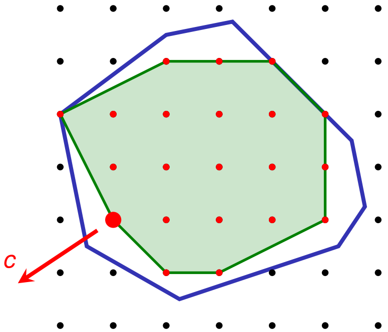
A Strongly Polynomial Algorithm for Bimodular Integer Programming

Rico Zenklusen

ETH Zurich

joint work with Stephan Artmann and Robert Weismantel

Toward general classes of efficiently solvable ILPs



Integer Linear Program (ILP)

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\},$$

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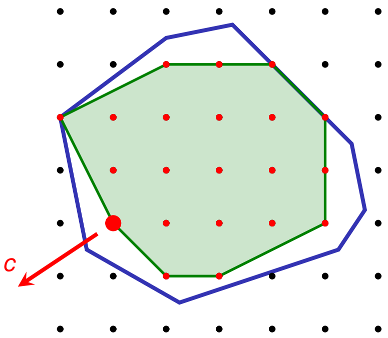

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Two classes of efficiently solvable ILPs

- If $n = O(1)$ or $m = O(1)$
  $\rightarrow$ Lenstra's Algorithm. (Lenstra [1983])
- If A is totally unimodular (TU)
  $\rightarrow$ Relaxation is naturally integral.

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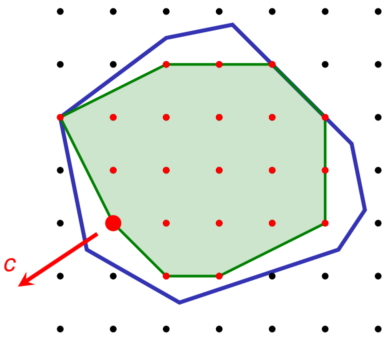

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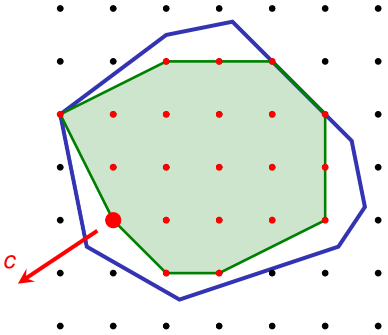

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What if minors, in absolute value, are still bounded, but not by 1?

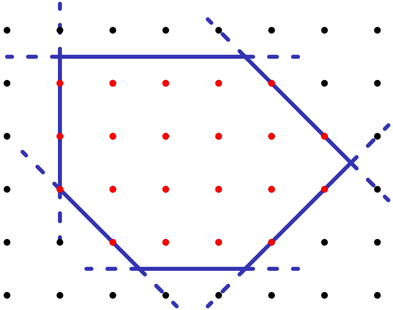
- One can show that for any $\epsilon > 0$, if minors are of order n^ϵ, then ILP gets NP-hard.
  (see, e.g., Burch et al. [2003], Chestnut, Z. [2016])

Beyond TU-ness: Bimodular integer programs

Definition: Bimodular Integer Program (BIP)

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}, \quad \text{where}$$

(i) All $n \times n$ minors of A are $\in \{-2, -1, 0, 1, 2\}$.
(ii) $\text{rank}(A) = n$.



$$\begin{pmatrix} 0 & -2 \\ 1 & -1 \\ 1 & 1 \\ 0 & 2 \\ -1 & 0 \\ -1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} -1 \\ 4 \\ 9 \\ 9 \\ -1 \\ -3 \end{pmatrix}$$

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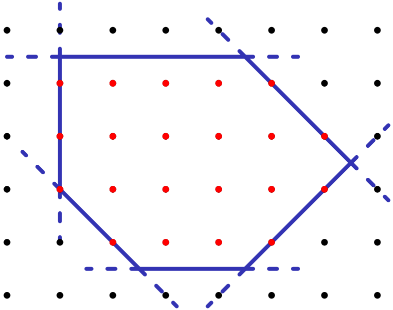
Any ILPs s.t. all minors of A are $\in \{-2, -1, 0, 1, 2\}$ can easily be reduced to BIP.



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Theorem AWZ [2017]

There is a strongly polynomial algorithm to solve BIP.

