bach, Sra '13; Nishihara, Jegelka, Jordan '14]

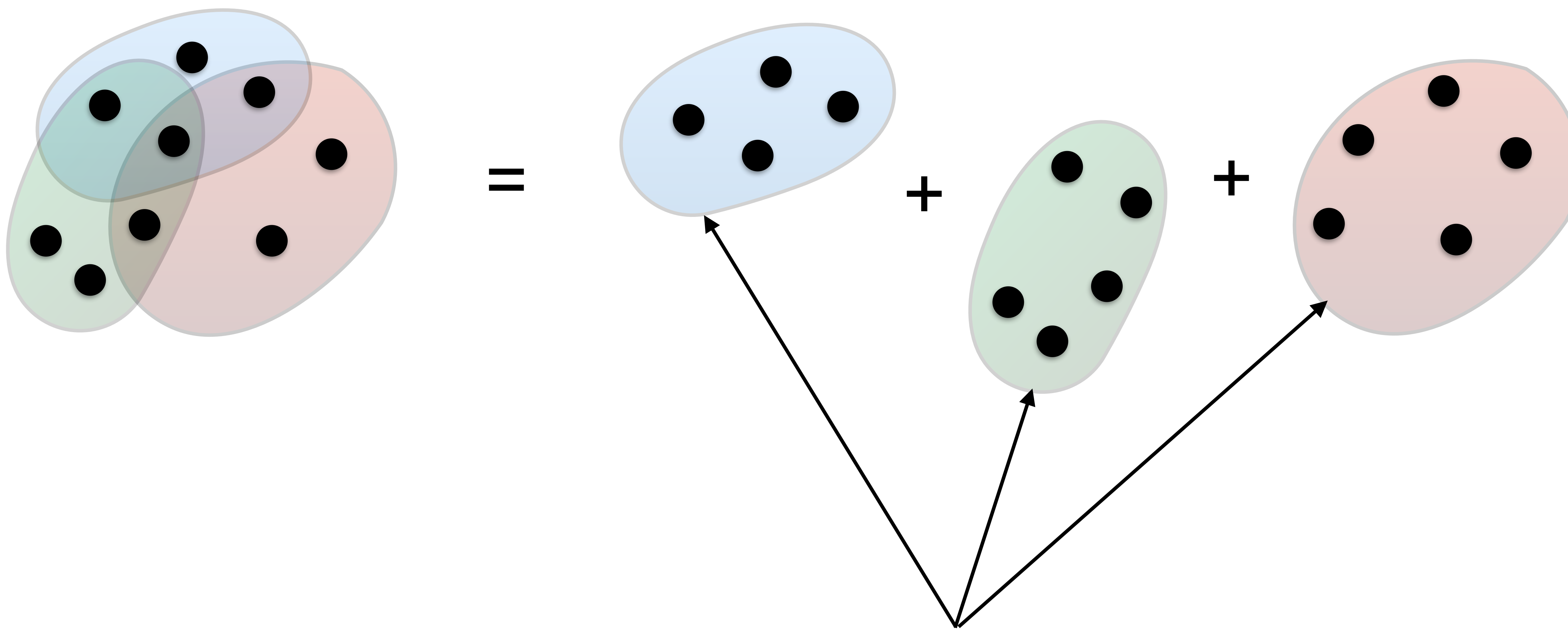
Decomposable Structure



$$f(S) = \sum_{e \in E} f_e(S)$$

Cut function of a single edge

Decomposable Structure



hyperedges, local functions, …

This Talk

Sum-of-Submodular / Decomposable SFM

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

Continuous algos based on gradient descent

Combinatorial algos based on max flow approaches

Interplay between continuous optimization
and combinatorial structure

Main Combinatorial Player

Key concept underpinning submodular algorithms

Base Polytope

$$B(f) = \{y \mid \langle y, \mathbf{1}_S \rangle \leq f(S) \;\; \forall S \subseteq V, \; \langle y, \mathbf{1}_V \rangle = f(V)\}$$

Think of y as a linear function on V

Continuous Algorithms

From Submodular to Convex Optimization

[Edmonds '70, Lovasz '83, Fujishige '84]
Submodular Minimization can be reduced to finding a point of minimum norm in the base polytope

Discrete Problem

$$\min_{S \subseteq V} f(S)$$

→

Exact Convex Formulation

$$\min_{y \in B(f)} \|y\|^2$$

Discrete Problem

$$\min_{S \subseteq V} f(S)$$

Exact Convex Formulation

$$\min_{y \in B(f)} \|y\|^2$$

Convex objective is very nice

Constraint is very complicated

Discrete Problem

$$\min_{S \subseteq V} f(S)$$

→

Exact Convex Formulation

$$\min_{y \in B(f)} \|y\|^2$$

Convex objective is very nice

Constraint is very complicated

Algorithms based on
Ellipsoid/Cutting Plane [GLS '84, LSW '15]

Provable polynomial running times

Running time prohibitively large in applications

Discrete Problem

$$\min_{S \subseteq V} f(S)$$

→

Exact Convex Formulation

$$\min_{y \in B(f)} \|y\|^2$$

Convex objective is very nice

Constraint is very complicated

Fujishige-Wolfe and Frank-Wolfe algorithms

Suitable for some applications

Pseudo-polynomial running times

Known to require large number of iterations

In the decomposable setting, can leverage more of the convex optimization toolkit

$$\textbf{Lemma:}\ \text{If } f = \sum_{i=1}^{r} f_i \text{ then } B(f) = \sum_{i=1}^{r} B(f_i)$$

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

→

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

f_i simple $\Rightarrow$ quadratic min over $B(f_i)$ easy

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

→

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

Decomposition allows for fast and practical algorithms based on gradient descent

[Stobbe, Krause '10; Jegelka, Bach, Sra '13; Nishihara, Jegelka, Jordan '14, E., Nguyen '15]

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

→

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

[E., Nguyen ICML '15]
Minimize the convex formulation using
Random Coordinate Gradient Descent

Also accelerated algorithm

Fastest running times currently known

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

$$g(y) := \left\| \sum_{i=1}^{r} y_i \right\|^2$$

Random Coordinate Descent Algorithm

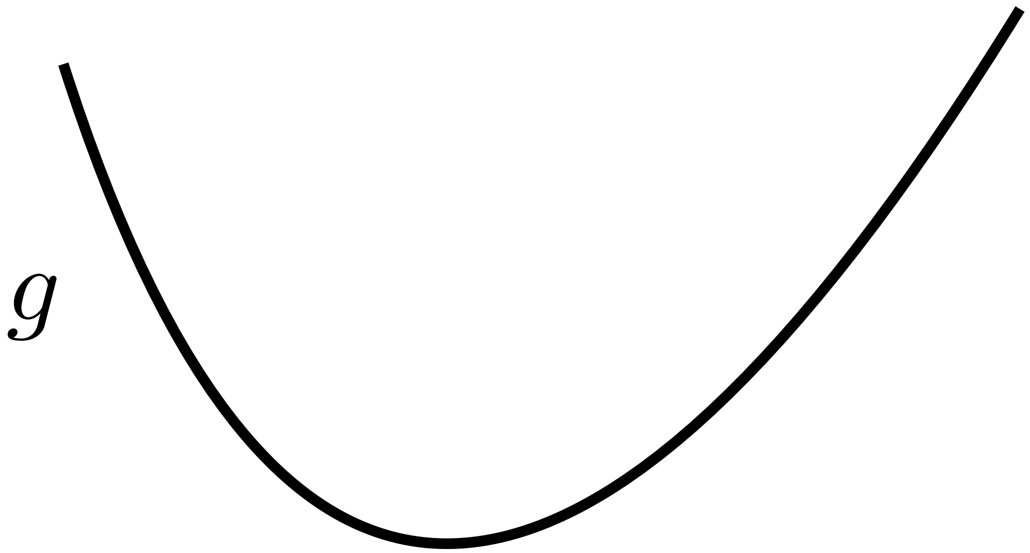
Start with $y = (y_1, \ldots, y_r)$, where $y_i \in B(f_i)$

In each iteration

Pick an index $i \in \{1, 2, ..., r\}$ uniformly at random

$\langle\!\langle$ *Update block i* $\rangle\!\rangle$

$$y_i' \leftarrow \operatorname*{argmin}_{z_i \in B(f_i)} \left(g(y) + \langle \nabla_i g(y), z_i - y_i \rangle + \|z_i - y_i\|^2 \right)$$



