A constant-factor approximation algorithm for the asymmetric travelling salesman problem

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Joint work with

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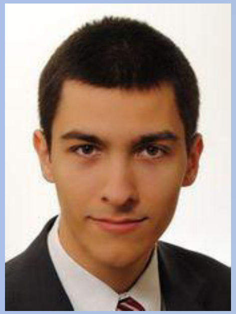
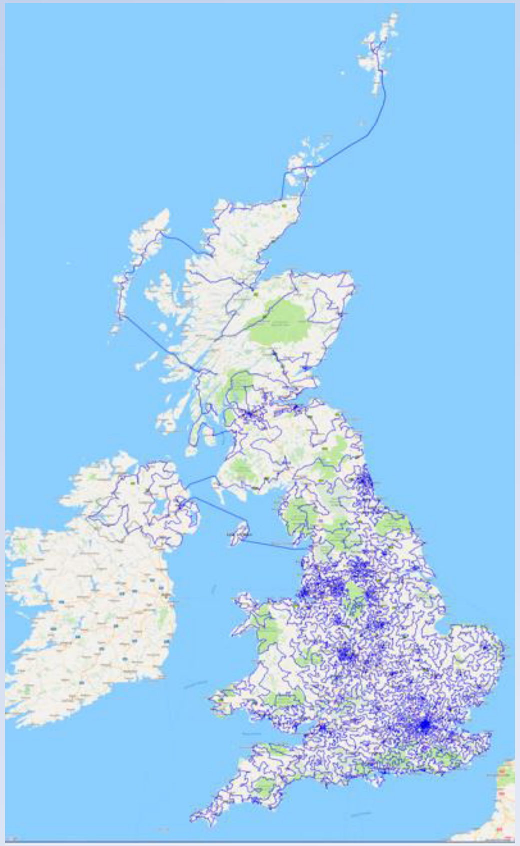
École Polytechnique Fédérale de Lausanne



Travelling salesman problem

Given n cities and their pairwise distances,
find a shortest tour visiting all n cities.

- One of the best known NP-hard optimization problems
- Studied since the 19th century
- Symmetric TSP: $d(i,j) = d(j,i) \quad \forall i,j$
- Asymmetric TSP: $d(i,j) \neq d(j,i)$ is possible



UK pub tour
[Cook et al., 2015]

Triangle inequality:
$d(i,j) \leq d(i,k) + d(k,j) \ \ \forall i,j,k$

Symmetric vs Asymmetric TSP

Symmetric TSP

- 1.5-approximation algorithm [Christofides ’76]
- Graphic TSP: unweighted graph shortest path metric
  - Current best 1.4 [Sebő & Vygen ’14], following
    [Oveis Gharan, Saberi & Singh ’11]
    [Mömke & Svensson ’11]
    [Mucha ’12]

Symmetric vs Asymmetric TSP

Asymmetric TSP

- $\log_2 n$-approximation algorithm [Frieze, Galbiati & Maffioli ’82]
- $0.99 \log_2 n$ [Bläser ’03]
- $0.84 \log_2 n$ [Kaplan, Lewenstein, Shafrir & Sviridenko ’03]
- $0.67 \log_2 n$ [Feige & Singh ’07]

- $O\left(\frac{\log n}{\log \log n}\right)$ [Asadpour, Goemans, Mądry, Oveis Gharan & Saberi ’10] via thin trees.

Asymmetric TSP – recent developements

- $O(poly \log \log n)$ bound on integrality gap of LP [Anari & Oveis Gharan '15]

Constant-factor approximations:

- Bounded genus graphs [Oveis Gharan & Saberi '11]
- Node-weighted graphs [Svensson '15]
- Graphs with 2 edge weights [Svensson, Tarnawski & V. '16]

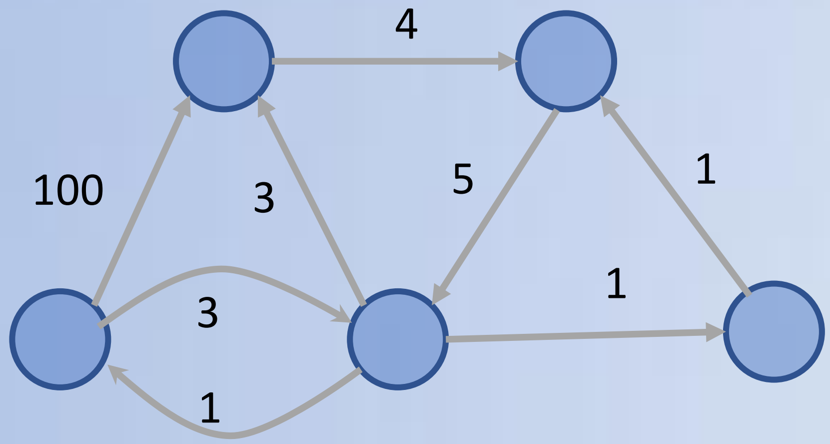
Our result: constant-factor approximation for general ATSP with respect to the Held-Karp relaxation.

ATSP – Graphic formulation

Input: directed graph $G = (V, E)$, edge weights $w: E \rightarrow \mathbb{R}_+$
Find a minimum weight **tour** F.

- Tour = closed walk visiting every vertex at least once =
  = Eulerian and connected edge multiset
- Eulerian: $\delta_F^{in}(v) = \delta_F^{out}(v)\ \forall v \in V$
- Subtour = closed walk (not necessarily connected)

In-degree & out-degree in F



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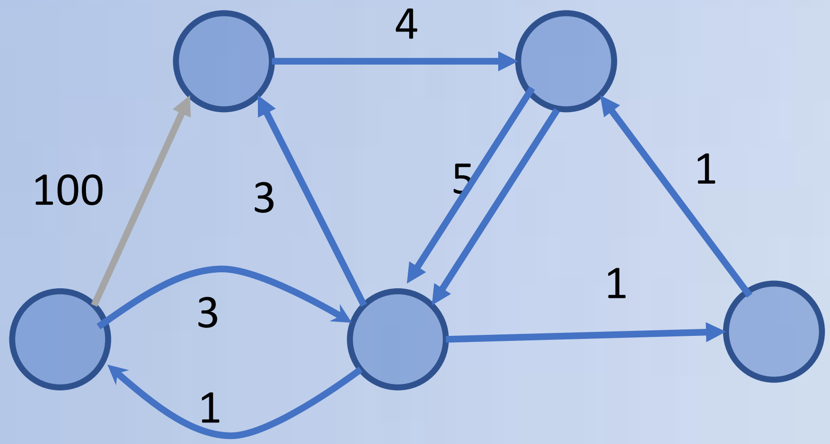

Held-Karp relaxation

- Input: $G = (V, E)$, edge weights $w: E \to \mathbb{R}_+$.
- Variables $x_e: \mathrm{E} \to \mathbb{R}_+$: multiplicity of selecting edge e.

minimize $\quad w^\top x$

subject to

$$x\left(\delta^{in}(v)\right) = x(\delta^{out}(v)) \quad \forall v \in V$$

$$x\left(\delta(S)\right) \geq 2 \qquad \forall S \subsetneq V, S \neq \emptyset$$

$$x \geq 0$$

Eulerian degree constraints

Subtour elimination constraints

Undirected degree:
$\delta(S) = \delta^{in}(S) + \delta^{out}(S)$

Held-Karp relaxation

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\begin{aligned}
\text{minimize} \quad & w^\top x \\
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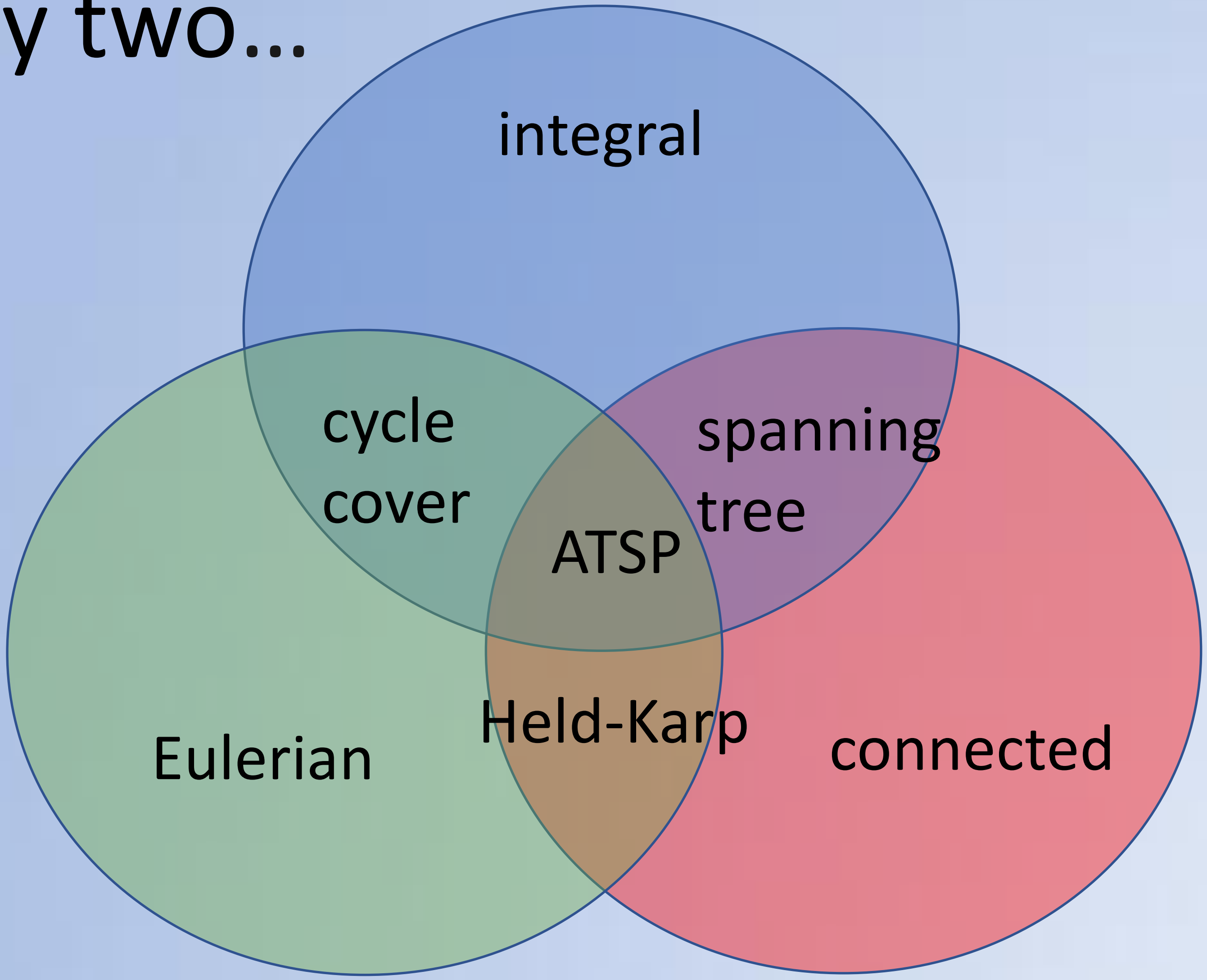
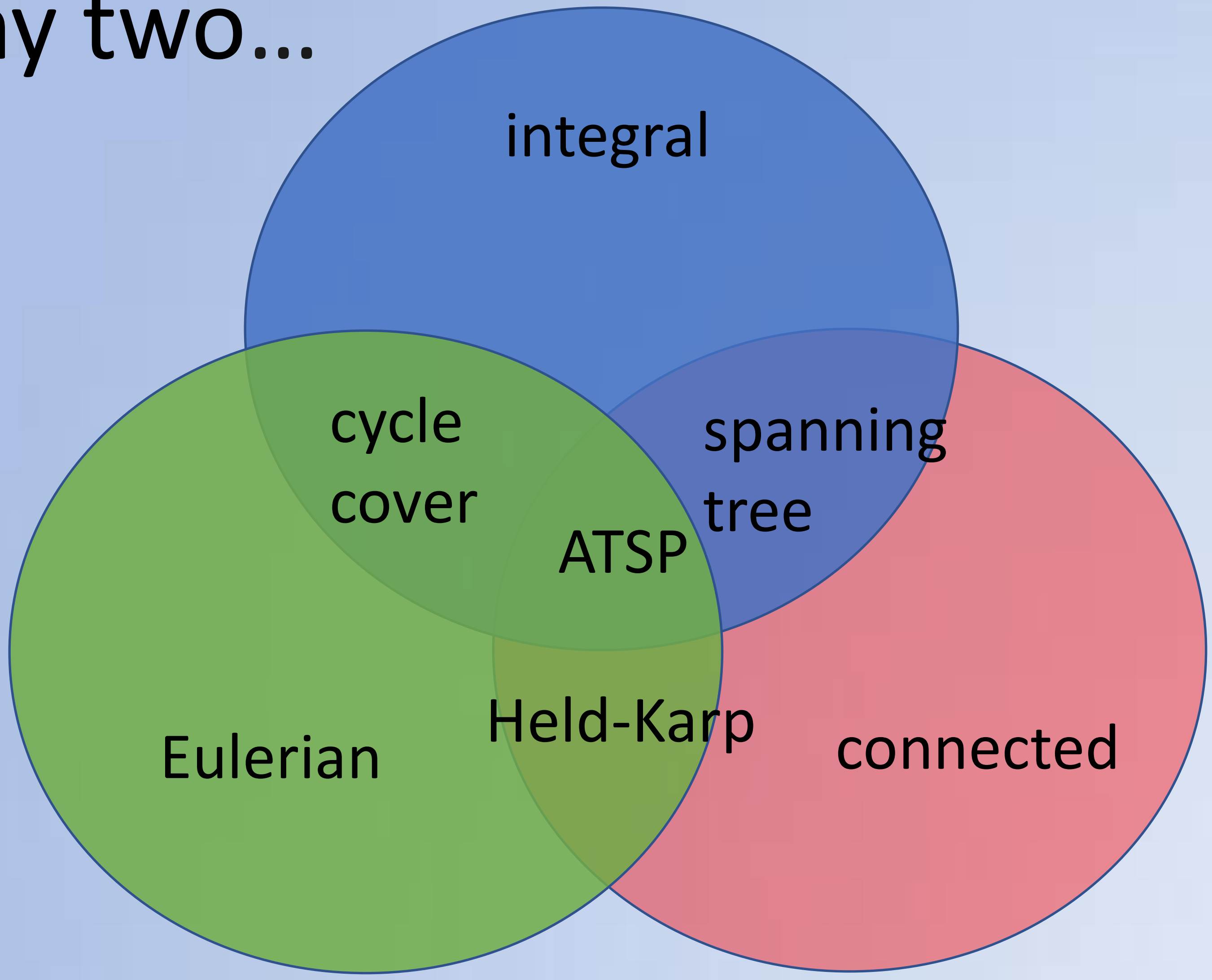
& x\big(\delta(S)\big) \geq 2 && \forall S \subsetneq V, S \neq \emptyset \\
& x \geq 0
\end{aligned}
$$