Some comments and gained insights

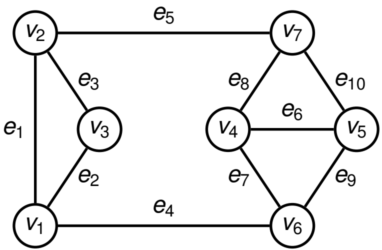
- BIP is equivalent to "parity-constrained TU ILPs".
- We heavily use Seymour's TU decomposition.
- Crucial role play parity-constrained combinatorial problems, like the T-cut problem.

  A useful tool: parity-constrained submodular minimization
  (Grötschel, Lovász, Schrijver [1981], Goemans and Ramakrishnan [1995]):

$$\min\{f(S) \mid S \subseteq N, |S| \text{ odd}\} \ .$$

submodular set function (↑ f)

Small minors

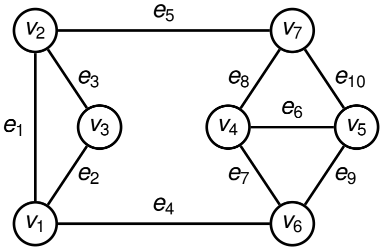


$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Largest minor of M in abs. value $= 2^{\mathrm{ocp}(G)}$, where $\mathrm{ocp}(G)$ is odd cycle packing number.

If $\mathrm{ocp}(G) = 1$, then M is tot. bimodular $\to$ can efficiently find max weight stable set through BIP.

Small minors



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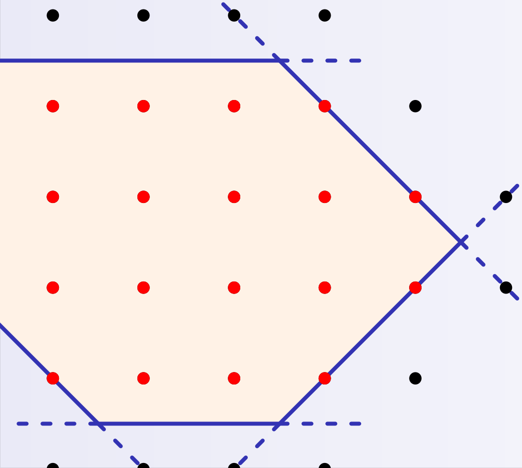
Some optimization questions studied in context of minors

- Odd cycle packing number. Kawarabayashi & Reed [2010], Bock, Faenza, Moldenhauer & Ruiz-Vargas [2010]
- Diameter of polyhedra and efficient simplex-type algorithms. Bonifas, Di Summa, Eisenbrand, Hähnle & Niemeier [2014], Eisenbrand & Vempala [2017]
- Computing largest minor. Summa, Eisenbrand, Faenza & Moldenhauer [2015], Nikolov [2015]
- Efficient minimization of seperable convex functions. Hochbaum & Shanthikumar [1990]



Our approach

Our approach on a high level

BIP

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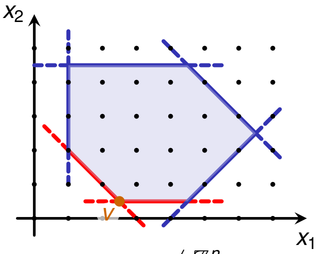
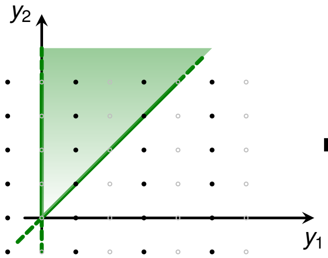
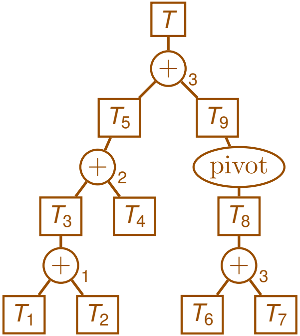
where A is bimodular.



we assume $v \notin \mathbb{Z}^n$





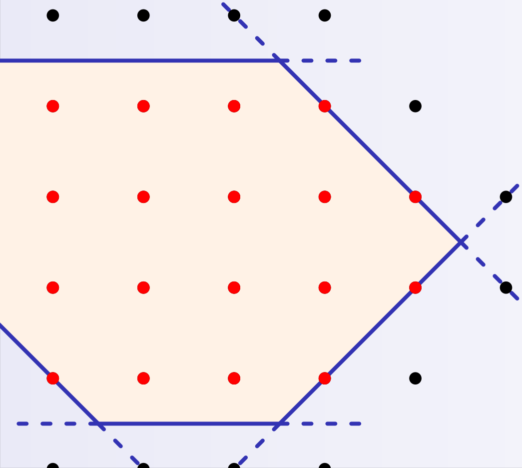
Conic parity TU problem (CPTU)

$$\max\{\tilde{c}^T y \mid Ty \le 0, y \in \mathbb{Z}^n_{\ge 0}, y(S) \text{ odd}\},$$

where T is TU, and $S \subseteq [n]$.

