Approximating the Permanent of Positive Semidefinite Matrices

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Joint work with

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Determinant

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) M_{1,\sigma(1)} \dots M_{n,\sigma(n)}$$

Permanent

$$\operatorname{per}(M) = \sum_{\sigma \in S_n} M_{1,\sigma(1)} \dots M_{n,\sigma(n)}$$

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2×2 Example

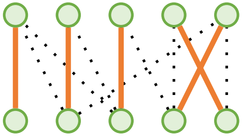
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(M) = ad - bc$$

$$\mathrm{per}(M) = ad + bc$$

Complexity of Permanent

- #P-hard to compute $\mathrm{per}(M)$ for $0/1$ matrices [Valiant'79].



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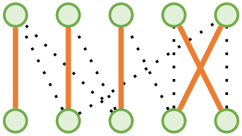

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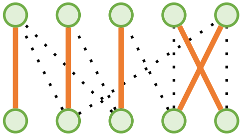
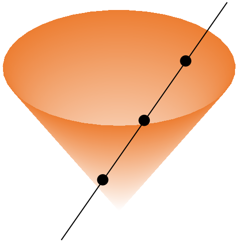


- #P-hard to compute sign of $\mathrm{per}(M)$ [Aaronson'11].
- #P-hard to compute $\mathrm{per}(M)$ for $M \succeq 0$ [Grier-Schaeffer'16].



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Additive $\pm\epsilon \|M\|^n$ approximation [Gurvits'05].

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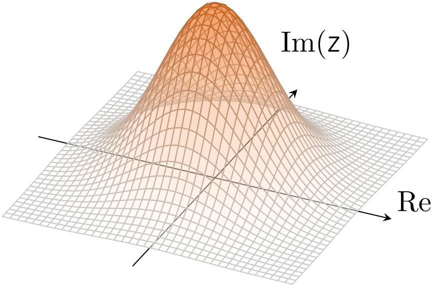
- Improved to $\frac{n!}{k!^{n/k}}$-approximation in time $2^{O(k+\log(n))}$ [Lieb'66].

Theorem [A-Gurvits-Oveis Gharan-Saberi'17]

The permanent of PSD matrices $M \in \mathbb{C}^{n\times n}$ can be approximated, in deterministic polynomial time, within

$$(e^{\gamma+1})^n \simeq 4.84^n.$$

Complex Gaussians



$$z \sim \mathbb{C}\mathcal{N}(0,1)$$

$$\mathbb{P}[z] = \frac{1}{\pi} e^{-|z|^2}$$

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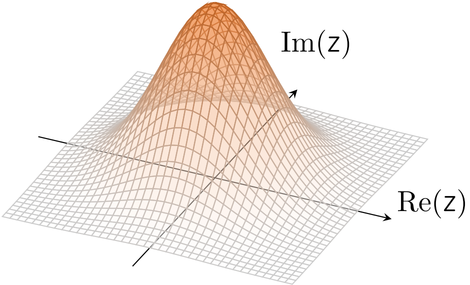

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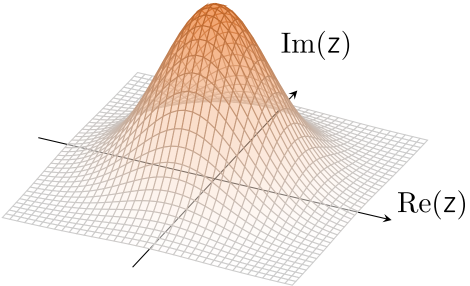

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- General (circularly-symmetric) complex Gaussian:

$$g = Cz,$$

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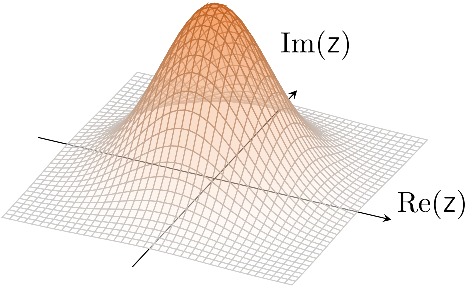