- Can be solved in polynomial time
- Integrality gap ≥ 2 [Charikar, Goemans & Karloff ’06]

Pick any two…


integral
cycle cover
spanning tree
ATSP
Held-Karp
Eulerian
connected

Pick any two…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Repeated cycle cover algorithm [Frieze, Galbiati & Maffioli ’82]

Relaxing connectivity:

1. Find minimum weight cycle cover
2. Contract and repeat

- Each cycle cover has cost $\leq OPT$
- Overall $\log_2 n$ rounds
- $\log_2 n$ approximation

Node-weighted case [Svensson'15]

Directed graph $G = (V, E)$, node weights $h: V \to \mathbb{R}_+$

$$w(u, v) = h(u) + h(v) \quad \forall u, v \in E$$

Local-Connectivity ATSP: relaxing connectivity constraints to "local"

α-light algorithm for Local-Connectivity ATSP

$(9 + \varepsilon)\alpha$-approximation for ATSP

Theorem [Svensson'15]
There exists a polytime $(27 + \varepsilon)$-approximation for node-weighted ATSP.

Roadmap

General ATSP

LP duality + uncrossing

Laminarly weighted ATSP

Graph theory: contractions

Irreducible instances

Node weighted algorithm + contractions

Vertebrate pairs

[Svensson ’15]

Local-connectivity ATSP

O(1)-light lcATSP algorithm in vertebrate pairs

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Dual of the Held-Karp relaxation

minimize $w^\top x$

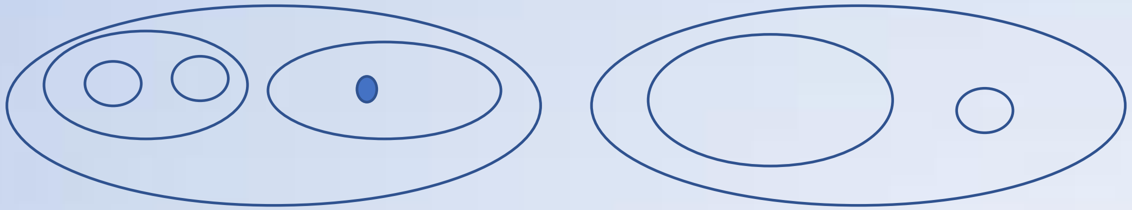
subject to

$$x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V$$
$$x(\delta(S)) \geq 2 \quad \forall \emptyset \neq S \subsetneq V$$
$$x \geq 0$$

maximize $2 \sum_{\emptyset \neq S \subsetneq V} y_S$

subject to

$$\sum_{S:(u,v)\in\delta(S)} y_S + \alpha_u - \alpha_v \leq w(u,v) \quad \forall (u,v) \in E$$
$$y \geq 0$$



- Dual can be solved in polynomial time.
- One can efficiently find an optimal (α, y) such that the support of y is a laminar family of sets.
  Efficient uncrossing [Karzanov'96]

Laminarly weighted ATSP: $\mathcal{I} = (G, \mathcal{L}, x, y)$

minimize $\quad w^\top x$

subject to

$$x(\delta^{in}(v)) = x(\delta^{out}(v)) \quad \forall v \in V$$
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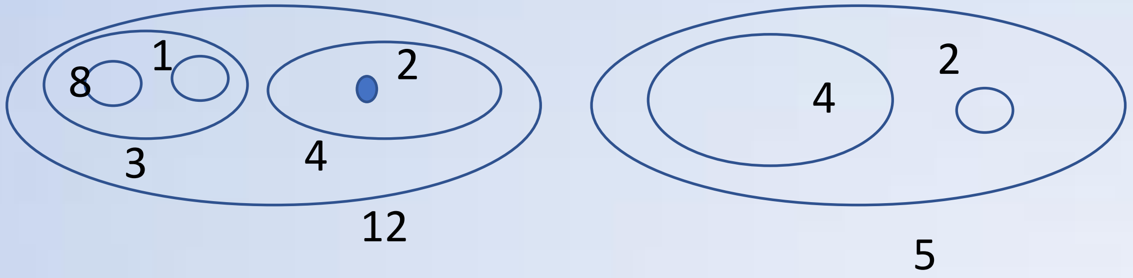

- G: directed graph
- $\mathcal{L}$: laminar family of sets
- x: feasible Held-Karp solution
  tight on every set in $\mathcal{L}$: $x(\delta(S)) = 2 \ \forall S \in \mathcal{L}$
- $y: \mathcal{L} \to \mathbb{R}_+$

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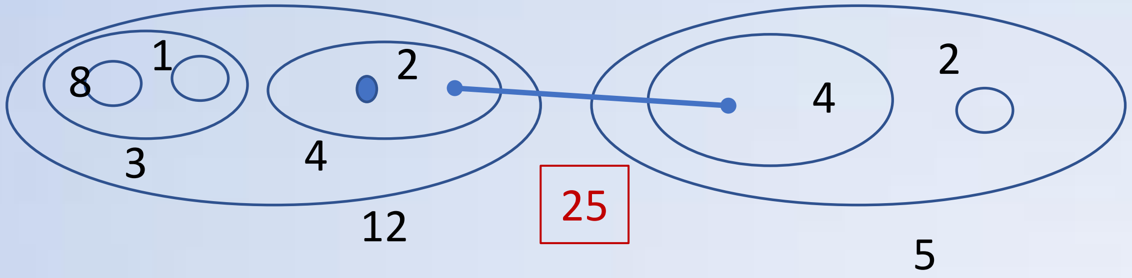

Induced weight function: $w(u,v) = \sum_{S:(u,v)\in\delta(S)} y_S$

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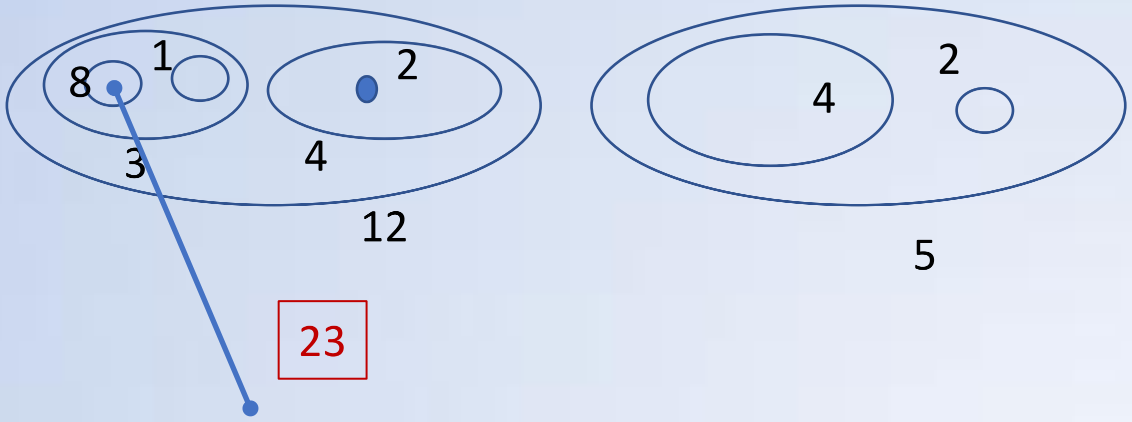


Induced weight function: $w(u,v) = \sum_{S:(u,v)\in\delta(S)} y_S$

Reduction to laminarly weighted ATSP

- Start with any G and w.
- Compute Held-Karp optimal solution x and dual y supported on laminar family $\mathcal{L}$
- Delete all edges with $x_e = 0$.

maximize $\quad 2\sum_{\emptyset\neq S\subsetneq V} y_S$

subject to

$$\sum_{S:(u,v)\in\delta(S)} y_S + \alpha_u - \alpha_v \leq w(u,v) \quad \forall (u,v)\in E$$
$$y \geq 0$$

Observations:

- Optimal solutions and optimum value are the same for w and for
$$w'(u,v) = w(u,v) + \alpha_v - \alpha_u$$
- All remaining edges have
$$w'(u,v) = \sum_{S:(u,v)\in\delta(S)} y_S$$

Roadmap

General ATSP

LP duality + uncrossing

Laminarly weighted ATSP

Graph theory: contractions

Irreducible instances

Node weighted algorithm + contractions

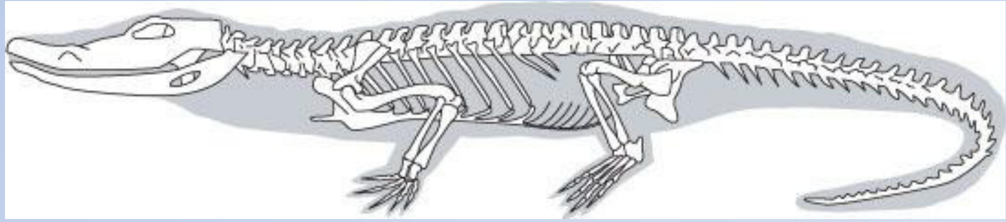
Vertebrate pairs

[Svensson ’15]