Seymour's TU decomposition

- Decompose T into base blocks (leaves).
- Solve CPTU by solving CPTUs on base blocks and propagating solutions up.



From BIP to CPTU

Prior work by Veselov and Chirkov [2009]

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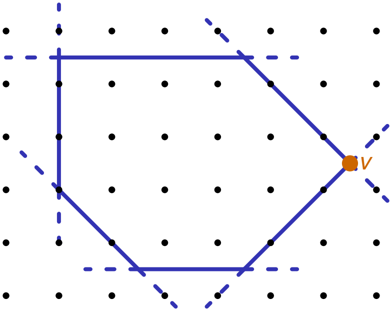
Let $\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\}$ be a BIP, $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, $v \in \text{vertices}(P)$, and let $\overline{A}x \leq \overline{b}$ be the v-tight subsystem of $Ax \leq b$.

Then each vertex of $C = \text{conv}(\{x \in \mathbb{Z}^n \mid \overline{A}x \leq \overline{b}\})$ lies on an edge of P.



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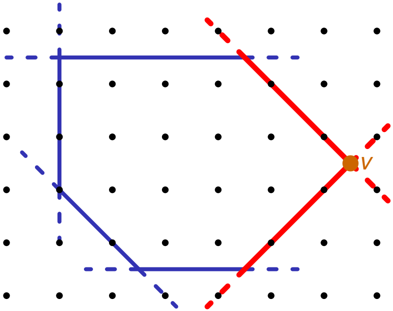

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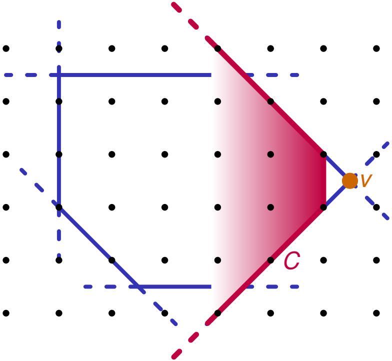

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- All vertices of C are feasible for BIP.
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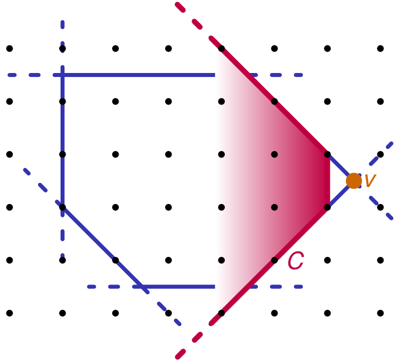

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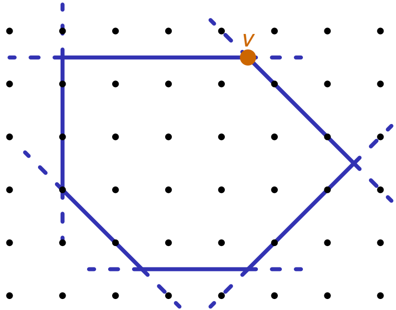

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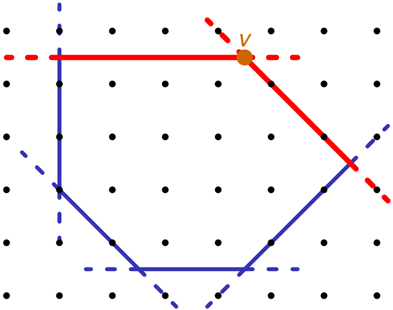

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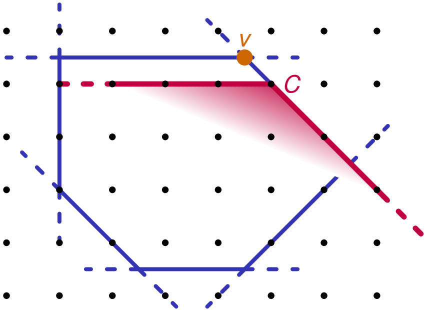

