Dealing with Constraints via Random Permutation

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Motivation

Optimization for Large-scale Problems

- How to solve large-scale constrained problems?
- Popular idea: solve small subproblems
  - CD (Coordinate Descent)-type: $\min f(x_1, \ldots, x_N)$.
    $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_N$

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  - CD (Coordinate Descent)-type: $\min f(x_1, \ldots, x_N)$.
    $x_1 \to x_2 \to \cdots \to x_N$
  - SGD (Stochastic Gradient Descent): $\min \sum_i f_i(x)$.
    $f_1 \to f_2 \to \cdots \to f_N$
- Widely (and wildly) used in practice: deep learning, glmnet for LASSO, libsvm for SVM, recommendation systems, EM

Optimization for Large-scale Problems

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- Widely (and wildly) used in practice: deep learning, glmnet for LASSO, libsvm for SVM, recommendation systems, EM
- Compared to other ideas, e.g., first-order methods and sketching:
  - Similar cheap iteration idea
  - “Orthogonal” to other ideas, so can combine

Go Beyond Unconstrained Optimization

- Many problems have (linear) constraints
- Classical convex optimization, e.g., linear programming.
  - Combinatorial optimization (this workshop)
  - Operations research problems
- Machine learning applications, e.g., structured sparsity and deep learning
- Can we apply the *decomposition* idea? Turn out to be tricky!
- Algorithm: CD + multiplier $\longrightarrow$ ADMM (Alternating Direction Method of Multipliers)

Multi-block ADMM

- Consider a linearly constrained problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^N} \quad & f(x_1, x_2, \ldots, x_n) \\
\text{s.t.} \quad & Ax \triangleq A_1 x_1 + \cdots + A_n x_n = b, \\
& x_j \in \mathcal{X}_j \subseteq \mathbb{R}^{d_j},\ j = 1, \ldots, n.
\end{aligned}
\tag{1}
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- Augmented Lagrangian function:

$$L_\gamma(x_1, \ldots, x_n; \lambda) = f(x) - \langle \lambda, \sum_i A_i x_i - b \rangle + \frac{\gamma}{2} \| \sum_i A_i x_i - b \|^2.$$

- **Multi-block ADMM** (primal CD, dual ascent)

$$\begin{cases} x_1 \longleftarrow \arg\min_{x_1 \in \mathcal{X}_1} L_\gamma(x_1, \ldots, x_n; \lambda), \\ \quad\quad\quad \vdots \\ x_n \longleftarrow \arg\min_{x_n \in \mathcal{X}_n} L_\gamma(x_1, \ldots, x_n; \lambda), \\ \lambda \longleftarrow \lambda - \gamma(Ax - b), \end{cases} \tag{2}$$

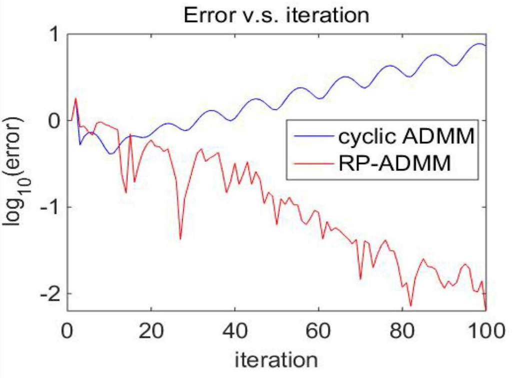
Divergence of 3-block ADMM

- 2-block ADMM converges [Glowinski-Marroco-1975], [Gabay-Mercier-1976].
- 3-block ADMM may diverge [Chen-He-Ye-Yuan-13].
- Example: solve 3×3 linear system

$$
\begin{aligned}
\min_{x_1, x_2, x_3} \quad & 0, \\
\text{s.t.} \quad & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,
\end{aligned} \tag{3}
$$

Random Permutation Helps

- RP-ADMM: Randomly permute update order $(312), (123), (213), \ldots$



- New outlet?