Local-connectivity ATSP

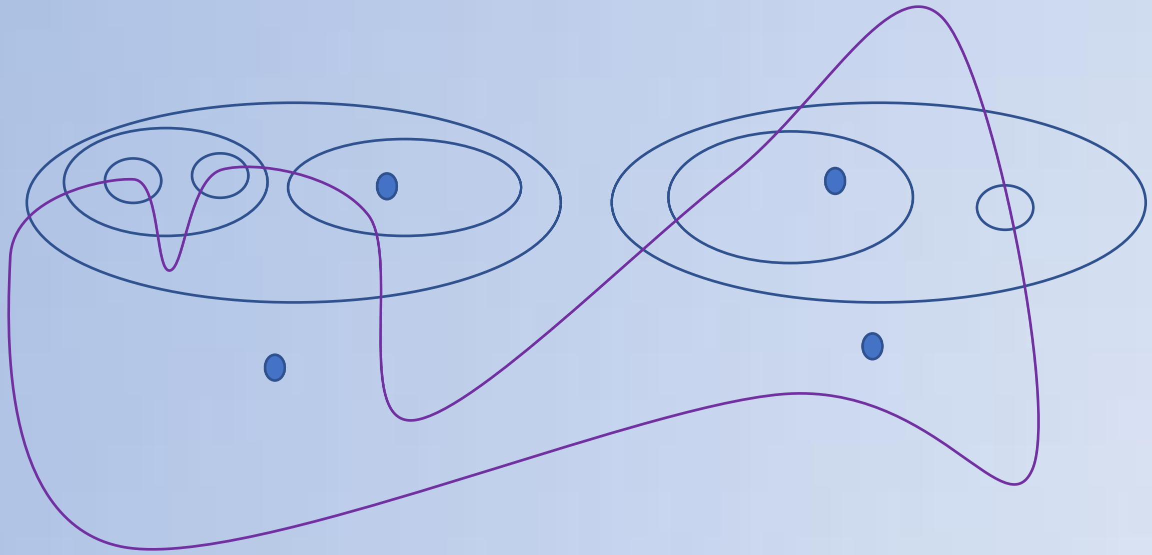
O(1)-light lcATSP algorithm in vertebrate pairs

Vertebrate pairs



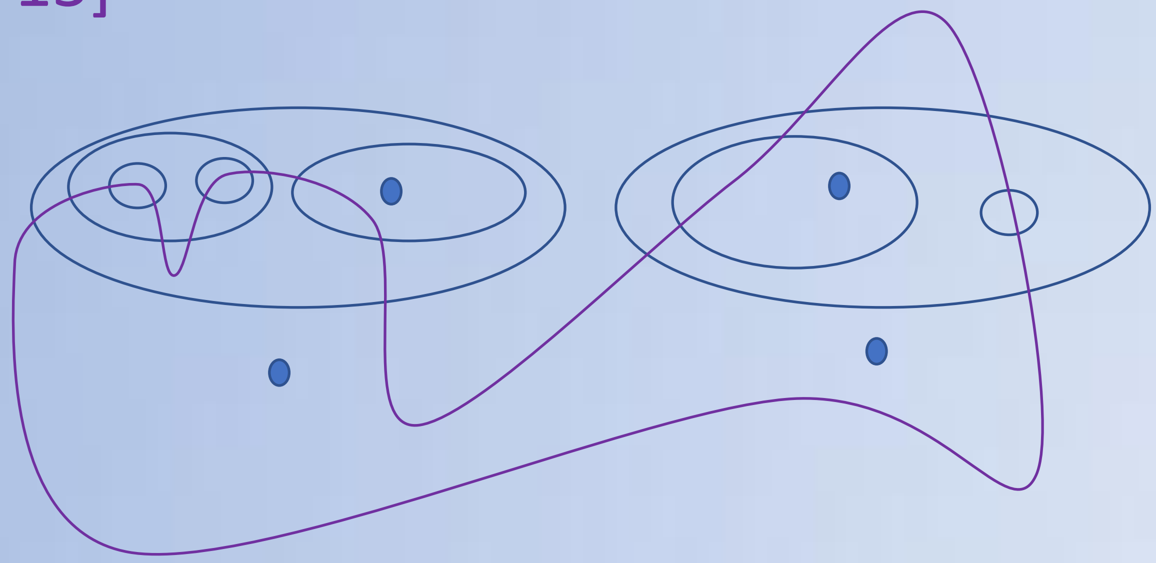
Vertebrate pair $(\mathcal{I}, B)$

- $\mathcal{I} = (G, \mathcal{L}, x, y)$ instance
- B: backbone = subtour that crosses every nonsingleton set in $\mathcal{L}$



Vertebrate pairs

- We will reduce general ATSP to solving ATSP for a vertebrate pair $(\mathcal{I}, B)$ with $w(B) = \Theta(OPT)$ (more or less…)
- Solve Local-Connectivity ATSP on such instances, and apply [Svensson'15]

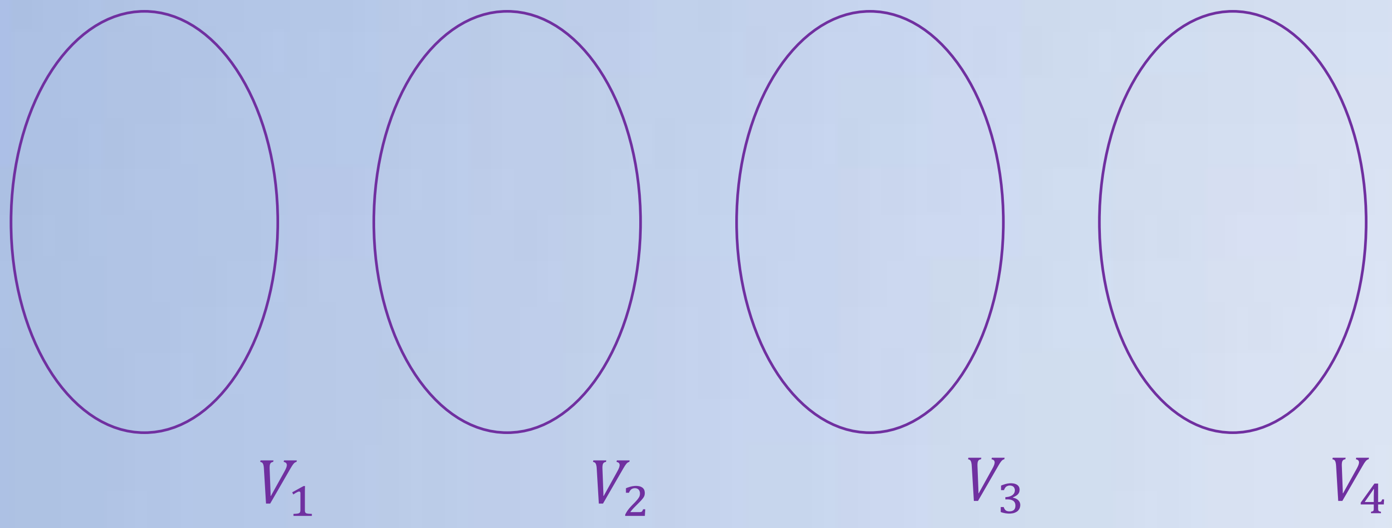


Local-Connectivity ATSP [Svensson'15]

Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$ with induced weights $w: E \to \mathbb{R}_+$

Lower bound function $\text{lb}: V \to \mathbb{R}_+$ with $\sum_{v \in V} \text{lb}(v) = OPT$

Input: partition of the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$



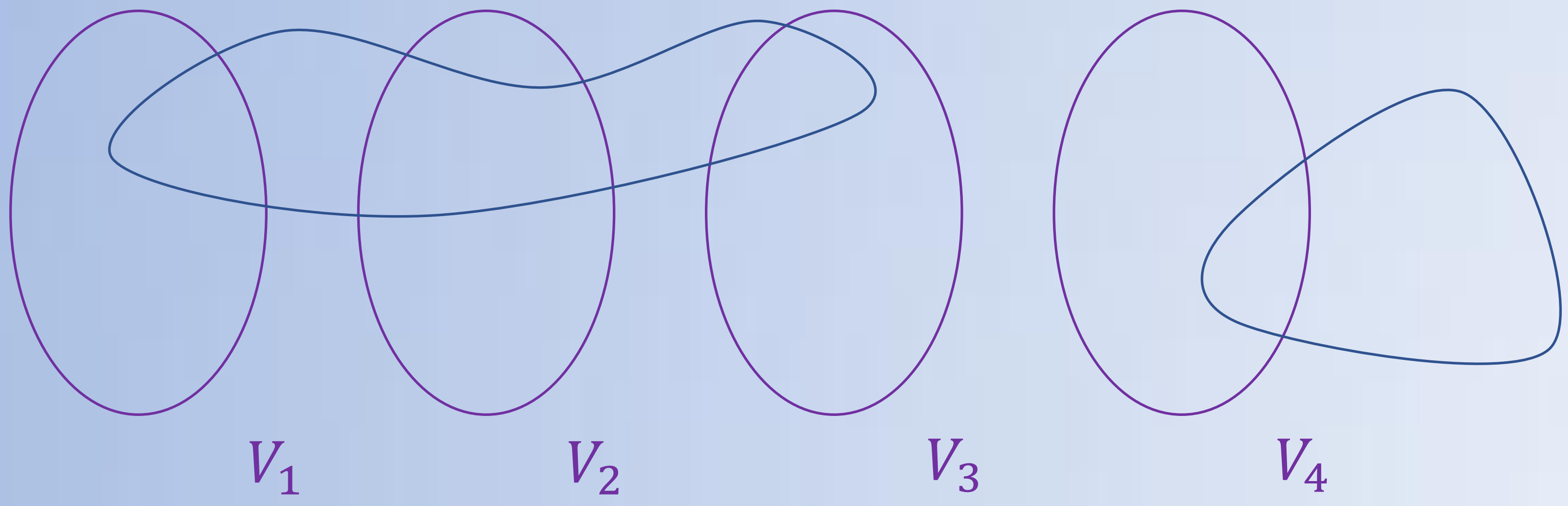
Local-Connectivity ATSP [Svensson'15]

Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$ with induced weights w: $E \to \mathbb{R}_+$

Lower bound function lb: $V \to \mathbb{R}_+$ with $\sum_{v \in V} \text{lb}(v) = OPT$

Input: partition of the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$

Output: Eulerian edge set F with $|\delta(V_i) \cap F| > 0$ for each V_i



Local-Connectivity ATSP [Svensson'15]

Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$ with induced weights $w\colon E \to \mathbb{R}_+$

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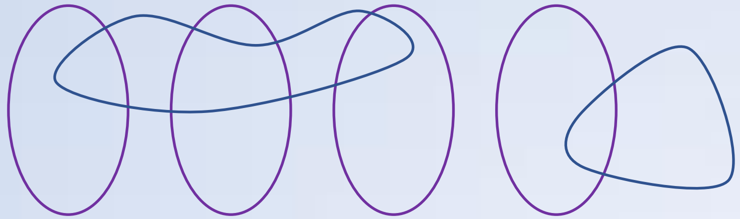
Input: partition of the vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_k$

Output: Eulerian F with $|\delta(V_i) \cap F| > 0$ for each V_i

α-light algorithm: for every component C of F,

$$\frac{w(E(C))}{\mathrm{lb}(V(C))} \leq \alpha$$



"Every component pays for itself locally"

Local-Connectivity ATSP [Svensson'15]

α-light algorithm for Local-Connectivity ATSP

$(9+\varepsilon)\alpha$-approximation for ATSP

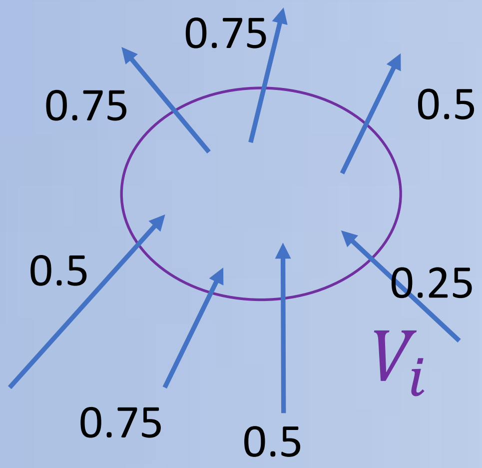
Theorem [Svensson'15]
There exists a polytime $(27+\varepsilon)$-approximation for node-weighted ATSP.