Smoothness

$$\|\nabla_i g(y) - \nabla_i g(z)\| \le \|y_i - z_i\|$$

y and z differ only in block i

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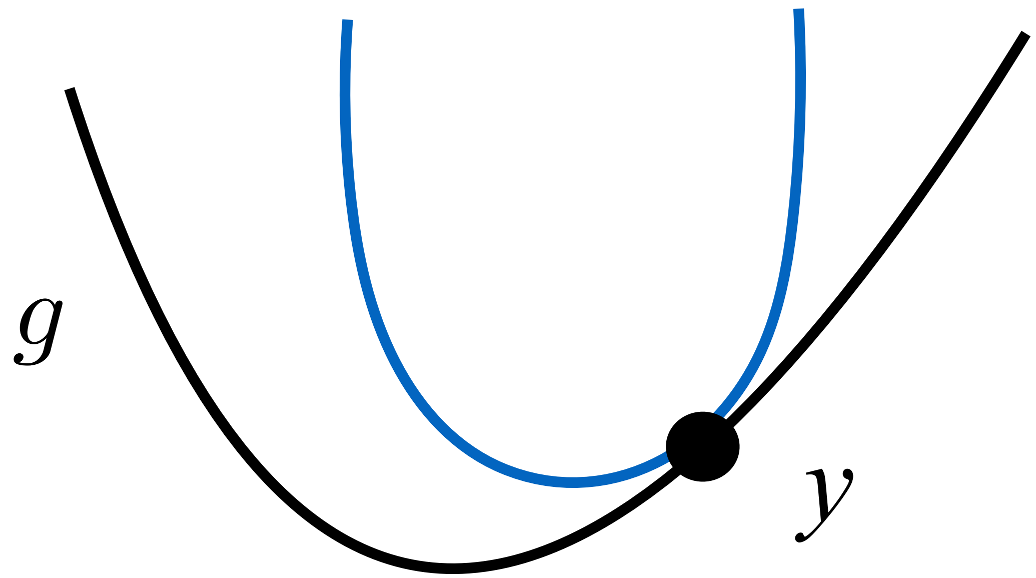

Smooth functions have quadratic upper bounds

Random Coordinate Descent Algorithm

Start with $y = (y_1, \ldots, y_r)$, where $y_i \in B(f_i)$

In each iteration

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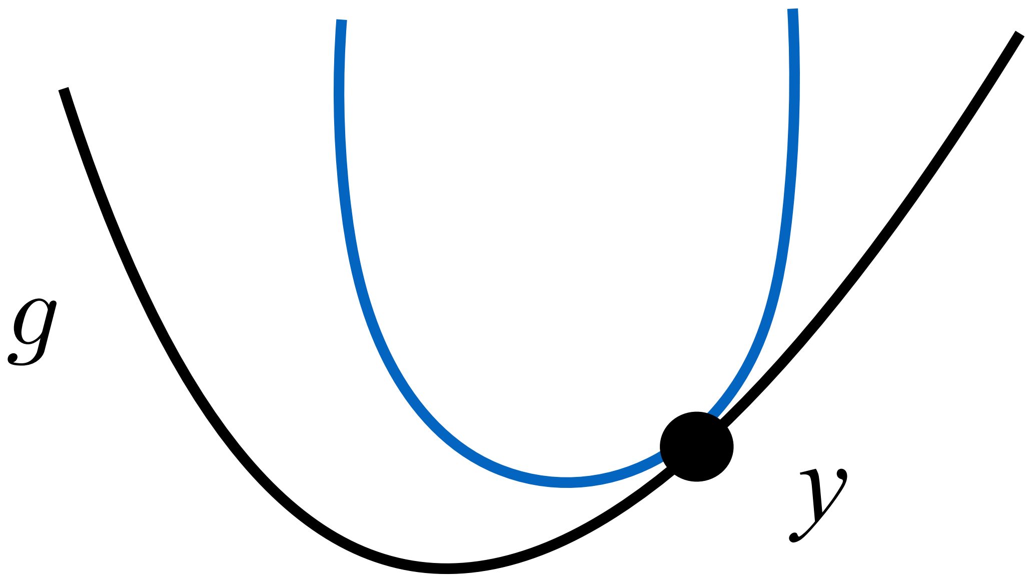
$\langle\langle$*Update block* $i\rangle\rangle$

$$y_i' \leftarrow \underset{z_i \in B(f_i)}{\operatorname{argmin}} \left(g(y) + \langle \nabla_i g(y), z_i - y_i \rangle + \|z_i - y_i\|^2 \right)$$

$$\phi(z) = g(y) + \langle \nabla_i g(y), z_i - y_i \rangle + \|z_i - y_i\|^2$$



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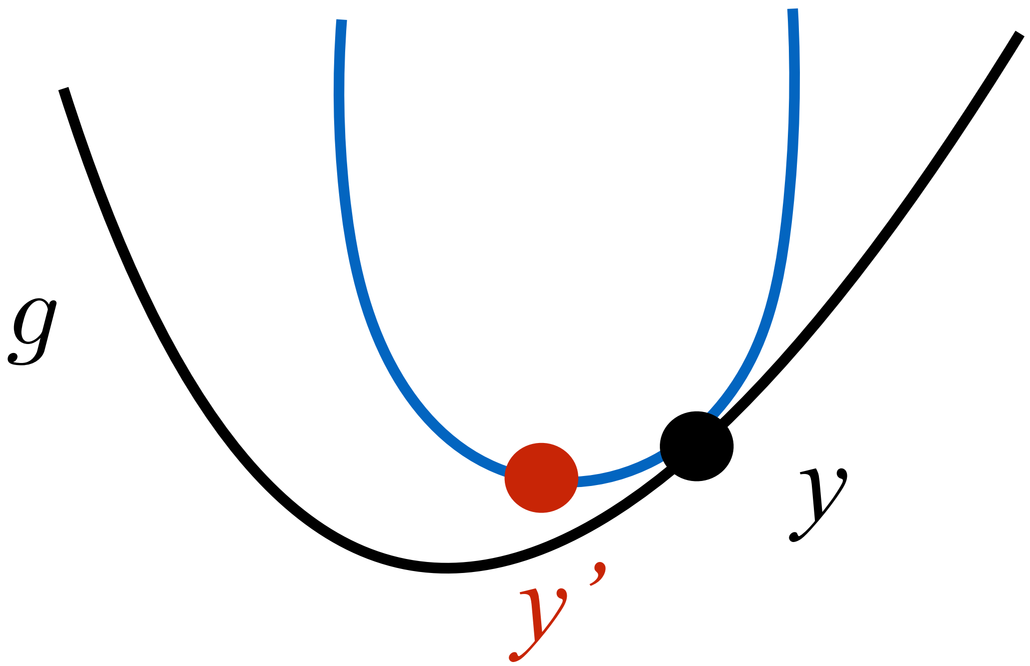

z and y differ only in block i

Minimize the upper bound

Random Coordinate Descent Algorithm

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In each iteration

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Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

$$g(y) := \left\| \sum_{i=1}^{r} y_i \right\|^2$$

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Efficient update for simple functions

Exact Convex Formulation

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Also accelerated version building on [Fercoq-Richtárik '13]

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

[E., Nguyen ICML '15]

Minimize the convex formulation using

Random Coordinate Gradient Descent

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

Exact Convex Formulation

$$\min_{y_i \in B(f_i)} \left\| \sum_{i=1}^{r} y_i \right\|^2$$

[E., Nguyen ICML '15]

Minimize the convex formulation using

Random Coordinate Gradient Descent

Running time (with acceleration) [ENV '17]

$$O\left(nr \log\left(\frac{F_{\max}}{\epsilon}\right) Q\right)$$

Q: time for one update via oracle for f_i

Discrete Problem

$$\min_{S \subseteq V} \sum_{i=1}^{r} f_i(S)$$

Exact Convex Formulation

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[E., Nguyen ICML '15]
Minimize the convex formulation using
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Running time (with acceleration) [ENV '17]

$$O(nr \log(nF_{\max})\, Q)$$

Q: time for one update via oracle for f_i

Discrete Algorithms

The quest for combinatorial algos for submodular min

[Cunningham '85]
It is an outstanding open problem to find a practical combinatorial algorithm to minimize a general submodular function, which also runs in polynomial time.

The quest for combinatorial algos for submodular min

[Cunningham '85]
It is an outstanding open problem to find a practical combinatorial algorithm to minimize a general submodular function, which also runs in polynomial time.

[Bixby et al. '85, Cunningham '85]
Framework for designing combinatorial algorithms for submodular minimization

Basic framework for all combinatorial algorithms

Cunningham Framework

Maintain a point y in the base polytope

Iteratively update y using "flow-style" updates

How to represent a point in the base polytope?

Basic framework for all combinatorial algorithms

Cunningham Framework

Maintain a point y in the base polytope

Iteratively update y using "flow-style" updates

How to represent a point in the base polytope?

As a convex combination of vertices

Pro: Very general (works for any submodular fn)

Con: Difficult/expensive to maintain and update

For decomposable submodular functions simpler version of Cunningham's approach works!