Outline

Background

Two-block ADMM

- ADMM usually refers to **2-block ADMM**
  [Glowinski-Marroco-75], [Gabay-Mercier-76],
  [Boyd-Parikh-Chu-Peleato-Eckstein-11] (5800 citations)

$$\begin{aligned} \min_{x,y} \quad & f(x) + g(y) \\ \text{s.t.} \quad & Ax + By = c. \end{aligned} \tag{4}$$

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- Augmented Lagrangian function:
  $L(x, y; \lambda) = f(x) + g(y) - \langle \lambda, Ax + By - c \rangle + \frac{\gamma}{2} \|Ax + By - c\|^2.$
- Two-block ADMM:

$$\begin{cases} x \longleftarrow \arg\min_x L(x, y; \lambda), \\ y \longleftarrow \arg\min_y L(x, y; \lambda), \\ \lambda \longleftarrow \lambda - \gamma(Ax + By - c). \end{cases} \tag{5}$$

Variants of multi-block ADMM

- Multi-block cyclic ADMM may diverge
- **Question**: How to make multi-block ADMM converge?
- Approach 1: Change algorithm.
  - Gaussian substitution [He-Tao-Yuan-11] .

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- **Question**: How to make multi-block ADMM converge?
- Approach 1: Change algorithm.
  - Gaussian substitution [He-Tao-Yuan-11] .
- Approach 2: Change algorithm + problem.
  - Strong convexity + small stepsize $\gamma = O(\sigma/N)$ [Han-Yuan-12] .
- And many other related works [Deng-Lai-Peng-Yin-13], [Lin-Ma-Zhang-14], [Lin-Ma-Zhang-15], [Sun-Toh-Yang-14], [Li-Sun-Toh-15] ,etc.

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- And many other related works [Deng-Lai-Peng-Yin-13], [Lin-Ma-Zhang-14], [Lin-Ma-Zhang-15], [Sun-Toh-Yang-14], [Li-Sun-Toh-15] ,etc.
- What is a minimal modification + stepsize 1?

Apply Randomization Trick to ADMM

- We know:
  1) ADMM may diverge;
  2) Randomization helps CD/SGD [Strohmer-Vershynin-08], [Leventhal-Lewis-10], [Nesterov-11], [Roux et al-12], [Blatt et al-07]

- First idea: (independently) randomized ADMM
  $(x_3 x_1 x_1 \lambda), (x_1 x_3 x_2 \lambda), \ldots$
- **Bad news**: can diverge!
  - Diverge for Gaussian data
  - Converge for the counter-example in [Chen-He-Ye-Yuan-13]

- Second idea: random permutation
  $(x_3 x_1 x_2 \lambda), (x_2 x_1 x_3 \lambda), \ldots$.
  It always converges in the simulation.

Summarize ADMM Variants

- **Cyclic**: $(x_1 x_2 x_3 \lambda), (x_1 x_2 x_3 \lambda), \ldots$.
- **Random permutation** (RP): $(x_3 x_1 x_2 \lambda), (x_2 x_1 x_3 \lambda), \ldots$
- **Independently random** (IR): $(x_3 x_1 x_1 \lambda), (x_2 x_1 x_2 \lambda), \ldots$

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- **Independently random** (IR): $(x_3 x_1 x_1 \lambda), (x_2 x_1 x_2 \lambda), \ldots$
- Simulation: RP always converges, other two can diverge.

  RP $>$ IR, Cyclic.
- Wait...practitioners may not care? (divergence of cyclic ADMM is just worst-case?)

Numerical Experiments: Cyc-ADMM Often Diverges

Table 1: *Solve Linear Systems by Cyc-ADMM, RP-ADMM and GD*

N	Diverg. Ratio for Cyc-ADMM	Iterations for $\epsilon = 0.001$		
		CycADMM[i]	RPADMM	GD
Gaussian $N(0,1)$				
3	0.7%	3.2e01	8.8e01	1.4e02
100	3%	1.0e03	7.4e03	6.5e03
Uniform $[0,1]$				

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3	3.2%	7.0e01	2.6e02	6.0e02
100	100%	N/A	1.4e04	9.7e04

- Cyc-ADMM can diverge often; sometimes diverges w.p. 100%.
  - In fact, easy to diverge if off-diagonal entries are large. Cyc-ADMM is somewhat similar to Cyc-BCD.
- RP-ADMM converges faster than GD.