Local-Connectivity ATSP: node-weighted case

- Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$, with $\mathcal{L}$ containing only singletons (ignore B)

$$w(u,v) = y_{\{u\}} + y_{\{v\}}$$

- Define $\mathrm{lb}(u) = 2y_{\{u\}} \quad \forall u \in V$
- Partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ all strongly connected
- Modify G and x, and solve an integer circulation problem



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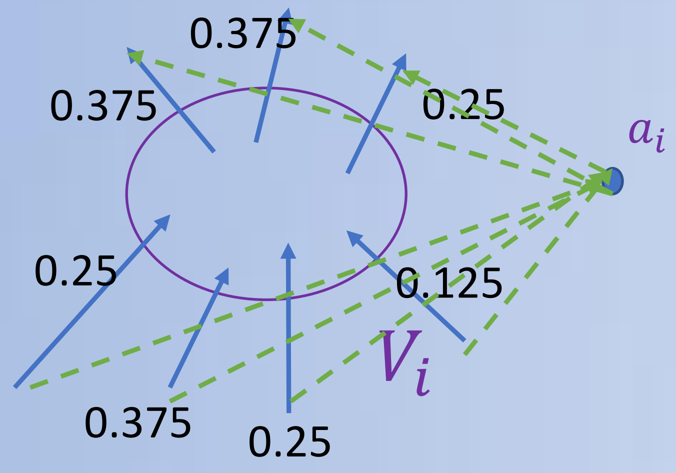
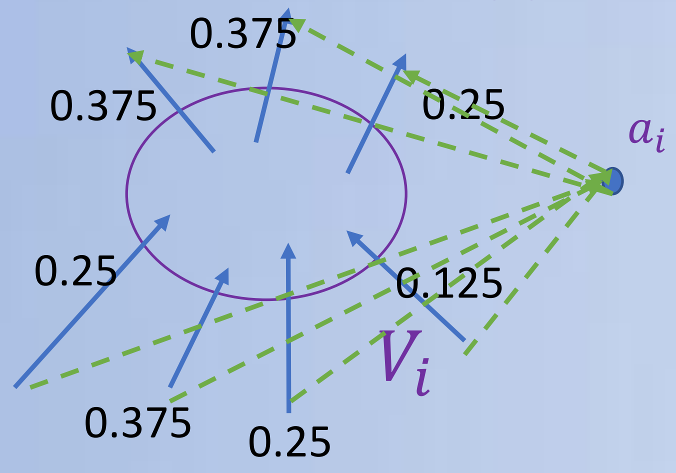


- For each V_i, create auxiliary vertex a_i
- Reroute 1 fractional unit of incoming and outgoing flow x to a_i
- Solve integer circulation problem routing =1 unit through each a_i
- Map back to original G

Local-Connectivity ATSP: node-weighted case

- The rerouted x is feasible to the circulation problem of weight OPT
- Flow integrality: there exists integer solution of weight $\leq OPT$
- After mapping back, every vertex with $y_v > 0$ has in-degree ≤ 2
- For a component C, $w(E(C)) = \sum_{(u,v)\in E(C)} y_{\{u\}} + y_{\{v\}} \leq 4\sum_{v\in C} y_{\{v\}}$
- $\mathrm{lb}(V(C)) = 2\sum_{v\in C} y_{\{v\}}$ $\Longrightarrow$ 2-light algorithm



Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- Vertebrate pair $(\mathcal{I}, B)$. Assume $\mathcal{L}$ has a single non-singleton component S. Thus,

$$w(u,v) = \begin{cases} y_{\{u\}} + y_{\{v\}} + y_S & if\, (u,v) \in \delta(S) \\ y_{\{u\}} + y_{\{v\}} & if\, (u,v) \notin \delta(S) \end{cases}$$

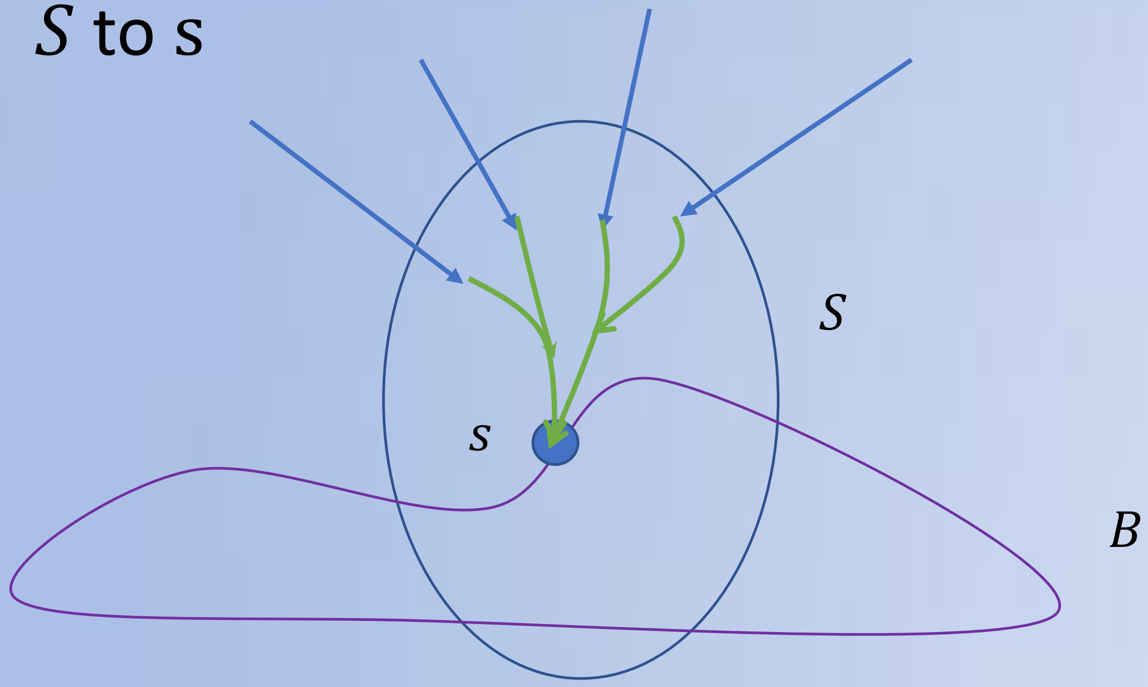
- Define

$$\mathrm{lb}(u) = \begin{cases} 2y_{\{u\}} & if\, u \in V \setminus V(B) \\ \dfrac{w(B)}{|V(B)|} & if\, u \in V(B) \end{cases}$$

- $\sum_{v \in V} \mathrm{lb}(v) = O(OPT)$, since $w(B) = \Theta(OPT)$

Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- By assumption, $x(\delta^{in}(S)) = x(\delta^{out}(S))$ =1
- Backbone property: there is a node $s \in V(B) \cap S$
- Simple flow argument: we can route the incoming 1 unit of flow to S to s



Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

- Partition $V = V_1 \cup V_2 \cup \cdots \cup V_k$
- Add backbone B into Eulerian set F.
- Via flow splitting, "force" all edges entering S to proceed to $s \in V(B)$
- Create auxiliary vertices a_i as before
- Solve integral circulation problem, and add solution to F.

Local-Connectivity ATSP: one nonsingular set in $\mathcal{L}$

Analysis

- For all components C not crossing S, $w(E(C))/\mathrm{lb}(V(C)) \leq 2$ exactly as in the node-weighted case
- Giant component C_0 containing B.
  - Contains all edges crossing S
  - Has lower bound $\mathrm{lb}(V(C_0)) \geq \mathrm{lb}(V(B)) = \Theta(OPT)$
  - $w(E(C_0)) \leq w(F) \leq O(OPT)$
- Therefore solution is $O(1)$-light.
- Same approach extends to arbitrary $\mathcal{L}$: enforce that every subtour crossing a set in $\mathcal{L}$ must intersect the backbone.

Roadmap

General ATSP

LP duality + uncrossing

Laminarly weighted ATSP

Graph theory: contractions

Irreducible instances

Node weighted algorithm + contractions

Vertebrate pairs

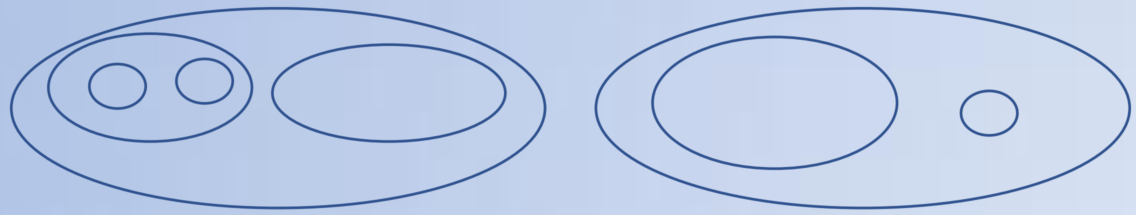
[Svensson'15]

Local-connectivity ATSP

O(1)-light lcATSP algorithm in vertebrate pairs

Motivation: reducing by contraction

- All sets in the family $\mathcal{L}$ are singletons: node-weighted ATSP
- Would like to reduce the problem by contracting nonsingleton sets in $\mathcal{L}$



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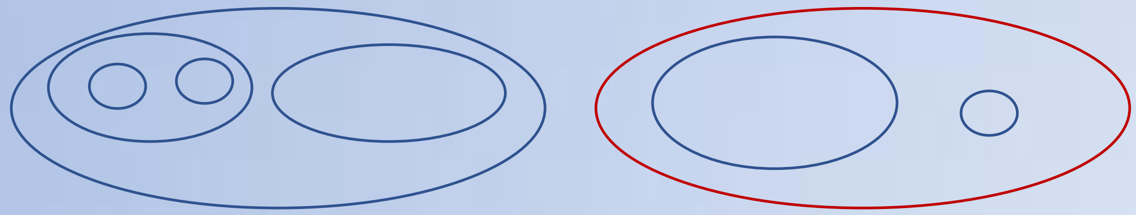