Recall: For $f = \sum_{i=1}^{r} f_i$ we have $B(f) = \sum_{i=1}^{r} B(f_i)$

Can represent $y \in B(f)$ as

$y = y_1 + \cdots + y_r$ where $y_i \in B(f_i)$

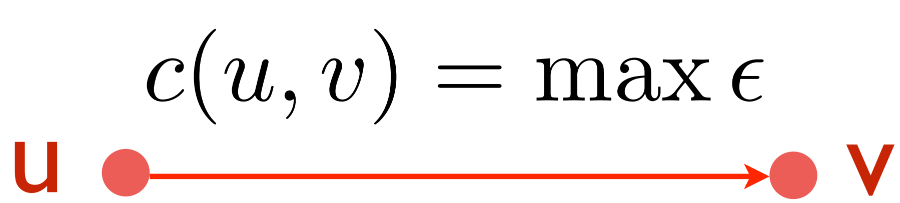
Verify $y_i \in B(f_i)$ using oracle for f_i

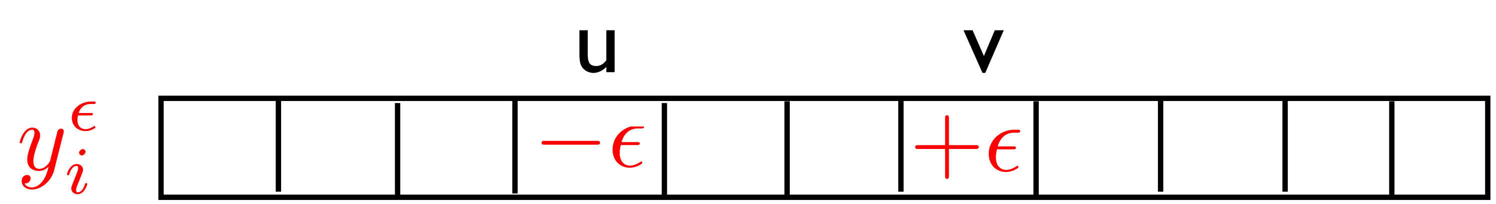
From decomposable functions to graphs

Maintain a point $y \in B(f)$ with a decomposition $y = y_1 + \cdots + y_r$ into points $y_i \in B(f_i)$

Update using auxiliary graph $G = (V, \cup_i E_i)$

$(u, v) \in E_i$ if $y_i^\epsilon \in B(f_i)$ for some $\epsilon > 0$

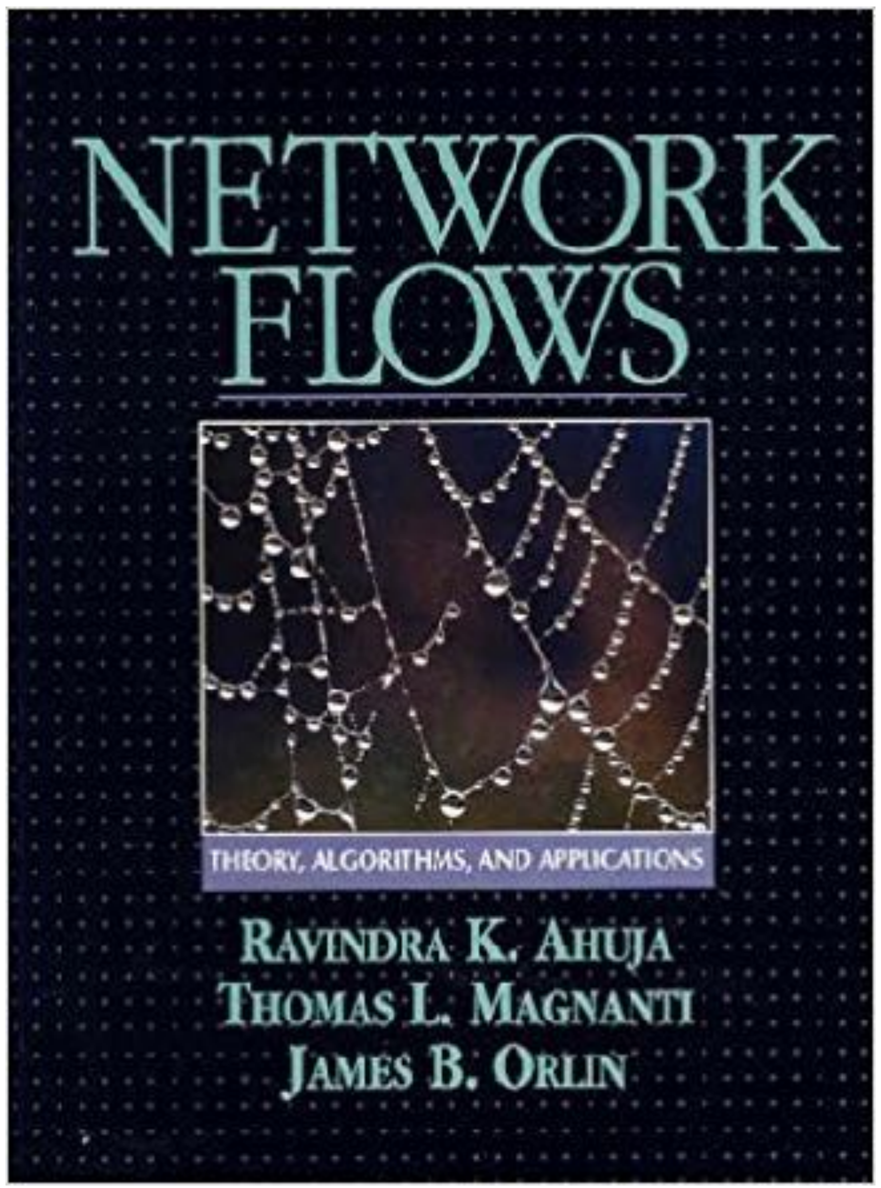


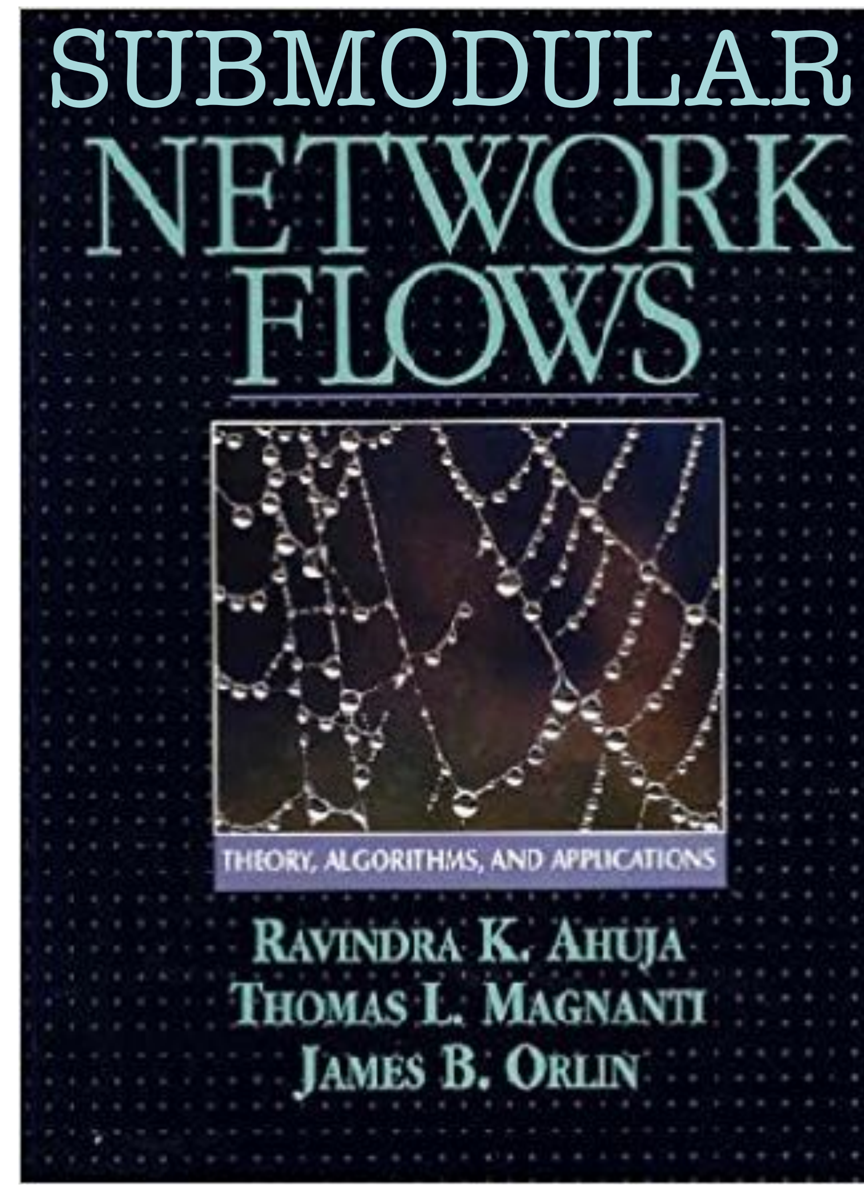


For graph cut instance, the auxiliary graph is the usual residual graph for network flows