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- Order (123) fails; maybe (231) works?

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- Order (123) fails; maybe (231) works?
- **Fact**: Any fixed order diverges.

Summary: Why We Want to Understand RP-ADMM

Theoretical Curiosity + Practical Need.

- **First**, decomposition idea can be useful for solving constrained problems
  - cyclic ADMM may not converge.
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Summary: Why We Want to Understand RP-ADMM

Theoretical Curiosity + Practical Need.

- **First**, decomposition idea can be useful for solving constrained problems
  - cyclic ADMM may not converge.
  - RP-ADMM: a simple solution
- **Second**, help understand RP-rule, e.g. RP-CD, RP-SGD.
  - Many people write IR papers.
  - Many people run RP experiments (default choice in deep learning package e.g. Torch)

Convergence Analysis of RP-ADMM

Solve Linear System

- Solve a square linear system of equations ($f_i = 0, \ \forall i$).

$$\begin{aligned} \min_{x \in \mathbb{R}^N} \quad & 0, \\ \text{s.t.} \quad & A_1 x_1 + \cdots + A_n x_n = b, \end{aligned} \tag{6}$$

where $A = [A_1, \ldots, A_n] \in \mathbb{R}^{N \times N}$ is full-rank, $x_i \in \mathbb{R}^{d_i}$ and $\sum_i d_i = N$.

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- Why linear system?
    - Basic constrained problem
    - Already difficult to analyze.

Main results

Theorem 1[ii]

The expected output of RP-ADMM converges to the solution of (6), i.e.

$$\{E_{\xi_k}(y^k)\}_{k\to\infty} \longrightarrow y^*. \tag{7}$$

Remark: Expected convergence $\neq$ convergence, but is a strong evidence for convergence.

Denote M as the expected iteration matrix of RP-ADMM.

Theorem 2

$\rho(M) < 1$, i.e. spectral radius of M is less than 1.

[ii] S, Luo, Yinyu Ye, "On the Expected Convergence of Randomly Permuted ADMM",

Why Spectral Analysis?

Meta-proof-frameworks in optimization don't work (or I don't know how).

Potential function.

- E.g. GD, C-CD or R-CD for $min_x x^T Ax$, the potential function is the (expected) objective.
- Our system: $E(y^{k+1}) = ME(y^k)$, but $\|M\| > 2.3$ for the counterexample. $y^T My$ is not a potential function.
- There exists P such that $P - M^T PM$ is PSD, and $y^T Py$ is a potential function. Hard to compute P.

Contraction: can prove convergence of 2-block ADMM.

- Again, how to distinguish between cyclic ADMM and PR-ADMM?
- Not a big surprise. 2-block is very special.

Switched Linear System

RP-ADMM can be viewed as switched linear systems:

$$y_{k+1} = M_k y_k,$$

where $M_k \in \{B_1, \dots, B_m\}$. For RP-ADMM, $m = n!$.

Our problem: each single B_i is not stable (corresponding to a single order), but randomly picking from $\{B_1, \dots, B_m\}$ makes the system stable.

Related to product of random matrices [Furstenberg-Kesten-60]; but hard to apply to our case.