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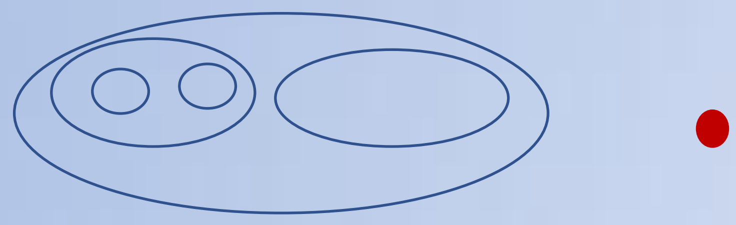
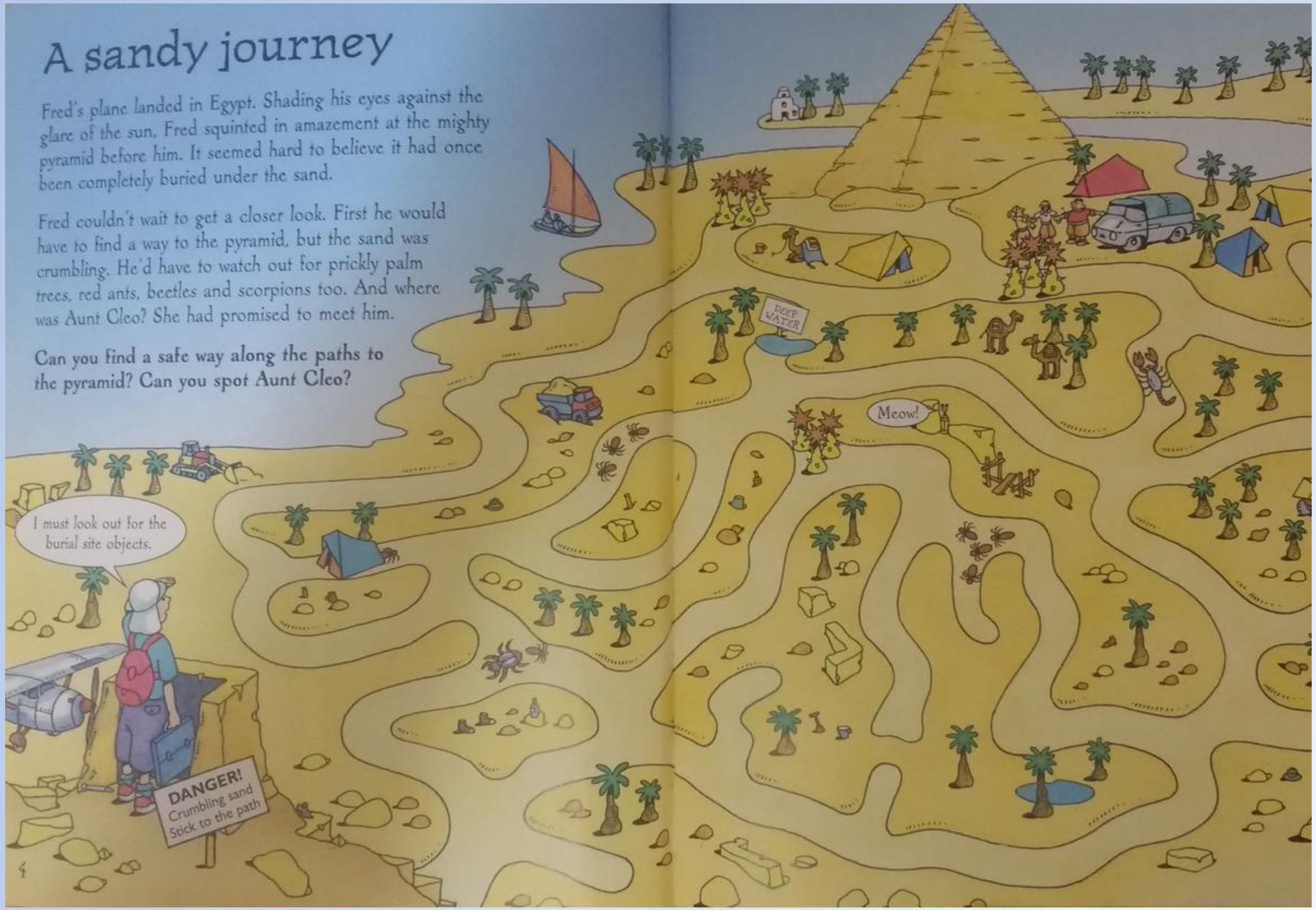


Irreducible instances

- Irreducible set $S \in \mathcal{L}$: There exists $u, v \in S$ such that the shortest path between u and v inside S visits *"almost all"* sets $X \subseteq S, X \in \mathcal{L}$
- Irreducible instance $\mathcal{I} = (G, \mathcal{L}, x, y)$: all sets in $\mathcal{L}$ are irreducible



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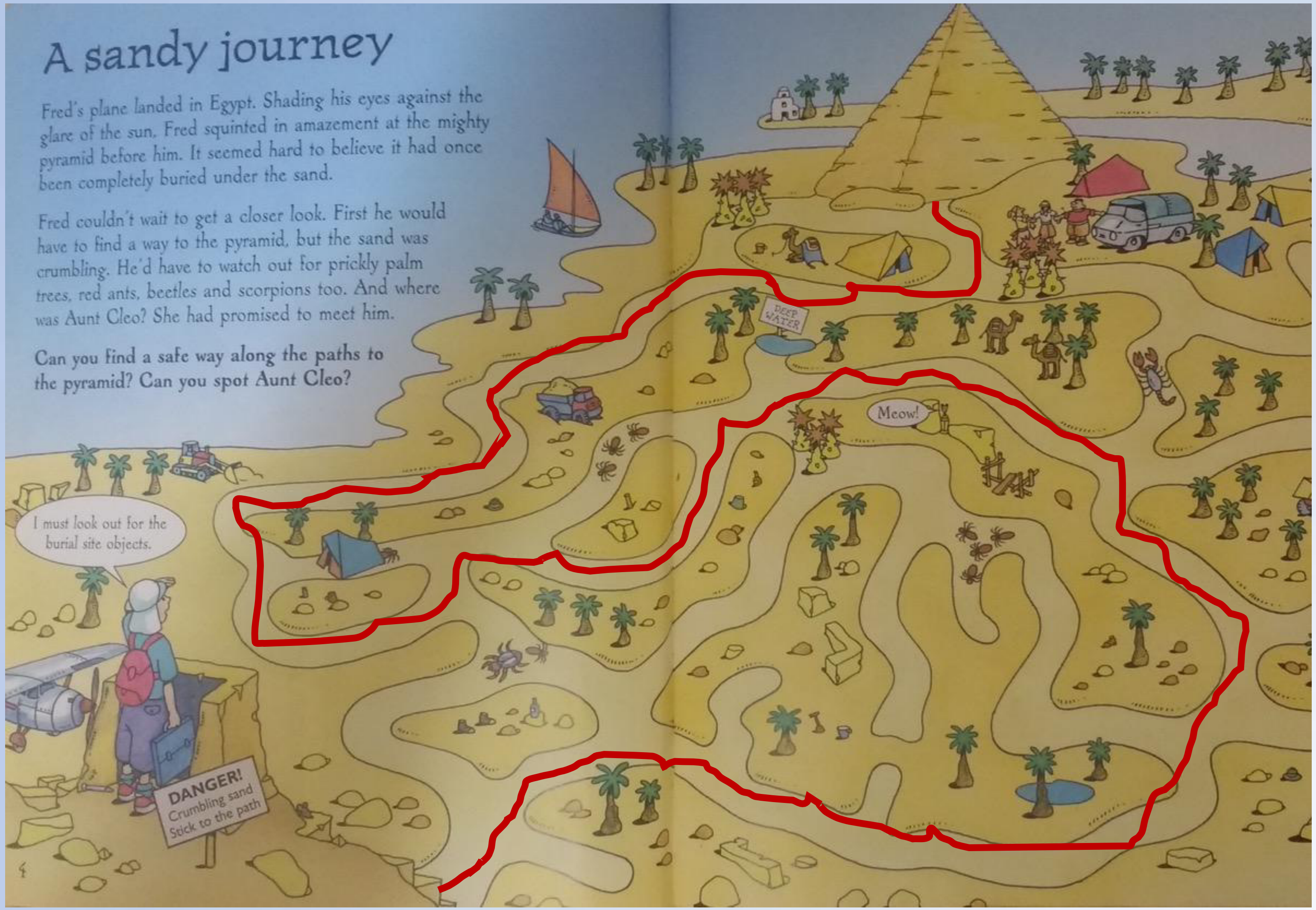

Irreducible instances

- Reducible set $S \in \mathcal{L}$: For every pair $u, v \in S$, there is a "cheap" path connecting them (if they are connected).
- Reducible sets can be contracted.

Theorem:

polytime ρ-approximation for irreducible instances

$$\Longrightarrow$$

polytime 8ρ-approximation for arbitrary instances

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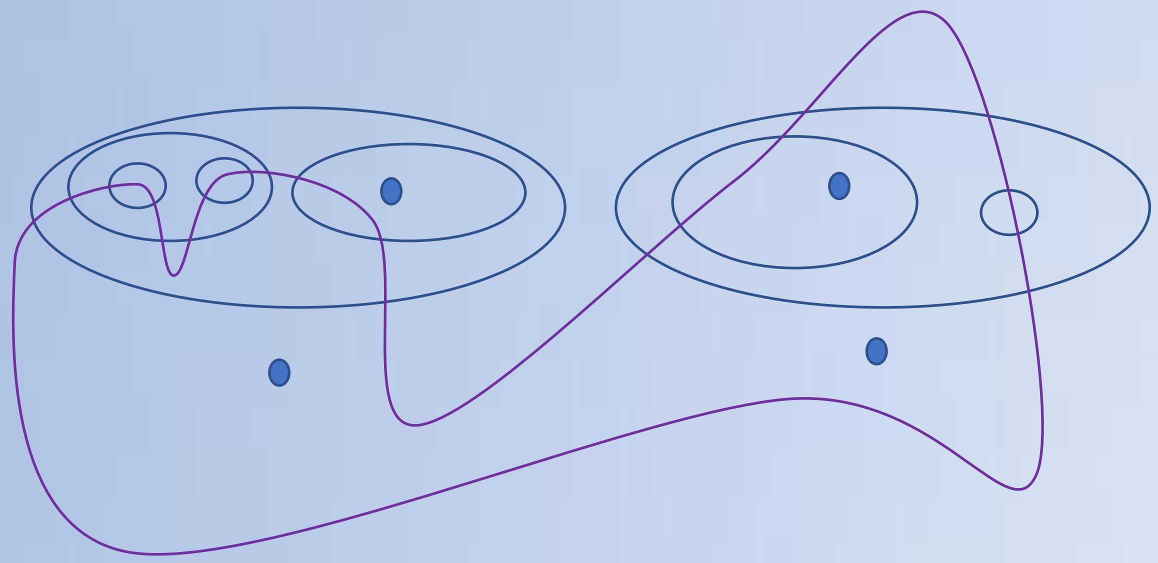
Local-connectivity ATSP

O(1)-light lcATSP algorithm in vertebrate pairs

Vertebrate pairs

Vertebrate pair $(\mathcal{I}, B)$

- $\mathcal{I} = (G, \mathcal{L}, x, y)$ instance
- B: backbone = subtour that crosses every nonsingleton set in $\mathcal{L}$



Finding a vertebrate pair in an irreducible instance $\mathcal{I} = (G, \mathcal{L}, x, y)$

1. Obtain a node-weighted instance by contracting all maximal sets in $\mathcal{L}$
2. Use [Svensson '15] to find a tour here, and blow it back to a subtour B in the original instance $\mathcal{I}$ in a pessimistic way:
   inside each maximal $S \in \mathcal{L}$, B crosses $\geq 0.75 value(S)$
3. If it crosses every set in $\mathcal{L}$, then $(\mathcal{I}, B)$ is a vertebrate pair
4. Otherwise, recurse by contracting all maximal sets in $\mathcal{L}$ not crossed by B.
   This works because their total weight is $\leq 0.25 value(\mathcal{I})$

Roadmap

General ATSP

LP duality + uncrossing

Laminarly weighted ATSP

Graph theory: contractions

Irreducible instances

Node weighted algorithm + contractions

Vertebrate pairs

[Svensson ’15]

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Summary

- Via all these reductions, we obtain an **5500**-approximation algorithm for ATSP.
- Squeezing the arguments a bit more and opening up black boxes, can be probably decreased to a few hundreds.
- Still very far from lower bound 2 on the integrality gap of Held-Karp

Open questions

- Improve to a constant < 100
- Thin tree conjecture is still open.
- Bottleneck ATSP.
- Better than 3/2 approximation for symmetric TSP.