Translate classical maxflow toolkit to submodular

Edmonds-Karp-Dinitz, Preflow-push, …





Translate classical maxflow toolkit to submodular

Edmonds-Karp-Dinitz, Preflow-push, …

Sample Result: If all f_i have small support preflow-push runs in $O(n^2 r)$ oracle calls

First shown by [Fujishige-Zhang '94] for r = 2

[Fix et al. '13] adapt IBFS algo. of [Goldberg et al. '11]

Translate classical maxflow toolkit to submodular

Edmonds-Karp-Dinitz, Preflow-push, …

[E., Nguyen, Vegh NIPS '17]

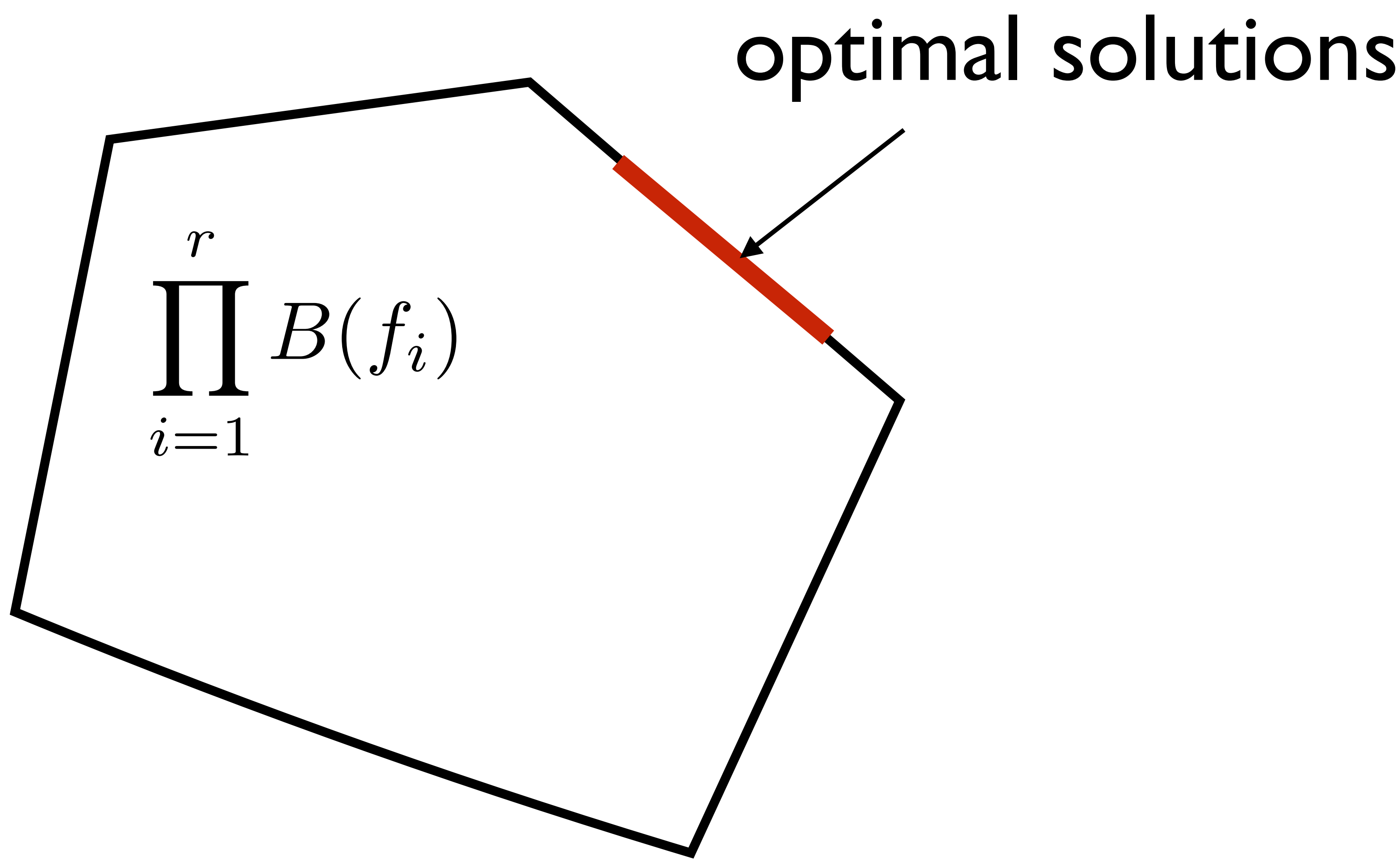
Discrete to Continuous

Tight running time analyses for the gradient descent algorithms

Discrete vs Continuous

Experimental comparison on computer vision tasks

Recall our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$

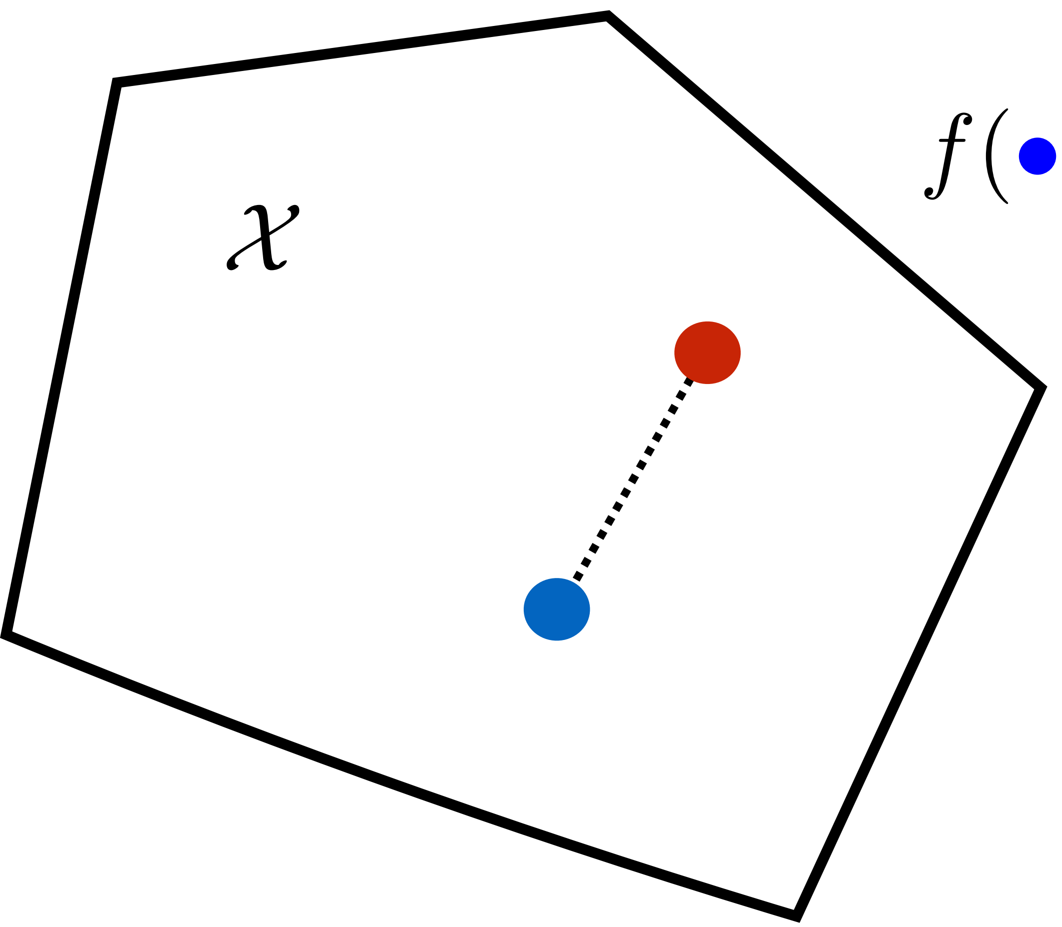


Minimize objective using random coordinate descent

Linear convergence rate despite lack of strong convexity

Use combinatorial structure instead of strong convexity

$$\min\left\{f(x) : x \in \mathcal{X}\right\}$$

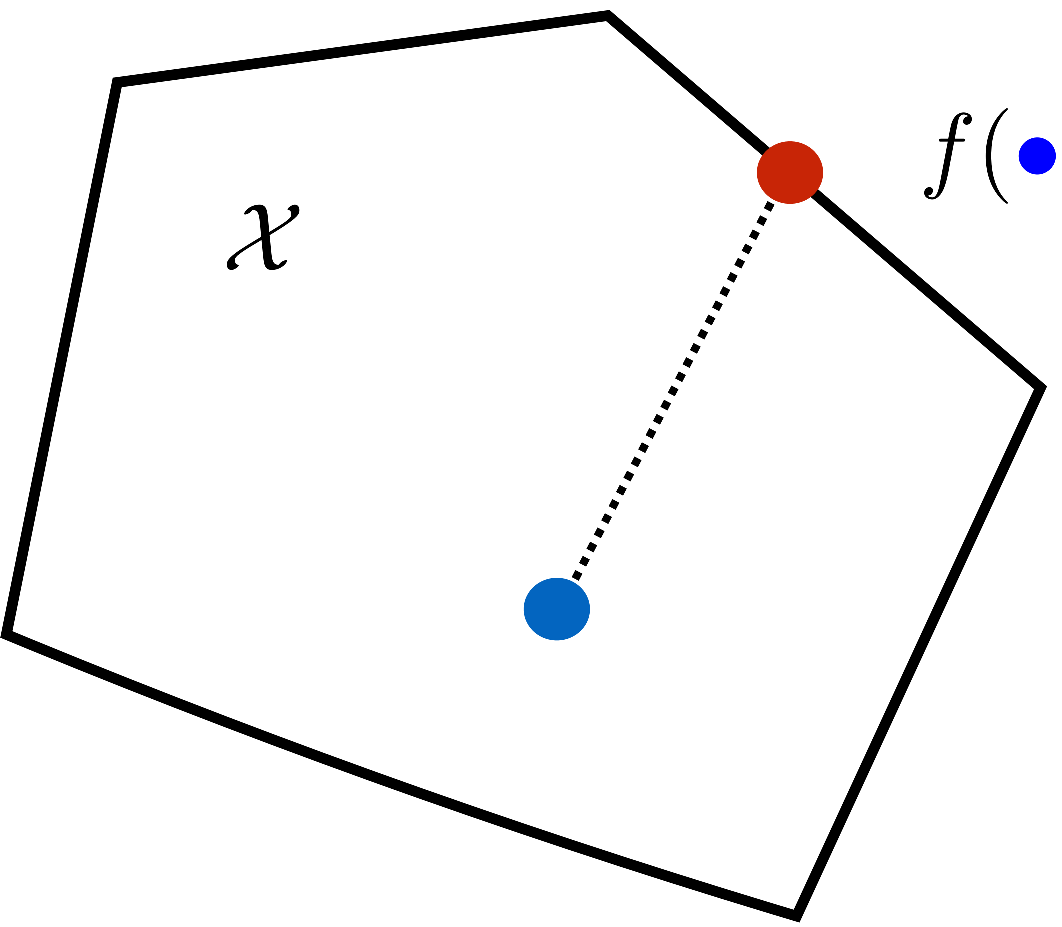