A useful first step is to find a convex combination of B_i's that is stable [Wicks et al.-94]

Theorem 2: a Pure Linear Algebra Problem

- Define matrix L_σ by deleting half off-diagonal entries of $A^T A$

$$L_\sigma[\sigma(i), \sigma(j)] \triangleq \begin{cases} A_{\sigma(i)}^T A_{\sigma(j)} & j \leq i, \\ 0 & j > i, \end{cases} \tag{8}$$

- Example:

$$L_{(231)} = \begin{bmatrix} 1 & A_1^T A_2 & A_1^T A_3 \\ 0 & 1 & 0 \\ 0 & A_3^T A_2 & 1 \end{bmatrix}.$$

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- Define $Q = E(L_\sigma^{-1})$. Compare: $E(L_\sigma) = \frac{1}{2}(I + A^T A)$.
- Theorem 2 claims $\rho(M) < 1$, with M being a function of A:

$$M = \begin{bmatrix} I - QA^T A & QA^T \\ -A + AQA^T A & I - AQA^T \end{bmatrix}. \tag{9}$$

Two Main Lemmas to Prove Theorem 2: Lemma 1

- **Step 1**: Relate M to a symmetric matrix AQA^T.

Lemma 1

$$\lambda \in \mathrm{eig}(M) \iff \frac{(1-\lambda)^2}{1-2\lambda} \in \mathrm{eig}(AQA^T). \tag{10}$$

When Q is symmetric, we have

$$\rho(M) < 1 \iff \mathrm{eig}(AQA^T) \subseteq (0, \frac{4}{3}). \tag{11}$$

- This lemma treats Q as a black box.

Lemma 2

- **Step 2**: Bound eigenvalues of AQA^T.

Lemma 2

For any non-singular A, let $Q = E(L_\sigma^{-1})$ where L_σ is given by (8), then

$$\mathrm{eig}(AQA^T) \subseteq (0, \frac{4}{3}). \tag{12}$$

- **Remark**: $4/3$ should be tight: we find examples > 1.33.

What is AQA^T

- AQA^T relates to RP-CD (quadratic): $x \longleftarrow (I - QA^T A)x$.
  - RP-CD converges $\Longleftrightarrow$ $\text{eig}(AQA^T) \in (0, 2)$.

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- Cyc-CD (quadratic): $x \longleftarrow (I - L_{12\ldots n}^{-1} A^T A)x$
  - Cyc-CD converges $\Longleftrightarrow$ $\mathrm{eig}(AL_{12\ldots n}^{-1} A^T) \in (0, 2)$.

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- Cyc-CD (quadratic): $x \longleftarrow (I - L_{12\ldots n}^{-1} A^T A)x$
  - Cyc-CD converges $\Longleftrightarrow$ $\mathrm{eig}(AL_{12\ldots n}^{-1} A^T) \in (0, 2)$.
- **Remark**: spectrum of RP-CD is "nicer" than Cyc-CD.
  - "Pre-assigned" space for RP-ADMM.

Proof Sketch of Lemma 2

- **Step 2.1**: Symmetrization $\Longrightarrow$ induction formula of $Q = E(L_\sigma^{-1})$.
- **Step 2.2**: Induction inequality of $\rho = \rho(QA^T A)$:

$$\rho \leq P(\hat{\rho}, \rho) \triangleq \max_{\theta \geq 0} \ \hat{\rho} + \theta \left(\frac{\rho}{4\rho - 4 + \theta} - 1 \right), \qquad (13)$$

where $\hat{\rho}$ is the $(n-1)$-block analog of $\rho(QA^T A)$.

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  where $\hat{\rho}$ is the $(n-1)$-block analog of $\rho(QA^T A)$.

- **Remark**: $\rho = 4/3$ is the fixed point of $\rho = P(\rho, \rho)$.
  - $P(\frac{4}{3}, \frac{4}{3}) = \frac{4}{3} + \max_{\theta \ge 0} \theta(\frac{\rho}{\rho + \theta} - 1) = \frac{4}{3} - \max_{\theta \ge 0} \frac{\theta^2}{\rho + \theta} = \frac{4}{3}$.

Variants of ADMM

Interesting Byproduct: New Randomization Rule

- Finding: 2-level symmetrization is enough.
- New algorithm: Bernolli randomization (BR).
  - Phase 1: sweep $1, \ldots, n$; for each block, update w.p. $1/2$;
  - Phase 2: sweep $n, \ldots, 1$; if previously not updated, now update.
- Examples of valid order: (2, 3; 4, 1), (1, 2, 4; 3).
  Non-examples: $(3, 4, 1, 2)$

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- Examples of valid order: (2, 3; 4, 1), (1, 2, 4; 3).
  Non-examples: (3, 4, 1, 2)
- **Proposition**: BR-ADMM converges in expectation.