ScaleOpt

Scaling Methods for Discrete and Continuous Optimization

- ERC Starting Grant 2018-22
- Openings for post docs and PhD students

  http://personal.lse.ac.uk/veghl/scaleopt.html

European Research Council

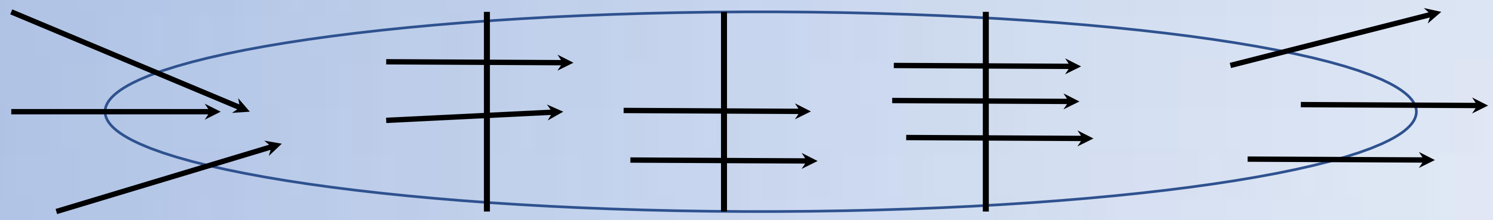
The London School of Economics and Political Science

Thank you!

Simplifying assumption *for the talk*

Assumption: all sets in the family $\mathcal{L}$ are strongly connected in G.

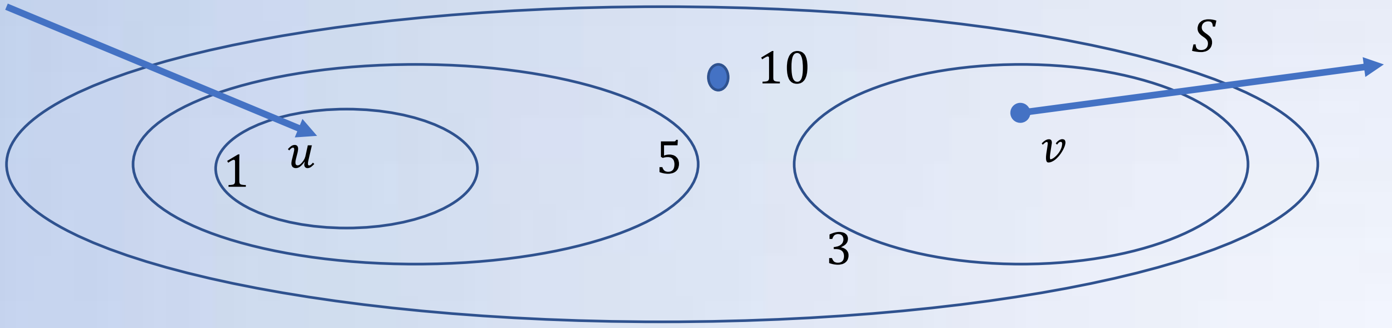
Not true in general, but the connected components have a nice path structure:



Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

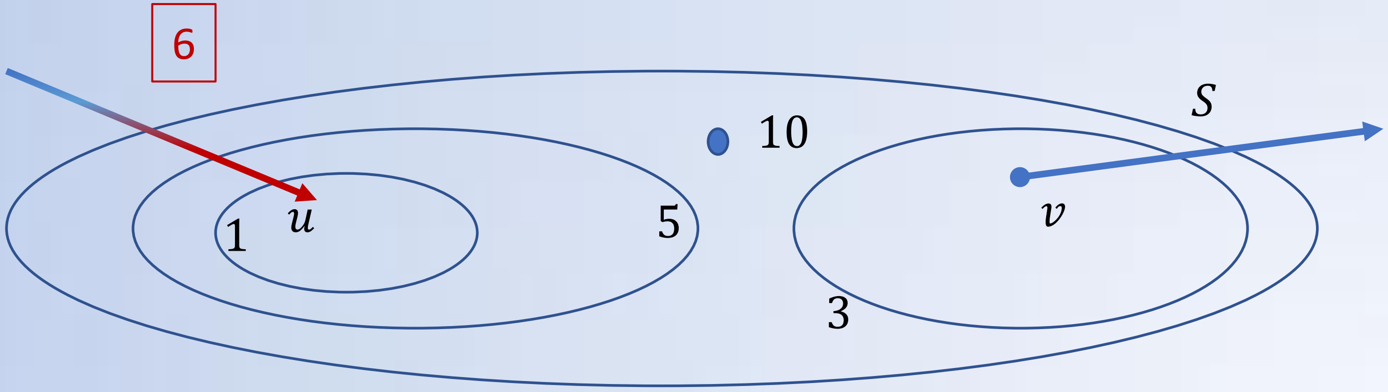
$$D_S(u, v) =$$



Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u,v) = \sum_{R: u \in R, R \subsetneq S} y_R$$

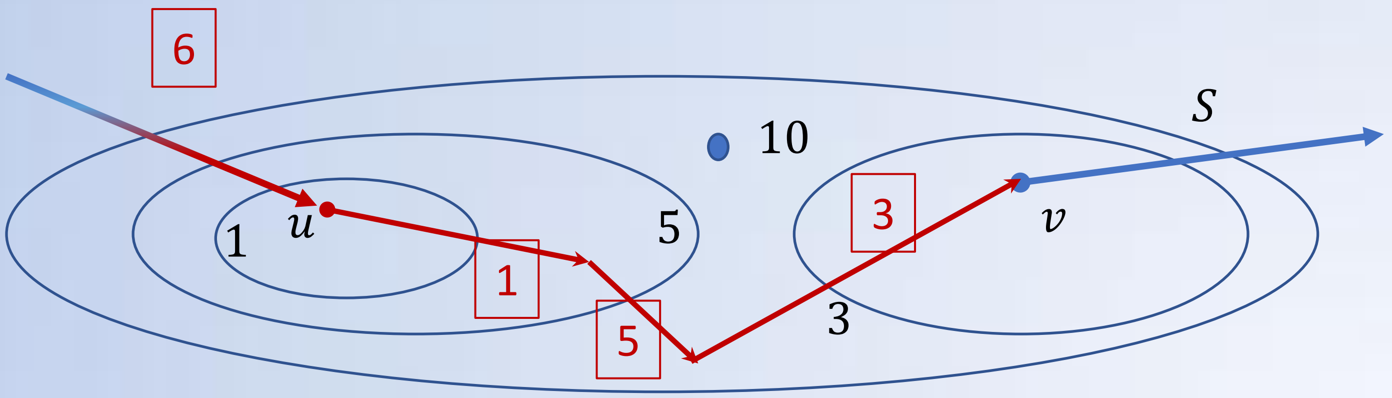


Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u,v) = \sum_{R:u \in R, R \subsetneq S} y_R + d_S(u,v)$$

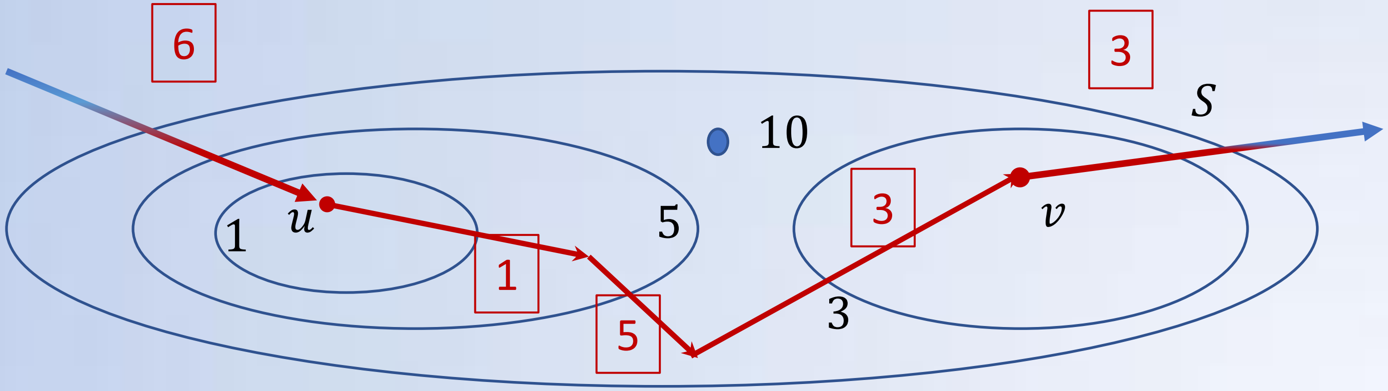
Min weight path inside S.



Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u,v) = \sum_{R: u \in R, R \subsetneq S} y_R + d_S(u,v) + \sum_{R: v \in R, R \subsetneq S} y_R = 18$$



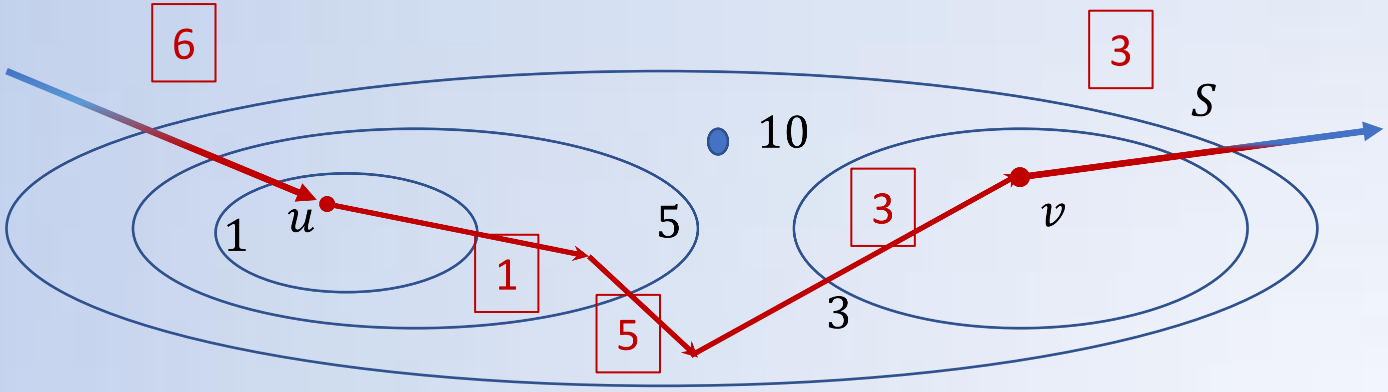
Paths traversing a set

- How much is the weight of connecting an incoming and an outgoing edge in a set $S \in \mathcal{L}$?

$$D_S(u,v) = \sum_{R:u\in R, R\subsetneq S} y_R + d_S(u,v) + \sum_{R:v\in R, R\subsetneq S} y_R = 18 \leq 38$$

Lemma:

$$D_S(u,v) \leq 2\sum_{R\subsetneq S} y_R = value(S)$$



Irreducible instances

- Reducible set $S \in \mathcal{L}$:

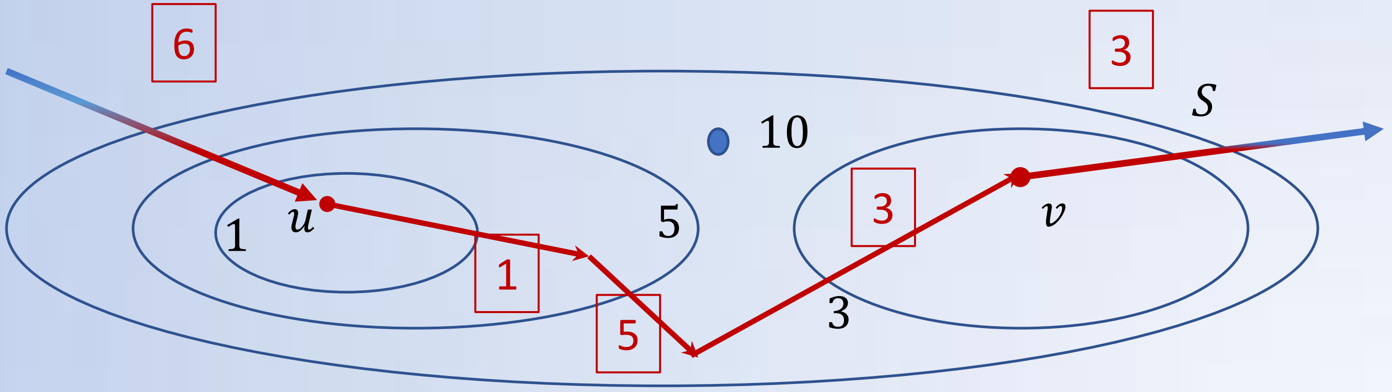
$$\text{Max}_{u,v \in S}\ D_S(u,v) \leq \frac{3}{4}\ value(S)$$

- Irreducible instance $\mathcal{I} = (G, \mathcal{L}, x, y)$: no set $S \in \mathcal{L}$ is reducible

Lemma:

$$D_S(u,v) \leq 2 \sum_{R \subsetneq S} y_R = value(S)$$

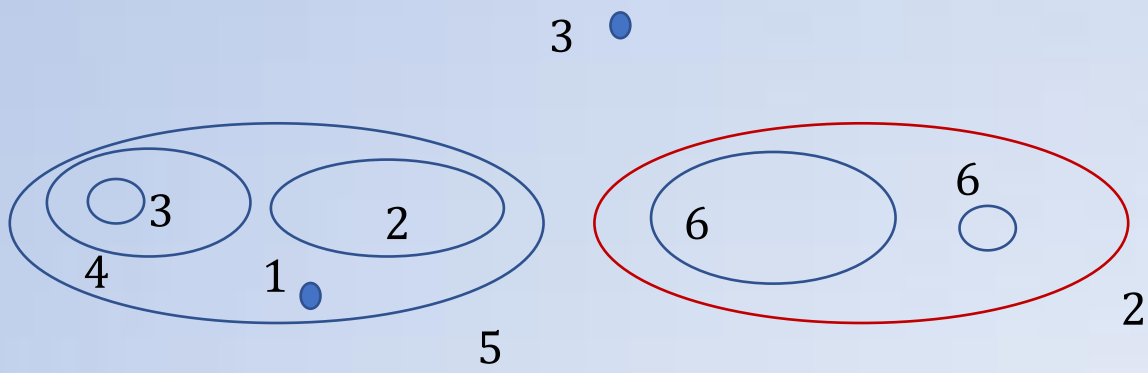
Theorem:
polytime ρ-approximation for irreducible instances $\Longrightarrow$ polytime 8ρ-approximation for arbitrary instances



Recursive algorithm via contractions

- Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$
- $value(\mathcal{I}) = 2\sum_{R \subsetneq V} y_R$
  $=$Held-Karp optimum
- S: minimal reducible set in $\mathcal{L}$.

$value(\mathcal{I}) = 64$

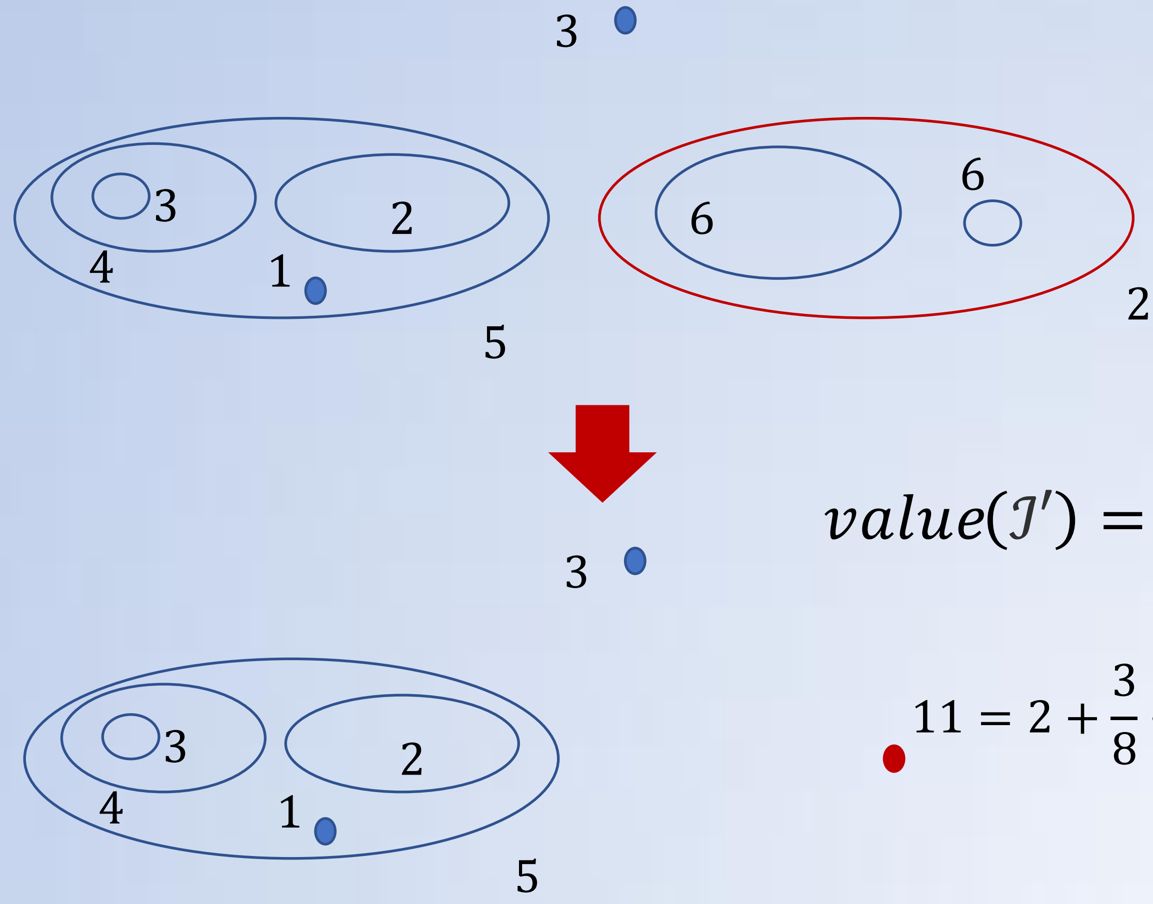


8ρ-approximation for $\mathcal{I}$ =
8ρ-approximation on instance by contracting S
+
ρ-approximation of irreducible instance “inside” S

Recursive algorithm via contractions

- Instance $\mathcal{I} = (G, \mathcal{L}, x, y)$
- $value(\mathcal{I}) = 2\sum_{R \subsetneq V} y_R$
  =Held-Karp optimum
- S: minimal reducible set in $\mathcal{L}$.
- $\mathcal{I}' = \mathcal{I}/S$: contract S in $\mathcal{I}$.
- $S \to s$
- $y_{\{s\}} = y_S + \frac{3}{8} value(S)$
- $value(\mathcal{I}') = value(\mathcal{I}) - \frac{1}{4} value(S)$