μ-Strong Convexity

$$f(\bullet) - f(\bullet) \geq \langle \nabla f(\bullet), \bullet - \bullet \rangle + \Omega_\mu(d(\bullet, \bullet)^2)$$

for any $\bullet, \bullet \in \mathcal{X}$

$$\min\left\{f(x) : x \in \mathcal{X}\right\}$$



μ-Strong Convexity

$$f({\color{blue}\bullet}) - f({\color{red}\bullet}) \geq \langle \nabla f({\color{red}\bullet}), {\color{blue}\bullet} - {\color{red}\bullet} \rangle + \Omega_\mu(d({\color{blue}\bullet}, {\color{red}\bullet})^2)$$

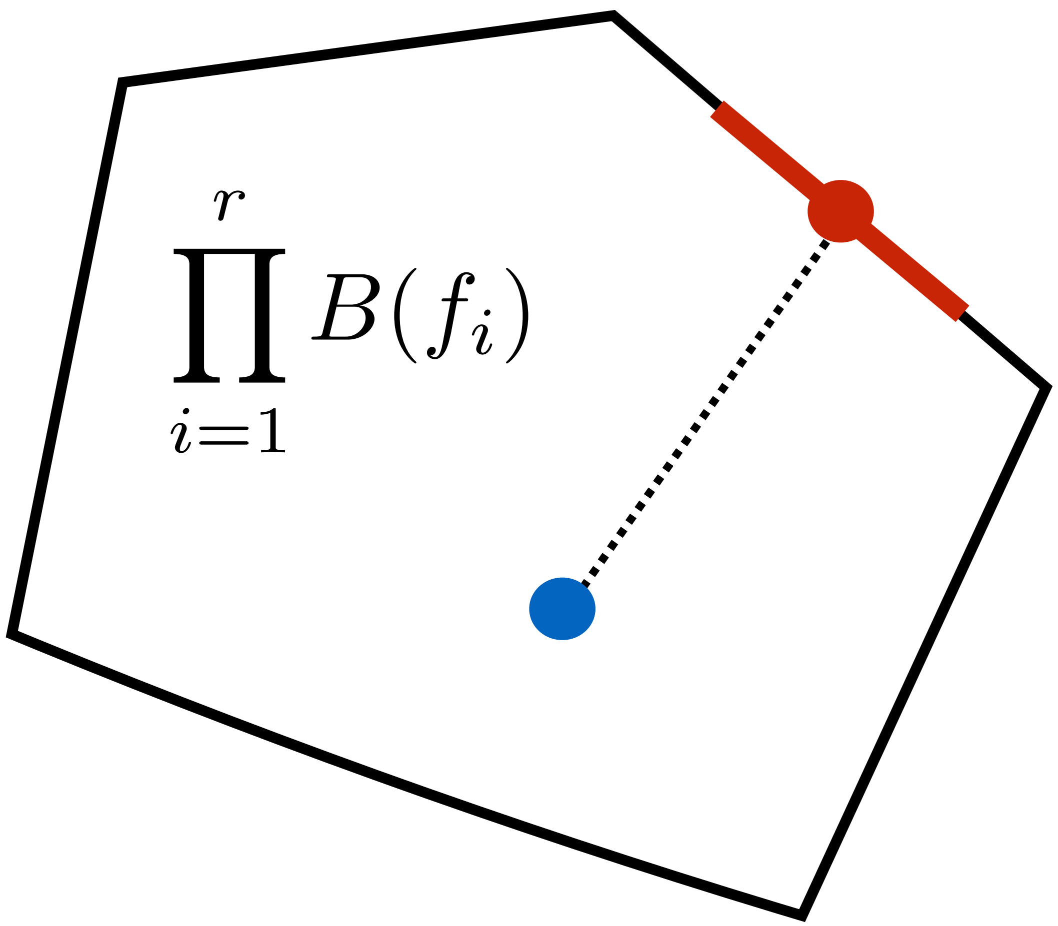
for any ${\color{blue}\bullet}, {\color{red}\bullet} \in \mathcal{X}$

What we really need

$$f({\color{blue}\bullet}) - f({\color{red}\bullet}) \geq \Omega_\mu(\min_{\color{red}\bullet} d({\color{blue}\bullet}, {\color{red}\bullet})^2)$$

where ${\color{red}\bullet}$ is an optimal point

Our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$

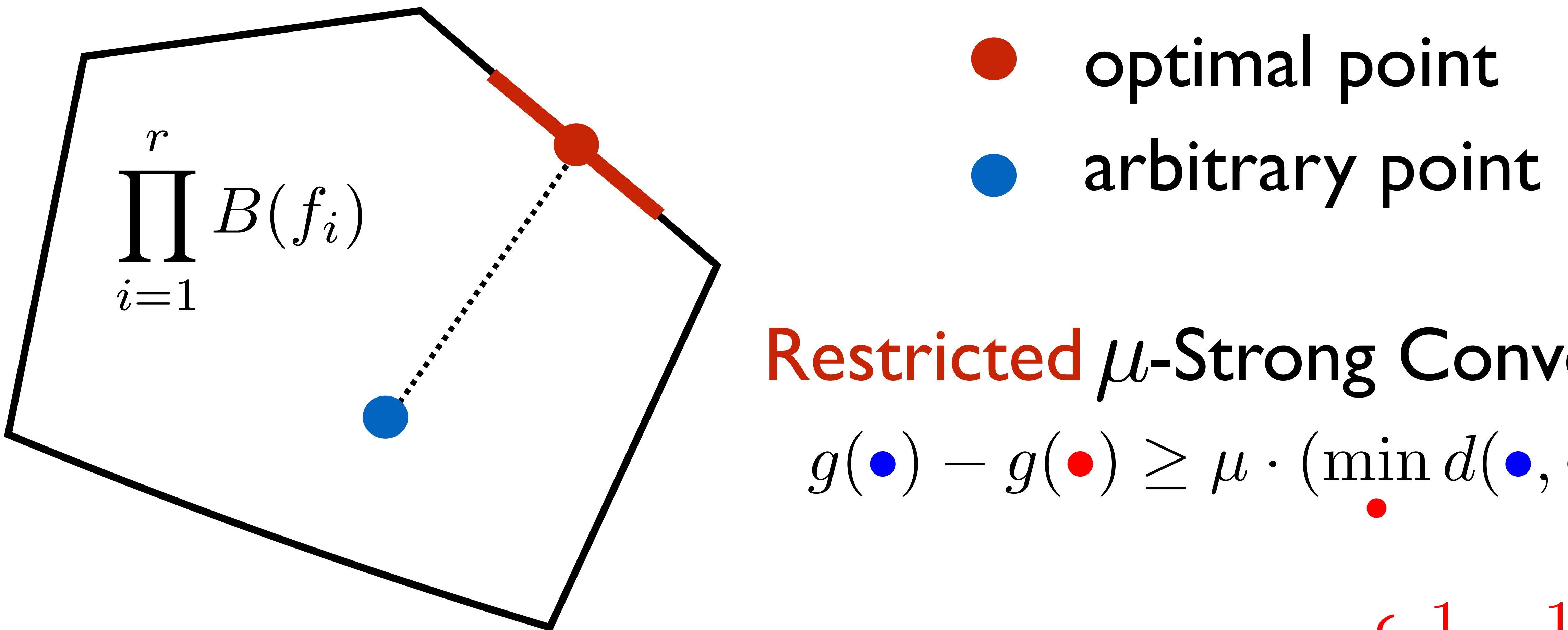


- optimal point
- arbitrary point

Restricted μ-Strong Convexity

$$g(\bullet) - g(\bullet) \geq \mu \cdot (\min_{\bullet} d(\bullet, \bullet)^2)$$