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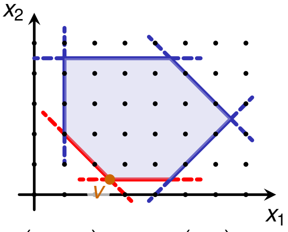
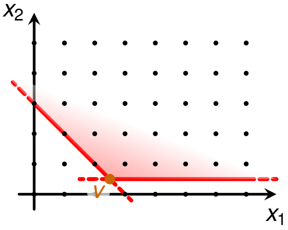
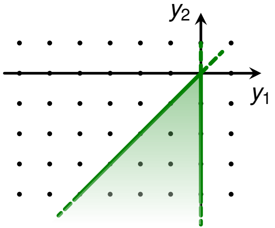
From BIP to CPTU (I)

$y = Q \cdot (x - v)$

$$\max\{c^T x \mid Ax \le b, x \in \mathbb{Z}^n\} = \max\{c^T Q^{-1} y \mid \overline{A} Q^{-1} y \le 0, Q^{-1}(b_Q + y) \in \mathbb{Z}^n\}$$







$\underbrace{\begin{pmatrix} \overline{A} \\ \\ \end{pmatrix}}_{A} \cdot x \le \underbrace{\begin{pmatrix} \overline{b} \\ \\ \end{pmatrix}}_{b}$

tight constraints
$\overline{A} \cdot v = \overline{b}$

$\underbrace{\begin{pmatrix} Q \\ \\ \end{pmatrix}}_{\overline{A}} \cdot x \le \underbrace{\begin{pmatrix} b_Q \\ \\ \end{pmatrix}}_{\overline{b}}$

full-rank square submatrix Q

$\underbrace{\begin{pmatrix} \mathrm{Id} \\ \\ \end{pmatrix}}_{\overline{A} Q^{-1} =: T} \cdot y \le 0$

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From BIP to CPTU (II)

$$\max\{\tilde{c}^T y \mid Ty \leq 0, Q^{-1}(b_Q + y) \in \mathbb{Z}^n\} = \max\{\tilde{c}^T y \mid Ty \leq 0, Q^{-1}(b_Q + y) \in \mathbb{Z}^n, y \in \mathbb{Z}^n\}$$

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Question

Given $w \in \mathbb{Z}^n$, when do we have $Q^{-1} w \in \mathbb{Z}^n$, where $Q \in \mathbb{Z}^{n \times n}$ with $\det Q \in \{-2, 2\}$?

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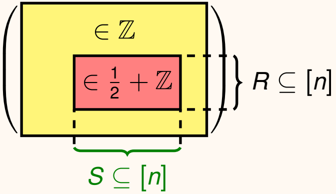
Up to row and column permutations, Q^{-1} looks as follows: $Q^{-1} =$



$\Longrightarrow \quad Q^{-1}w \in \mathbb{Z}^n \quad \Leftrightarrow \quad w(S)$ is even.

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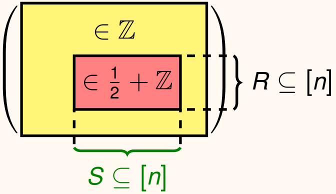

$\Longrightarrow \quad Q^{-1} w \in \mathbb{Z}^n \quad \Leftrightarrow \quad w(S)$ is even.

$\Longrightarrow \quad Q^{-1}(b_Q + y) \in \mathbb{Z}^n \quad \Leftrightarrow \quad (b_Q + y)(S)$ is even $\quad \Leftrightarrow \quad y(S)$ odd.

From BIP to CPTU (II)

$$\max\{\tilde{c}^T y \mid Ty \le 0, Q^{-1}(b_Q + y) \in \mathbb{Z}^n\} = \max\{\tilde{c}^T y \mid Ty \le 0, Q^{-1}(b_Q + y) \in \mathbb{Z}^n, y \in \mathbb{Z}^n\}$$
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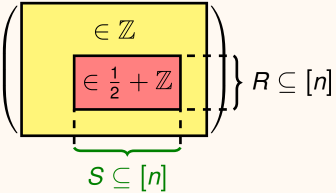
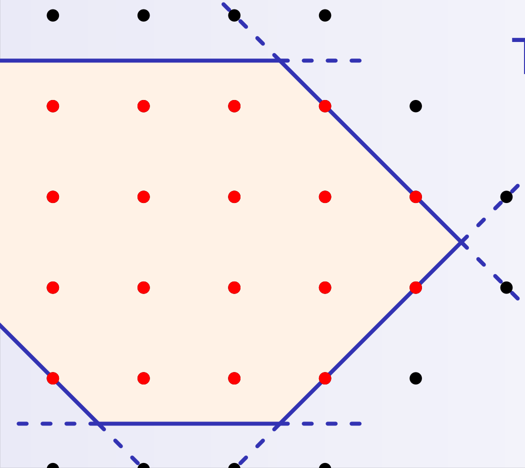