$value(\mathcal{I}) = 64$



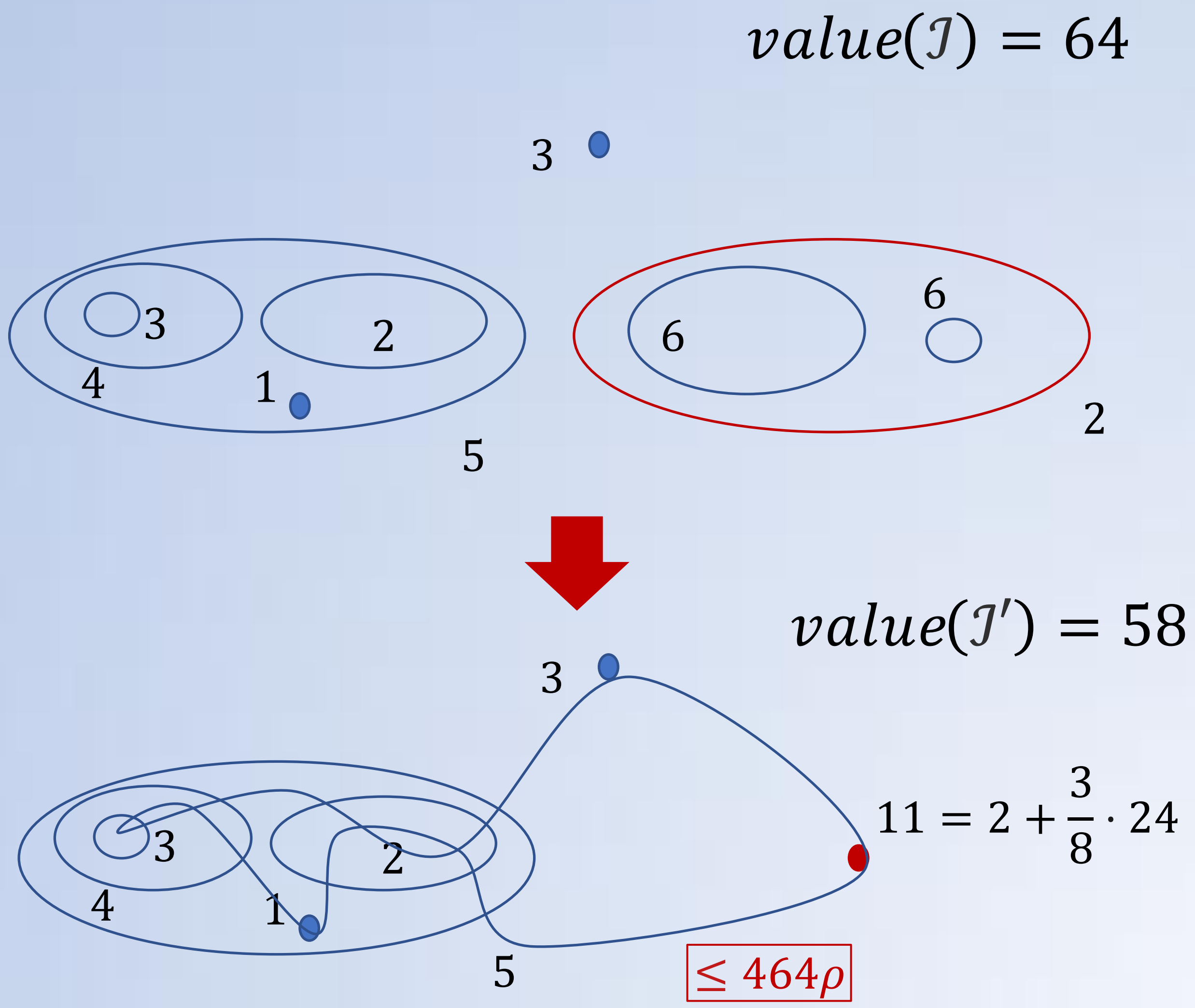
$value(\mathcal{I}') = 58$

$11 = 2 + \frac{3}{8} \cdot 24$

Recursive algorithm via contractions

Inductive assumption: We have a polytime 8ρ-approximation for smaller instances

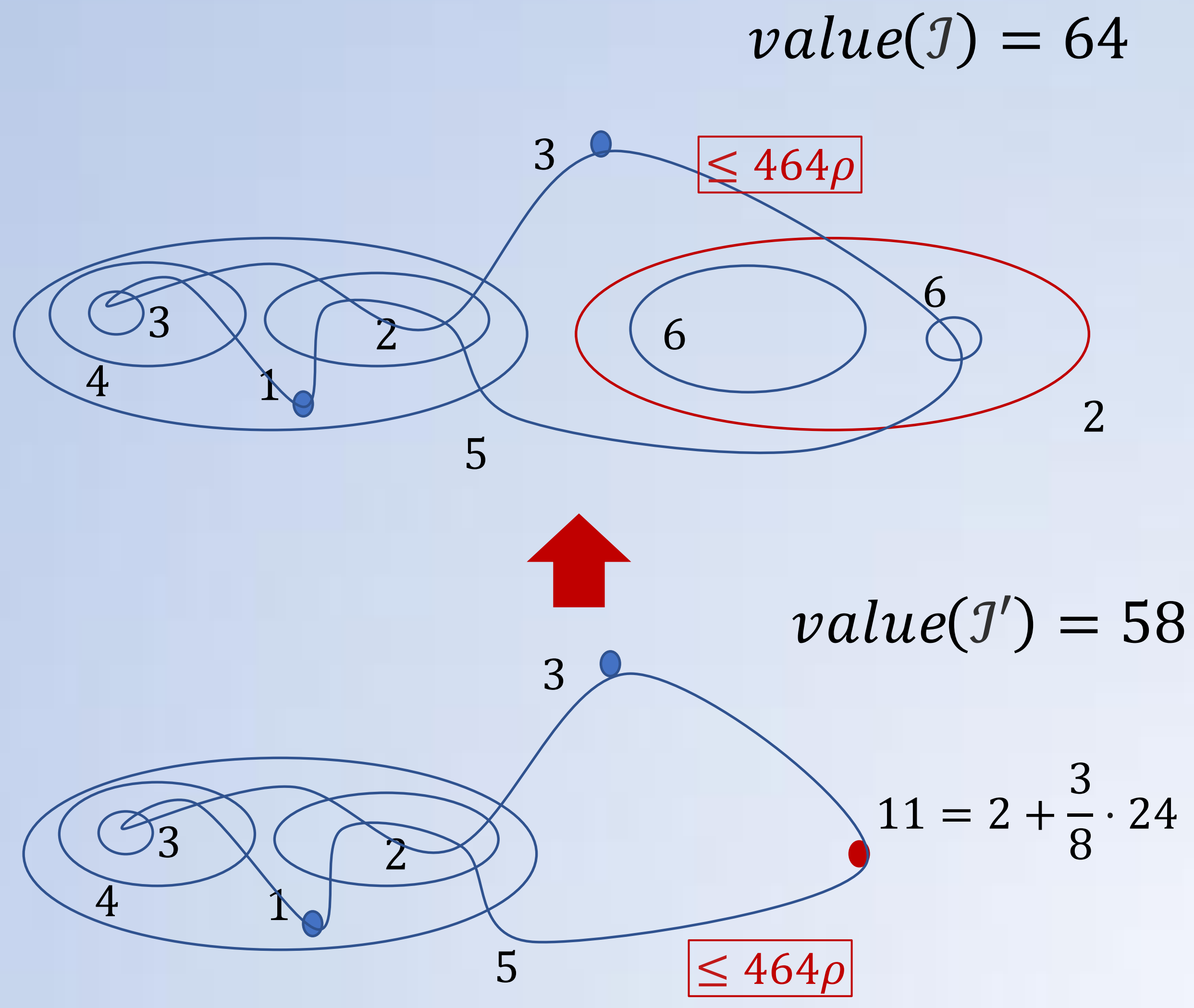
- Apply recursively on $\mathcal{J}'$ to obtain tour T'
  $w(T') \leq 8\rho value(\mathcal{J}')$
  $= 8\rho(value(\mathcal{J}) - \frac{1}{4} value(S))$



Contracting S

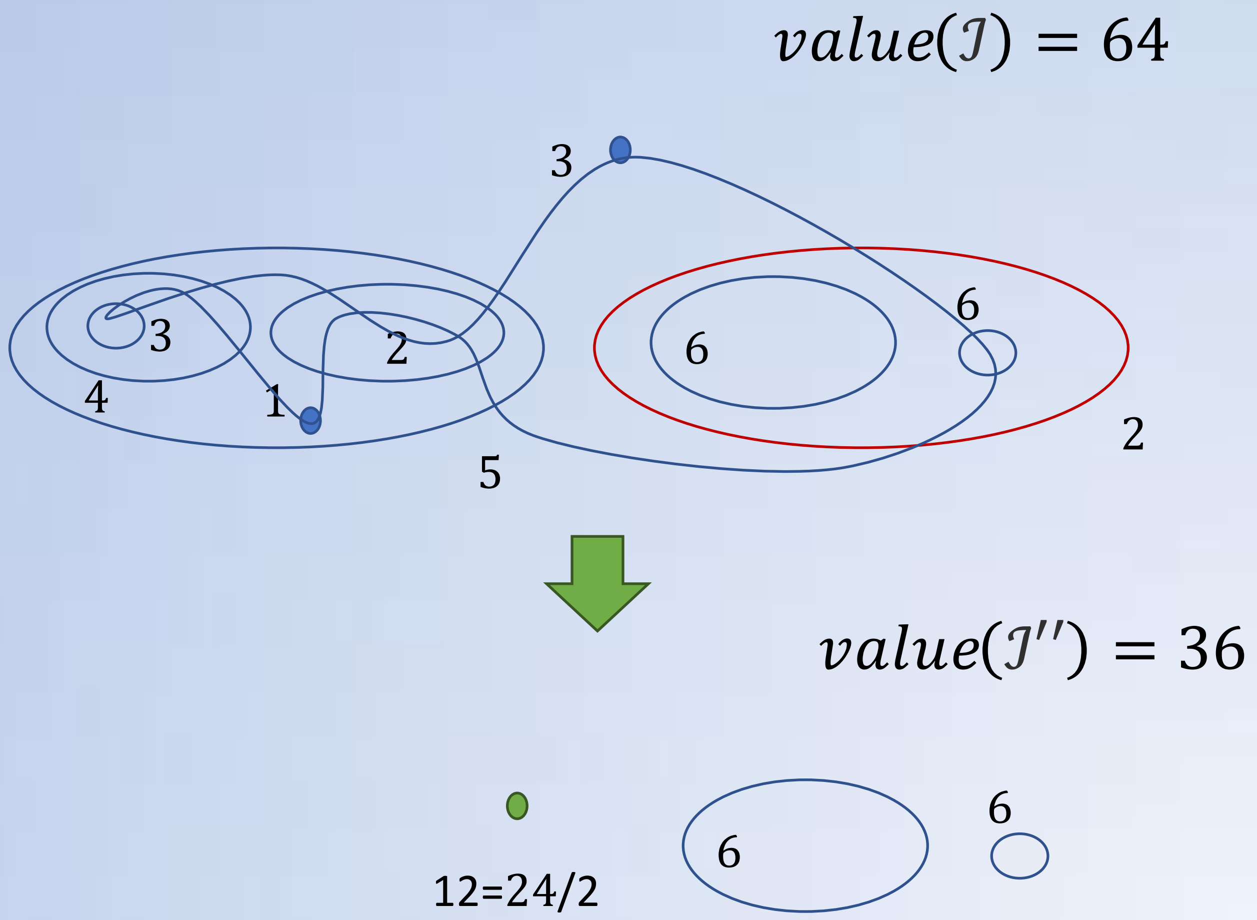
Inductive assumption: We have a polytime 8ρ-approximation for smaller instances

- Apply recursively on $\mathcal{J}'$ to obtain tour T'
  $w(T') \leq 8value(\mathcal{J}')$
  $= 8\rho(value(\mathcal{J}) - \frac{1}{4}value(S))$
- Map back to subtour T in $\mathcal{J}$ with $w(T) \leq w(T')$



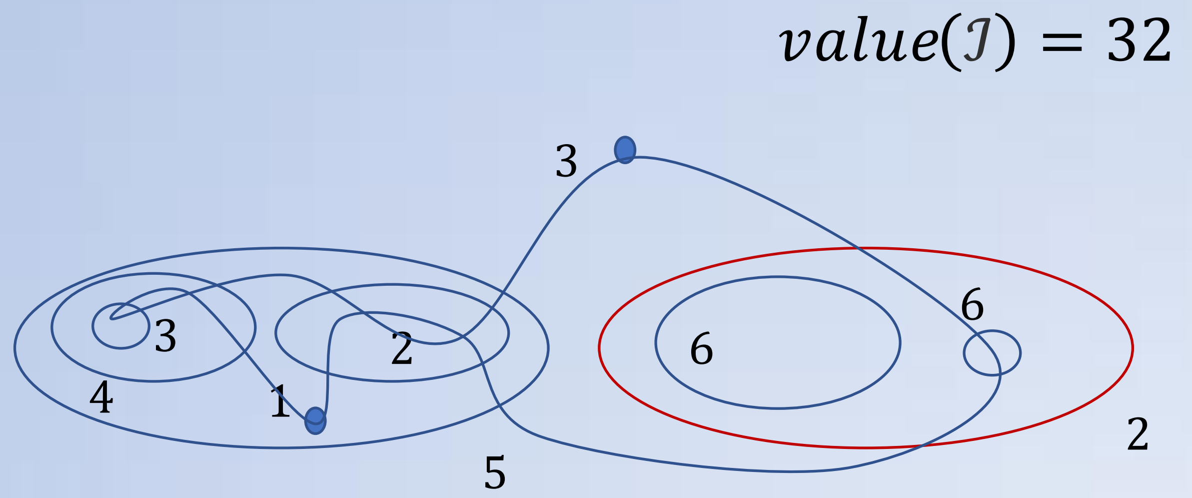
Inducing on S

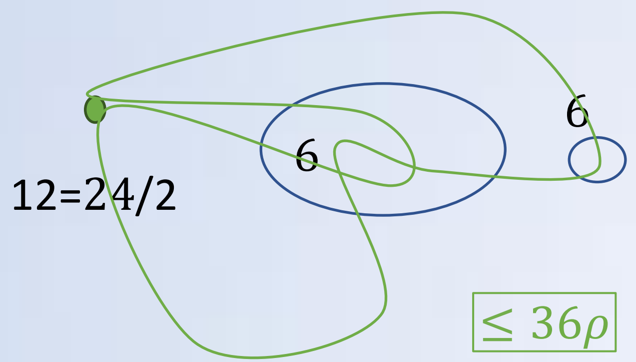
- We add a tour F_S inside S, using the ρ-approximation on irreducible instances.
- $\mathcal{J}''$: remove S, and contract $V \setminus S$ to $\bar{s}$, with
$$y_{\{\bar{s}\}} = value(S)/2$$
- $\mathcal{J}''$ is irreducible.



Inducing on S

- We add a tour F_S inside S, using the ρ-approximation on irreducible instances.
- $\mathcal{I}''$: remove S, and contract $V \setminus S$ to $\bar{s}$, with
$$y_{\{\bar{s}\}} = value(S)/2$$
- $\mathcal{I}''$ is irreducible.
- Find tour F'' in $\mathcal{I}''$ with weight
$$w(F'') \leq \rho value(\mathcal{I}'') = 2\rho value(S)$$





Inducing on S

- Find tour F'' in $\mathcal{J}''$ with weight
  $$w(F'') \leq \rho value(\mathcal{J}'') = 2\rho value(S)$$
- Map back F'' to F_S in $\mathcal{J}$ with
  $$w(F_S) \leq w(F) \leq 2\rho value(S)$$
- $T \cup F_S$ is a tour in $\mathcal{J}$

$$w(T \cup F_S) \leq 8\rho\left(value(\mathcal{J}) - \frac{1}{4}value(S)\right) + 2\rho value(S) = 8\rho value(\mathcal{J})$$

$value(\mathcal{J}) = 32$