Our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$

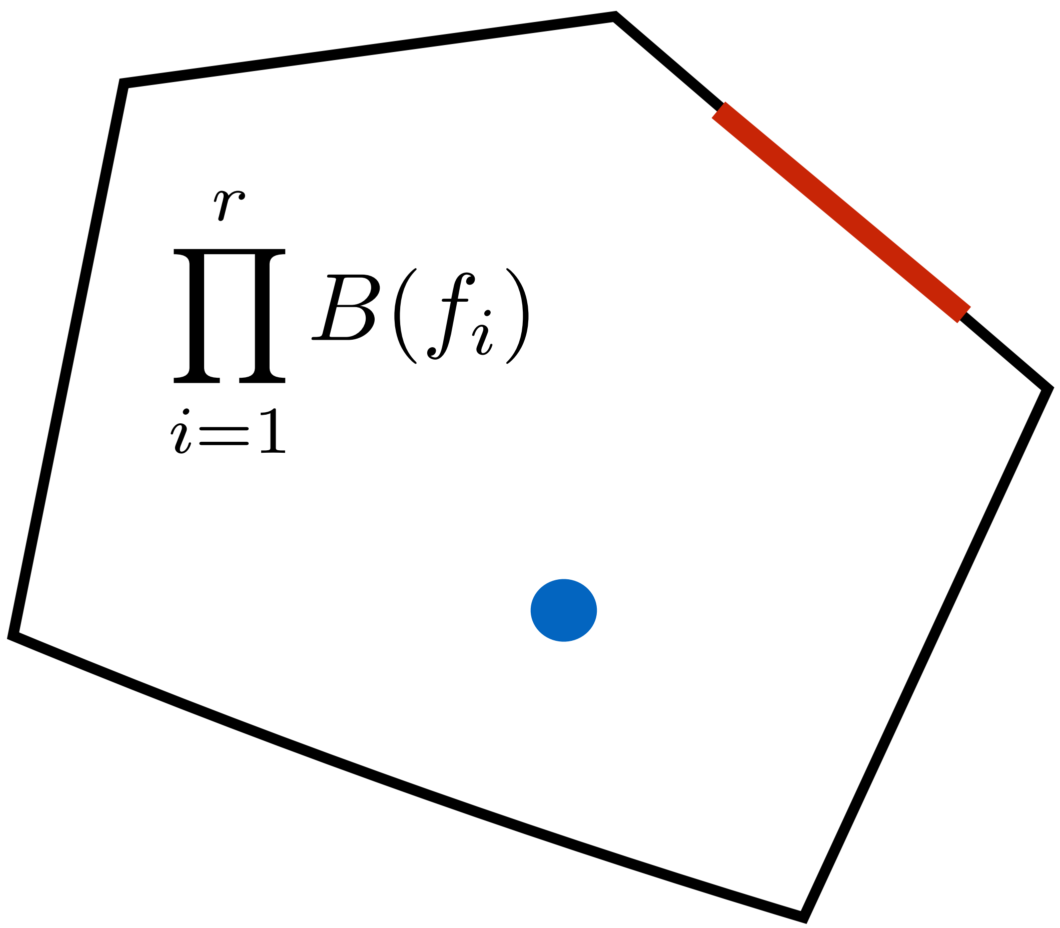


Restricted μ-Strong Convexity

$$g(\color{blue}{\bullet}) - g(\color{red}{\bullet}) \geq \mu \cdot (\min_{\color{red}{\bullet}} d(\color{blue}{\bullet}, \color{red}{\bullet})^2)$$

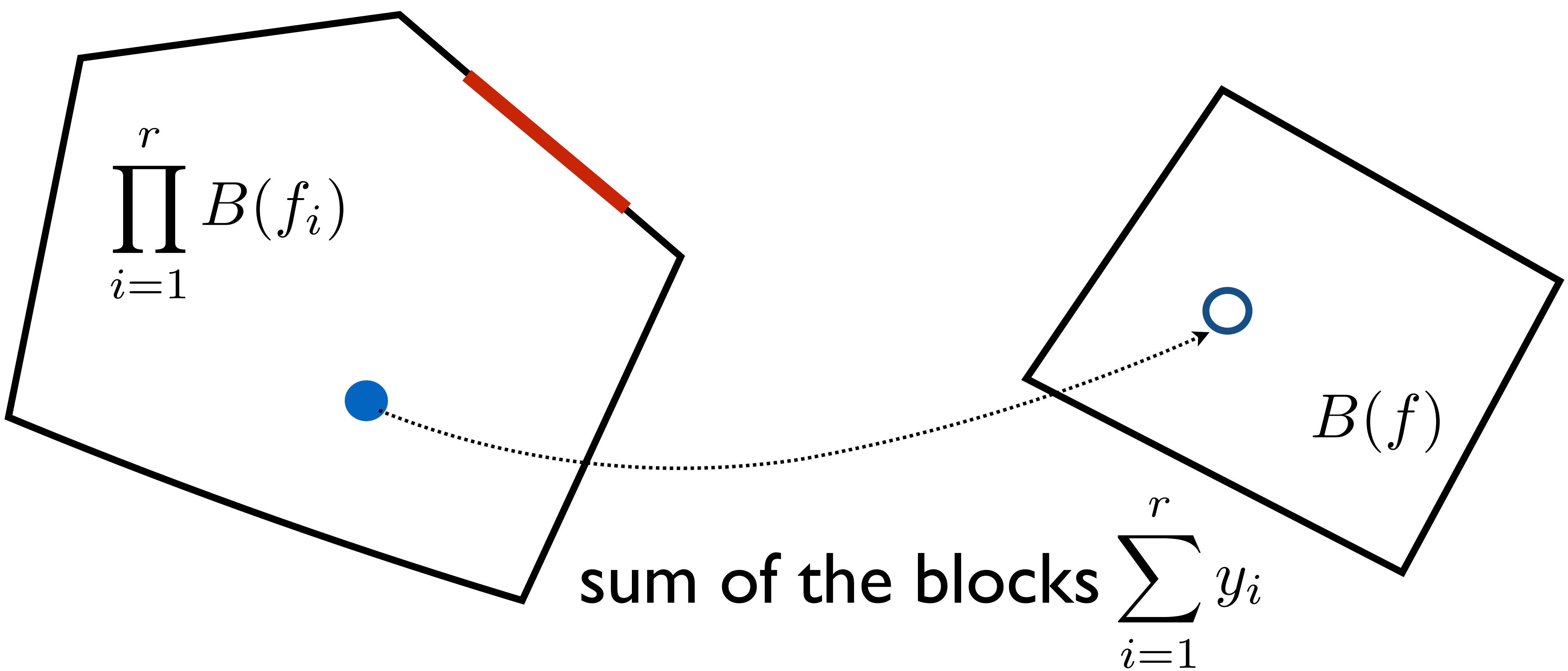
Can show $\mu \geq \max\left\{\frac{1}{nr}, \frac{1}{n^2}\right\}$