$\Rightarrow$ $Q^{-1} w \in \mathbb{Z}^n \quad \Leftrightarrow \quad w(S)$ is even.

$$\Rightarrow \quad Q^{-1}(b_Q + y) \in \mathbb{Z}^n \quad \Leftrightarrow \quad (b_Q + y)(S) \text{ is even} \quad \Leftrightarrow \quad y(S) \text{ odd.}$$



Toward simpler combinatorial problems via Seymour's TU decomposition

Seymour's TU decomposition (I)

Any TU matrix can be constructed from 3 basic types of TU matrices:

(i) Network matrices (gen. of incidence matrices),

(ii) transposes of network matrices,

(iii) the following two matrices:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

using the following operations:

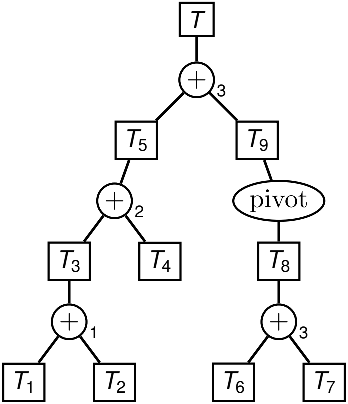
- *1-sum*: $L \oplus_1 R = \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}$,
- *2-sum*: $\begin{bmatrix} L & a \end{bmatrix} \oplus_2 \begin{bmatrix} d^T \\ R \end{bmatrix} = \begin{bmatrix} L & ad^T \\ 0 & R \end{bmatrix}$, and
- *3-sum*: $\begin{bmatrix} L & a & a \\ f^T & 0 & 1 \end{bmatrix} \oplus_3 \begin{bmatrix} 1 & 0 & d^T \\ g & g & R \end{bmatrix} = \begin{bmatrix} L & ad^T \\ gf^T & R \end{bmatrix}$,

where $\text{rows}(L) + \text{cols}(L) \geq 4$ and $\text{rows}(R) + \text{cols}(R) \geq 4$.

- permuting rows/columns,
- adding a row/column with at most 1 nonzero entry,
- negating a row/column,
- doubling a row/column,
- pivoting (think of simplex pivoting).

We slightly tweak Seymour's TU decomposition to get additional properties.

Key operations that have to be considered: 1-sums, 2-sums, 3-sums, and pivots.

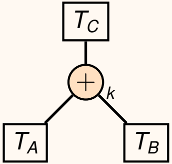


Using Seymour's decomposition to solve CPTU

CPTU problem: $\max\{c^T x \mid Tx \leq 0, x(S) \text{ odd}, x \in \mathbb{Z}^n_{\geq 0}\}$.

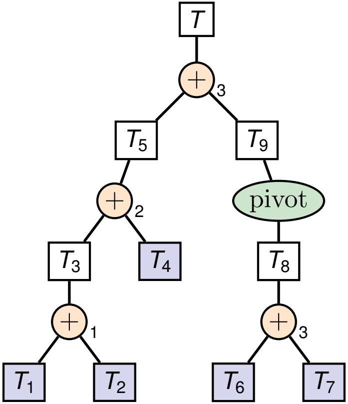
k-sums for $k \in \{1,2,3\}$

Efficient algo for CPTU wrt T_A, T_B implies efficient algo for CPTU wrt T_C.



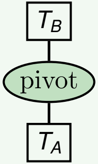
Base blocks

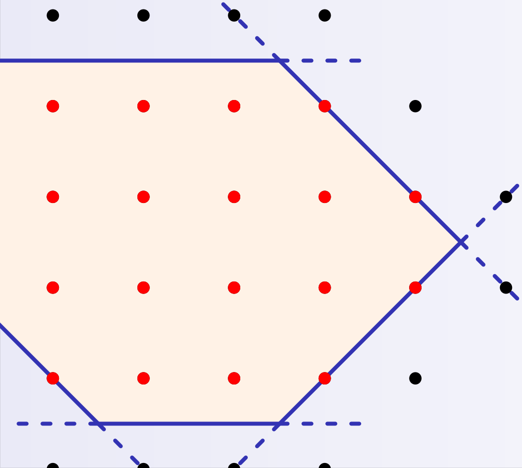
We can solve any CPTU for any base block matrix.