Another Way to Apply Decomposition to Constraints

The problem is still $\min_x f(x), \text{ s.t. } Ax = b$.

Original ADMM: each cycle is (x_1, x_2, x_3, λ).

Primal-dual ADMM: each cycle is $(x_1, \lambda, x_2, \lambda, x_3, \lambda)$.

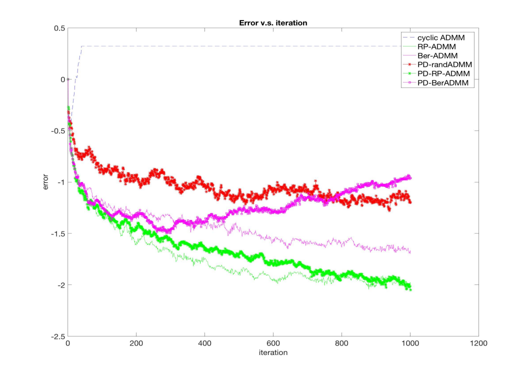
- Cyclic version still can diverge for the counter-example.
- Randomized version was proven to converge with high probability (e.g. [Xu-2017])

However, in simulation, randomized PD-ADMM is much slower than other versions (next page).

Comparison of Algorithms

Uniform [0,1] data:

- cyclic ADMM and primal-dual version of Bernolli randomization fail to converge.
- PD-rand-ADMM is much slower than others.

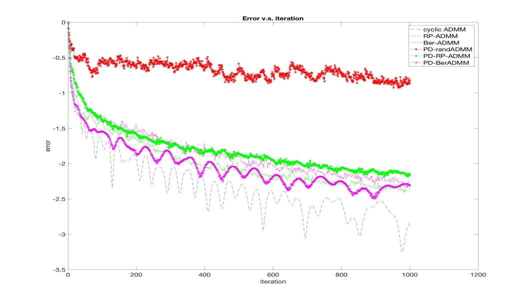


Comparison of Algorithms (cont'd)

Standard Gaussian data:

- PD-rand-ADMM is significantly slower than all other methods.
- Recall: randomized ADMM is the only method that diverges!

Strange issue: (independent) random rule is bad for Gaussian data.



Comparison of Algorithms (cont'd)

Simple summary of different methods (no stepsize tuning):

Update Order	Original Version	Primal-Dual Version
cyclic	Diverge	Diverge
indep. random	Diverge	Converge but very slow
Bernolli random	Converge	Diverge
random permutation	Converge [iii]	Converge? [iv]

Observation: random permutation is a universal "stabilizer".

Open question: Any convergence analysis of P-D version of RP-ADMM?

[iii] Only expected convergence for simple problems are proved.

[iv] Based on extensive simulation

Convergence Rate: Related Result and Discussion

Convergence Rate of Cyclic CD

- Status: many results on (independently) random rule; little understanding of RP/cyclic/whatever rule
  - A few works [Recht-Re-12], [Gurbuzbalaban-Ozdaglar-Parrilo-15], [Wright-Lee-17] studied random permutation, but why RP is better than IR in general is still unknown
  - Mark Schmidt talked about Gauss-Southwell rule this morning.
- Classical literature says: they are "essentially cyclic" rule, all converge for CD
- However, their convergence speed can be quite different

Convergence Rate of Cyclic CD

- **Question**: "true" convergence rate of cyclic CD or Gauss-Seidal method (Gauss 1823, Seidel 1874)?
- Why care cyclic order?
  - Understanding "non-independently-randomized" rule
  - Almost all convergence rate results on cyclic rule immediately apply to RP-rule
  - Randomization not available sometimes
- **Puzzle**: known rates can be sometimes n^2 times worse than R-CD for quadratic case [Beck-Tetruashvili-13], [Sun-Hong-15]
- Some claim cyclic order must be bad; an example given by Strohmer and Richtarik (independently) showed this.
  - Only $O(n)$ gap between C-CD and R-CD;
  - Only fails for some particular orders. Randomly pick order and fix, then becomes fast.