Our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$



Want: g(●) - g(●) $\geq \frac{1}{n^2} \cdot$ d(● , ●)^2

Our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$



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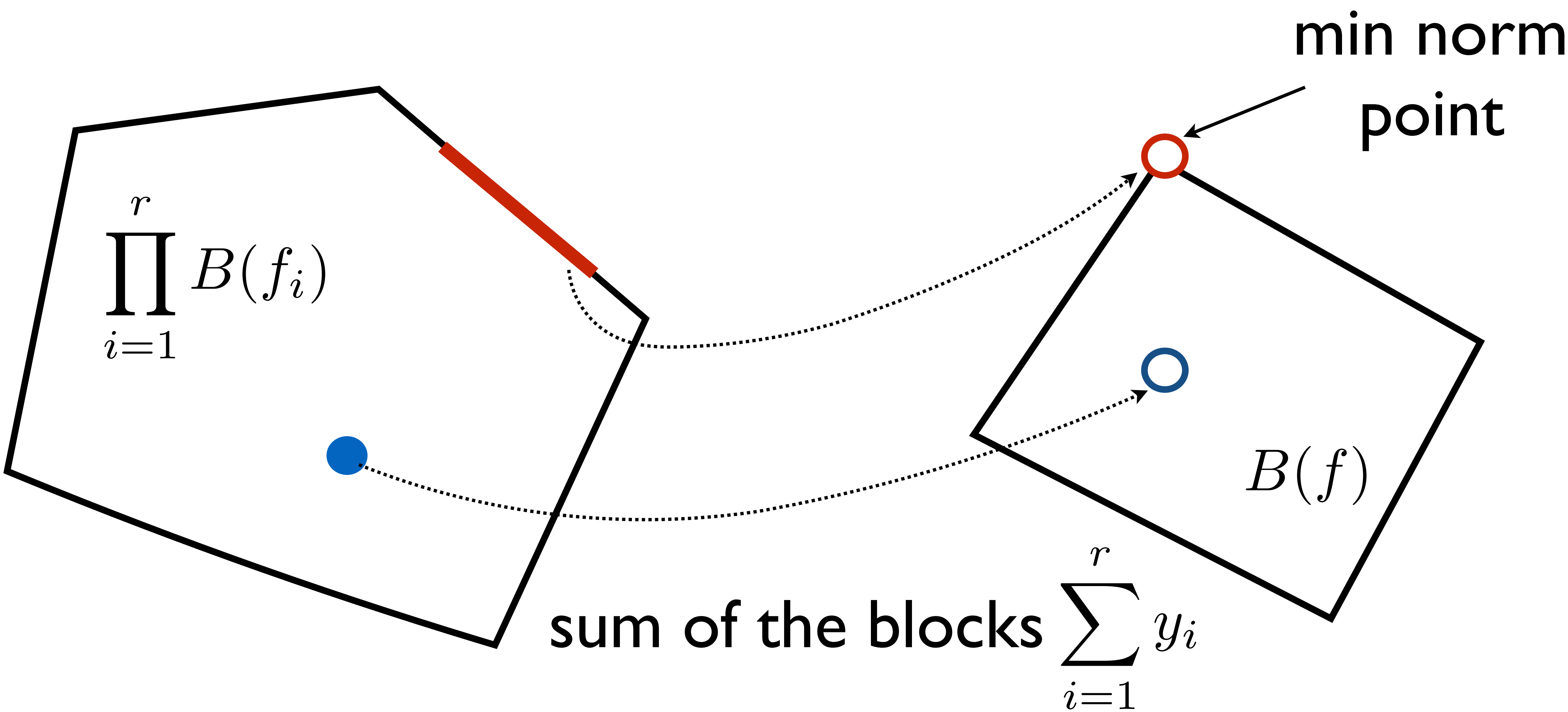

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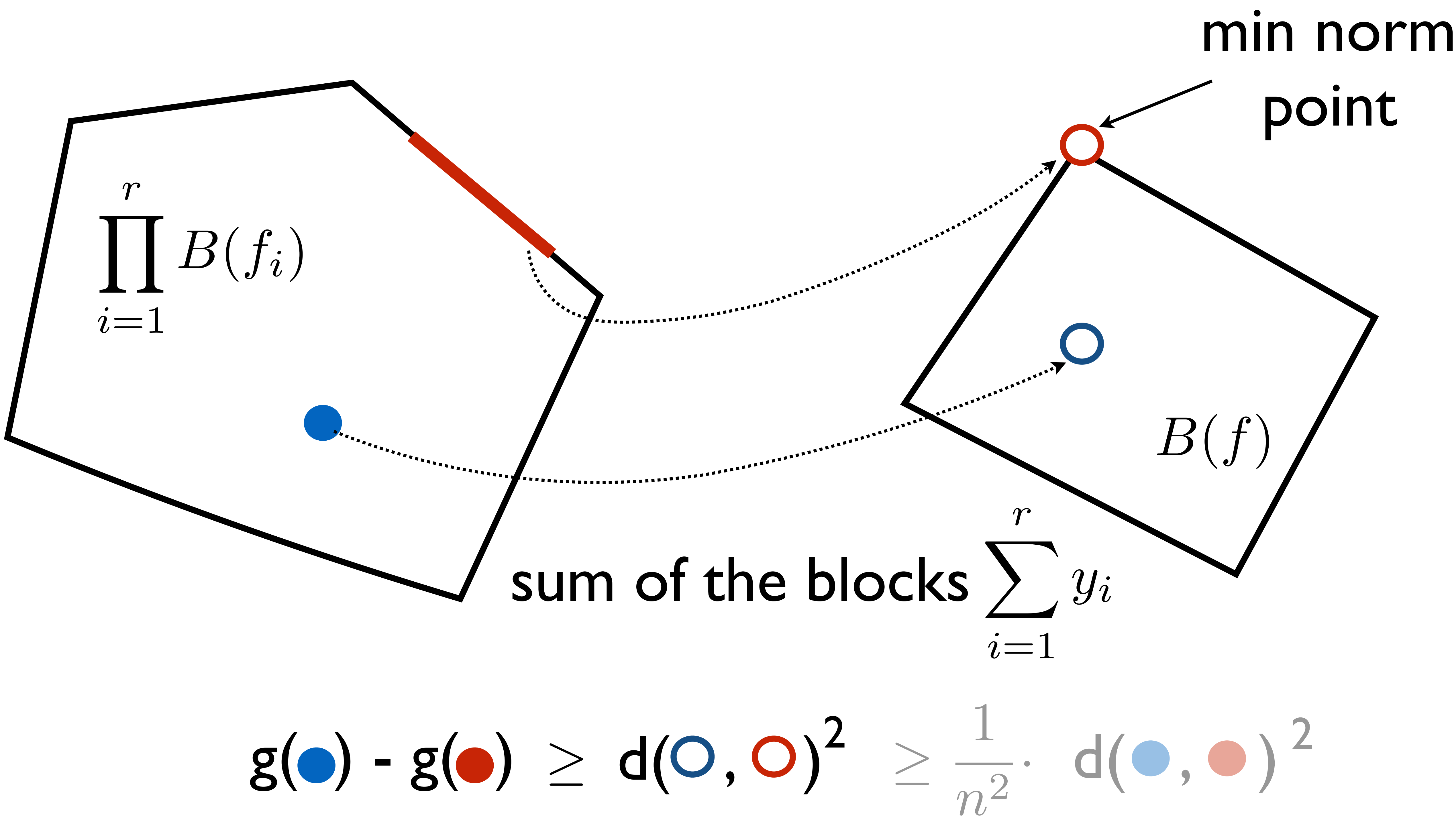
Our problem: $$\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$$