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Wick's Formula

$$\mathbb{E}\left[|g_1|^2 \ldots |g_n|^2\right] = \mathrm{per}(CC^\dagger).$$

Schur Power

The Schur power of an $n \times n$ matrix M is

$$n! \left\{ \underbrace{\begin{bmatrix} \vdots & \vdots & \vdots \\ \vdots & M_{\sigma(1),\tau(1)} \cdots M_{\sigma(n),\tau(n)} & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{n!} \right.$$

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Permanent is Loewner-Monotone

$$M_1 \succeq M_2 \succeq 0 \implies \mathrm{per}(M_1) \geq \mathrm{per}(M_2) \geq 0$$

Approximation using Monotonicity

- Permanent is monotone w.r.t. $\succeq$:

$$\mathsf{D} \succeq \mathsf{M} \succeq \mathsf{v}\mathsf{v}^\dagger \implies \mathrm{per}(\mathsf{D}) \geq \mathrm{per}(\mathsf{M}) \geq \mathrm{per}(\mathsf{v}\mathsf{v}^\dagger).$$

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Theorem [A-Gurvits-Oveis Gharan-Saberi'17]

For any $M \succeq 0$ there exist diagonal matrix D and rank-1 matrix $vv^\dagger$ such that

$$D \succeq M \succeq vv^\dagger,$$

and $\mathrm{per}(D) \leq 4.85^n \, \mathrm{per}(vv^\dagger)$.

Computing the Approximation

- Solve and output the following

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- Equivalently solve the convex program

$$\begin{array}{ll} \inf_{\mathsf{D}^{-1}} & \log(\mathrm{per}((\mathsf{D}^{-1})^{-1}), \\ \text{subject to} & \mathsf{M}^{-1} \succeq \mathsf{D}^{-1} \succeq 0. \end{array}$$

- No such convex program for the best rank-1 matrix.

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- Prove the "PSD Van der Waerden"

PSD Van der Waerden [A-Gurvits-Oveis Gharan-Saberi'17]

If B is a correlation matrix and P the orthogonal projection onto the image of B, then

$$\mathrm{per}(P) \geq 4.85^{-n}.$$

PSD Van der Waerden

- Given correlation matrix B (i.e. $\mathsf{B} \succeq 0$ and $\mathrm{diag}(\mathsf{B}) = 1$), want to show

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- Show for some unit vector $\mathsf{v} \in \mathrm{imag}(\mathsf{B})$

$$\mathrm{per}(\mathsf{v}\mathsf{v}^{\dagger}) \geq 4.85^{-\mathsf{n}}.$$

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- Show for some unit vector $v \in \mathrm{imag}(B)$

$$\mathrm{per}(vv^\dagger) \geq 4.85^{-n}.$$

- Let B be the Gram matrix of unit vectors $u_1, \dots, u_n$. Generate v by normalizing the projection vector of $u_1, \dots, u_n$ onto some direction g

$$v = \frac{[g^\dagger u_1 \dots g^\dagger u_n]}{|[g^\dagger u_1 \dots g^\dagger u_n]|}.$$

GM-AM Ratio

- Let u be a random vector (e.g., uniformly sampled from $u_1, \ldots, u_n$). Define the GM-AM ratio as:

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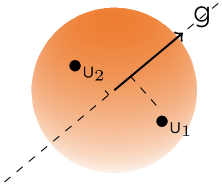
Lemma [A-Gurvits-Oveis Gharan-Saberi'17]

If u is a random unit vector, there exists g such that the GM-AM ratio of $g^\dagger u$ is at least

$$e^{-\gamma}.$$



Complex Gaussians Come Back

- Let g be a standard complex Gaussian. Then with positive probability we have:

$$\text{GM-AM}(g^\dagger u) \geq \frac{\mathbb{E}\left[e^{\mathbb{E}\left[\log(|g^\dagger u|^2)\right]}\right]}{\mathbb{E}\left[|g^\dagger u|^2\right]} \geq \frac{e^{\mathbb{E}\left[\log(|g^\dagger u|^2)\right]}}{\mathbb{E}\left[|g^\dagger u|^2\right]}$$

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- But

$$\mathbb{E}\left[\log(\left|g^\dagger u\right|^2)\right] = -\gamma,$$

and

$$\mathbb{E}\left[\left|g^\dagger u\right|^2\right] = 1.$$

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