Rate of Cyclic CD

- **Answer** [v]: up to n^2 times worse than R-CD, for equal-diagonal quadratic case.

Table 2: Complexity for equal-diagonal case (divided by $n^2 \kappa \log \frac{1}{\epsilon}$ and ignoring constants. $\tau = \lambda_{\max}/\lambda_{\text{avg}} \in [1, n]$)

	C-CD	GD	R-CD	SVRG
Lower bound	τ	1	$\frac{1}{\tau}$	$\frac{1}{\tau}$
Upper bound	$\min\{\tau \log^2 n, n\}$	1	$\frac{1}{\tau}$	$\frac{1}{\tau}$

- Lower bound is based on analyzing one example. Steven Wright mentioned the example in two talks starting from 2015 summer. We independently discover the example.

- Analysis: tricky issue on non-symmetric matrix update. (even more tricky than ADMM case)

[v] Sun, Ye, "Worst-case Convergence Rate of Cyclic Coordinate Descent Method: O(n^2) Gap with Randomized Versions", 2016.

Relation to Other Methods

- Same gap exists for Kaczmarz method and POCS (Projection onto Convex Sets).
- POCS, dating back to Von Neumann in 1930's, has been studied extensively. See a survey [Bauschke-Borwein-Lewis-1997]
- Convergence rate given by Smith, Solmon and Wagner in 1977. Still in textbook.
- Translate to CD: a rate dependent on all eigenvalues.
  - Turn out to be ∞-times worse than our bound for the example.
  - Always worse than our bound (up to $O(\log^2 n)$ factor)

Convergence Speed of RP-CD and AM-GM inequality

Random permutation was studied in [Recht-Re'2012], mainly for RP-SGD.

Conjecture: Matrix AM-GM inequality ([Recht-Re'2012])

Suppose $A_1, \ldots, A_n \succeq 0$, then

$$\|\frac{1}{n!} \sum_{\sigma \text{ is a permutation}} A_{\sigma_1} \ldots A_{\sigma_n}\| \leq \|\frac{1}{n}(A_1 + \cdots + A_n)\|^n.$$

If this inequality holds, then the convergence rate of RP-CD for quadratic problems is faster than R-CD.

Zhang gave a proof for $n = 3$; Duchi gave a proof for a variant, again for $n = 3$.

Another variant of matrix AM-GM inequality

Conjeture (variant of matrix AM-GM inequality): If P_i is a projection matrix, $i = 1, \dots, n$, then

$$\frac{1}{n!} \sum_{\sigma \text{ is a permutation}} P_{\sigma_1} \dots P_{\sigma_n} \preceq \frac{1}{n}(P_1 + \dots + P_n). \tag{14}$$

Claim: If matrix AM-GM inequality (14) holds, then combining with our result $\mathrm{eig}(QA^T A) \in (0, 4/3)$, RP-CD has better convergence rate than that of R-CD for convex quadratic problems.

We know $\mathrm{eig}(I - QA^T A) = \mathrm{eig}(M_{RP-CD}) \in (-1, 1)$.

- Our result is about the left end by improving -1 to $-1/3$.
- Matrix AM-GM inequality (14) is about the right end near 1

We have some results on the expected convergence rate of RP-CD and RP-ADMM. Skip here.

Summary

- **Main result**: convergence analysis of RP-ADMM.
  - **Implication 1** (problem): solver for constrained problems.
  - **Implication 2** (algorithm): RP better.
    Even much better than independently randomized rule.
- Implication for RP-CD: “truncate” one side spectrum.
- Tight analysis of “non-independent-randomization”: worst-case understanding of cyclic order, but more works are needed.
- Lots of open questions:
  - convergence of PD version of RP-ADMM
  - AM-GM inequality
  - Jacobi preconditioning

Thank You!