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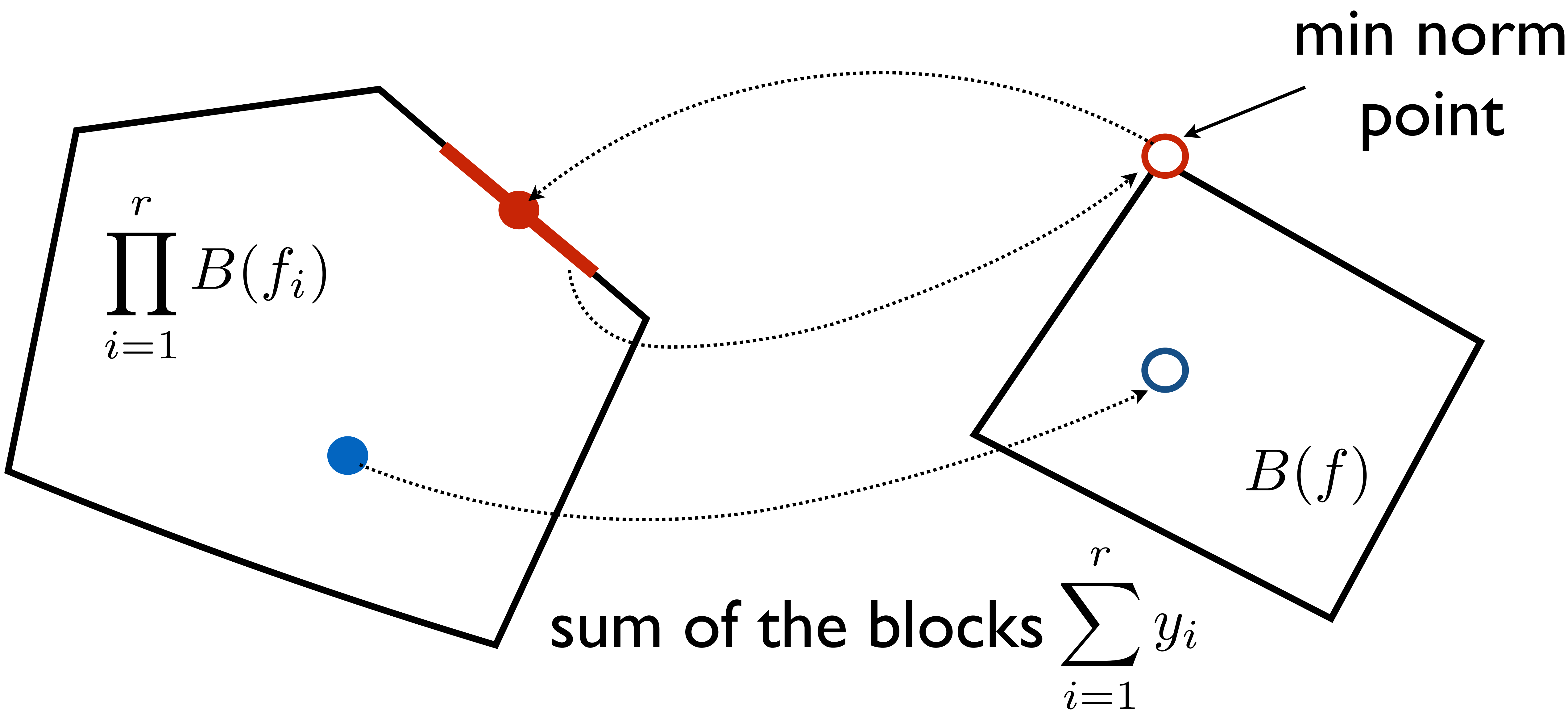
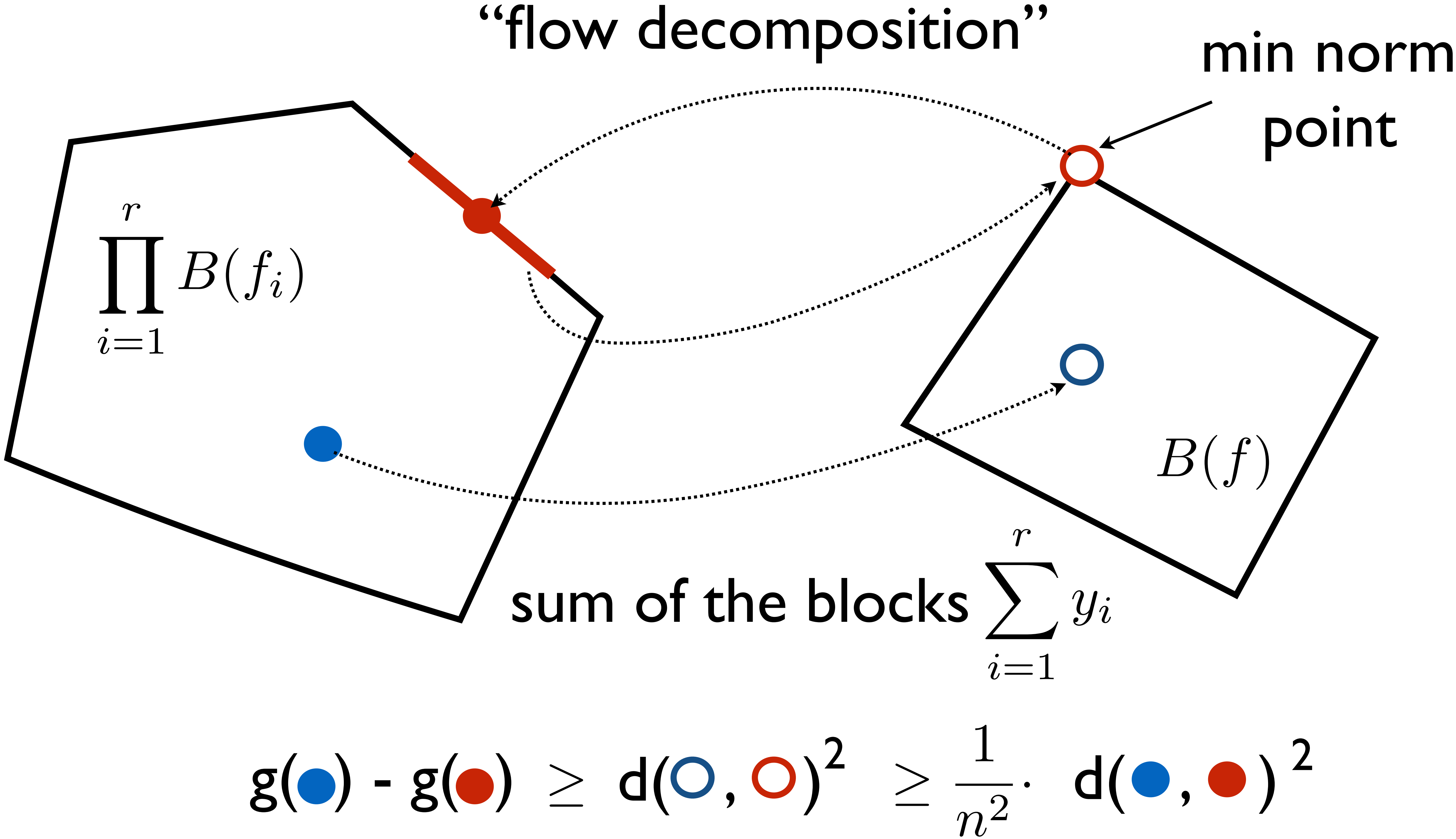
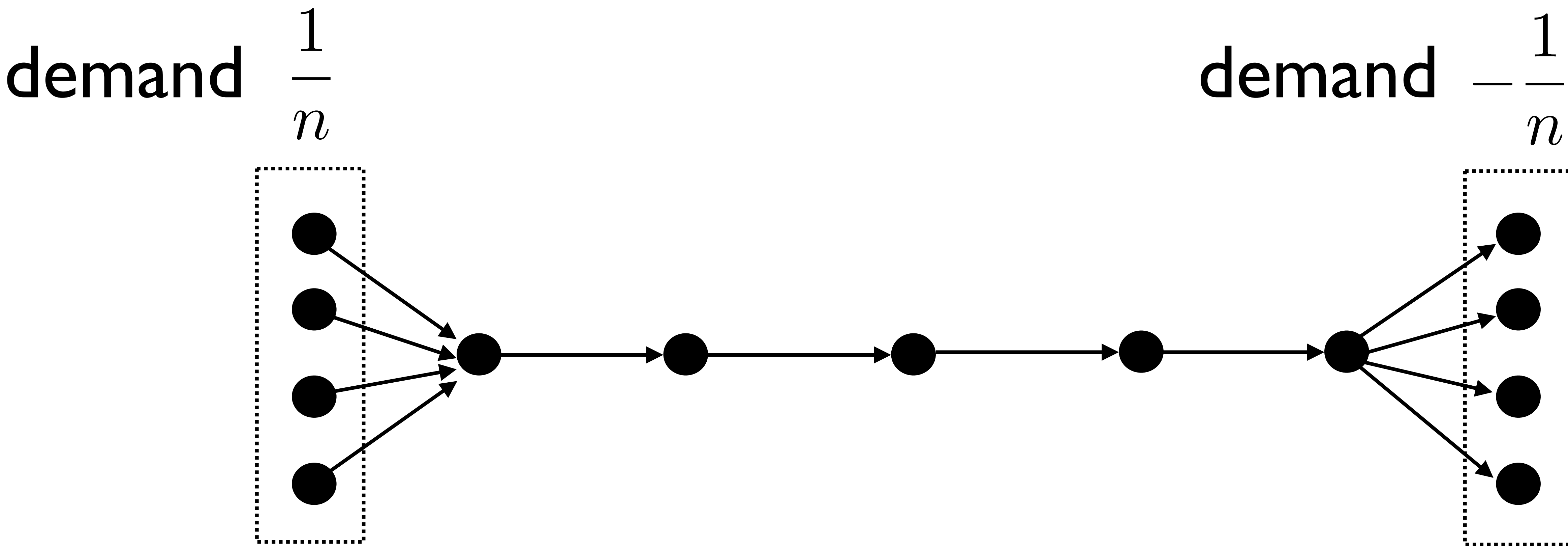


g(●) - g(●) $\geq$ d(○, ○)^2 $\geq \frac{1}{n^2} \cdot$ d(●, ●)^2

Our problem: $\min \left\{ g(y) = \left\| \sum_{i=1}^{r} y_i \right\|^2 : y \in \prod_{i=1}^{r} B(f_i) \right\}$





$$\ell_2^2 \text{ of demand} = \frac{1}{n^2} \cdot n$$

$$\ell_2^2 \text{ of flow} = n$$

$$\ell_2^2 \text{ of demand} = \frac{1}{n^2} \cdot (\ell_2^2 \text{ of flow})$$

Summary

Algorithms that leverage both convex optimization and the combinatorial structure (via graph max flow)