Pivots

Eficient algo for CPTU wrt T_A implies efficient algo for CPTU wrt T_B.





Propagation aspects on the example of 2-sums

CPTU: $\max\{c^T x \mid Tx \leq 0, x(S) \text{ odd}, x \in \mathbb{Z}^n_{\geq 0}\}$

Assume T can be written as a 2-sum:

$$T = \left(\begin{array}{c|c} L & ab^\top \\ \hline 0 & R \end{array}\right) = \begin{pmatrix} L & a \end{pmatrix} \oplus_2 \begin{pmatrix} b^\top \\ R \end{pmatrix}$$

$$S = S_L \,\dot{\cup}\, S_R$$

Lemma — **AWZ [2017]**

$\exists$ opt. sol. $x^* = \begin{pmatrix} x_L^* \\ x_R^* \end{pmatrix}$ to CPTU wrt T such that $b^T x_R^* \in \{-1, 0, 1\}$.

In what follows, assume $\text{rows}(R) \leq \text{rows}(L)$.

Assume you are given x_R^* with $b^T x_R^* \in \{-1, 0, 1\}$.
All one has to know to determine x_L^* is:

(i) value of $b^T x_R^* \in \{-1, 0, 1\}$, and

(ii) parity of $x_R(S_R) \in \{\text{even}, \text{odd}\}$.

For each of the 6 combinations of **(i)** and **(ii)** we construct an optimal x_R^*.

$$T = \left(\begin{array}{c|c} L & ab^\top \\ \hline 0 & R \end{array}\right) = \begin{pmatrix} L & a \end{pmatrix} \oplus_2 \begin{pmatrix} b^\top \\ R \end{pmatrix}$$

$$S = \underbrace{S_L}_{} \,\dot{\cup}\, \underbrace{S_R}_{}$$

For $\alpha \in \{-1, 0, 1\}$ and $\beta \in \{0, 1\}$, we compute:

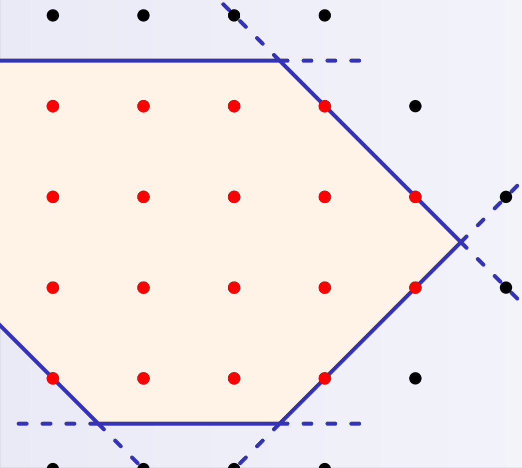
$$\rho(\gamma, \delta) := \max\{c_R^T x_R \mid R \cdot x_R \le 0,\ b^T x_R = \alpha,\ x_R(S_R) \equiv \beta \pmod 2,\ x_R \in \mathbb{Z}_{\ge 0}^{n_R}\} .$$

We incorporate these options into a problem involving L. We set:

J: components with $\beta = 1$

$$\begin{array}{rcl|cccccc} (\alpha, \beta) : & & & (-1,0) & (0,0) & (1,0) & (-1,1) & (0,1) & (1,1) \\ \overline{L} & = & \big[\ L & -a\ , & 0\ , & a\ , & -a\ , & 0\ , & a\ \big] \\ \overline{c}^T & = & \big[\ c_L^T & \rho(-1,0), & \rho(0,0), & \rho(1,0), & \rho(-1,1), & \rho(0,1), & \rho(1,1)\ \big] \end{array}$$

Combined problem to find x_L^*: $\max\{\overline{c}^T x \mid \overline{L}x \le 0, x \in \mathbb{Z}_{\ge 0}^{n_L+6}, x(S_L \cup J) \text{ odd}\}$



Conclusions

Our main result

- BIPs are efficiently solvable (even in strongly poly time).

Some natural open questions (. . . and things I am interested in)

- Recognition of bimodular matrices?
- Solve k-modular ILPs for $k = O(1)$, or even just determine feasibility?
- Reduction of k-modular ILP to modular optimization, e.g., TU problem with $x(S) \equiv 1 \pmod{k}$?
- Different approach to solve BIP not based on TU decomposition?
- Derive additional structural properties of k-modular matrices.