# Extended formulations (II): semidefinite programming lifts

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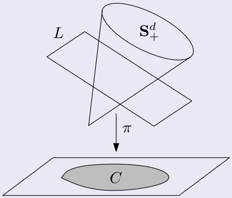
# Semidefinite programming extended formulations

$\mathbf{S}^d_+$ = cone of $d \times d$ positive semidefinite matrices

## Definition

Let $C$ be a convex set. We say that $C$ has an SDP extended formulation (or SDP lift) of size $d$ if we can write

$$C = \pi(\mathbf{S}^d_+ \cap L)$$



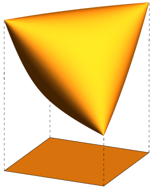

where

- $\pi$ is a linear map;
- and $L$ a linear subspace of $\mathbf{S}^d$

$xc_{SDP}(C)$ = smallest size of an SDP lift of $C$

# Examples of SDP lift

$$[-1,1]^2 = \left\{ (x,y) \in \mathbb{R}^2 : \exists u \in \mathbb{R} \text{ s.t. } \begin{bmatrix} 1 & x & y \\ x & 1 & u \\ y & u & 1 \end{bmatrix} \succeq 0 \right\}$$



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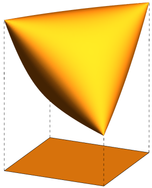

- Nuclear norm ball (cf. Pablo's talk)

$$\left\{ M \in \mathbb{R}^{n \times m} : \exists X, Y \text{ s.t. } \begin{bmatrix} X & M \\ M^T & Y \end{bmatrix} \succeq 0, \quad \frac{1}{2}(\mathrm{Tr}(X) + \mathrm{Tr}(Y)) \le 1 \right\}$$

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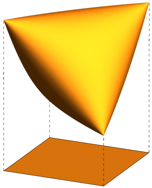

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- $STAB(G)$ for perfect graph $G$ (cf. Michel's talk)

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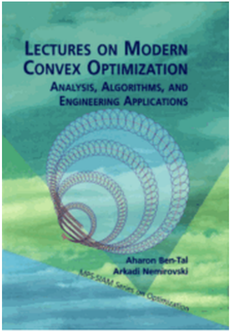

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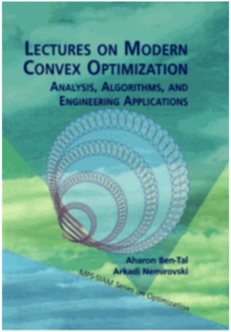


Gives SDP representations for various convex sets and functions:

- $\ell_p$ norm balls for $p \geq 1$ rational
- Nuclear norm / Schatten $\ell_p$ norms
- Sum of $k$ largest eigenvalues/singular values
- $\cdots$

Implemented in modeling tools like CVX and Yalmip

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- Scheiderer (2012): convex semialgebraic sets in the plane have SDP lift
- Scheiderer (2016): there are (many) convex semialgebraic sets that do not have an SDP representation

# Slack matrix

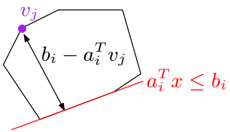
$P$ polytope in $\mathbb{R}^n$

Slack matrix of $P$: Nonnegative matrix $M$ of size #facets($P$) $\times$ #vertices($P$):

$$M_{i,j} = b_i - a_i^T v_j$$

where

- $a_i^T x \leq b_i$ are the facet inequalities of $P$
- $v_j$ are the vertices of $P$



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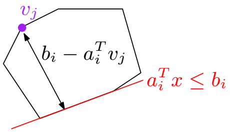


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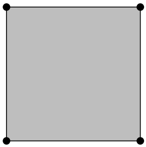

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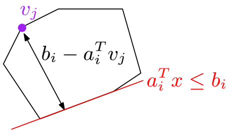


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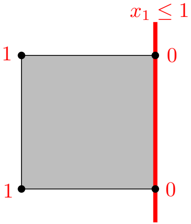

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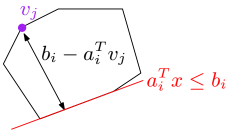
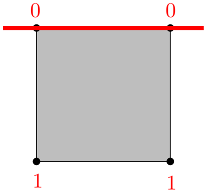



| | | | |
|---|---|---|---|
| 0 | 1 | 1 | 0 |
| 0 | 0 | 1 | 1 |



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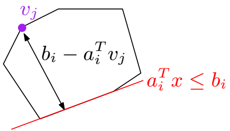


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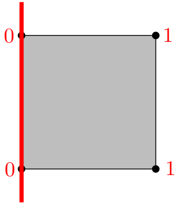

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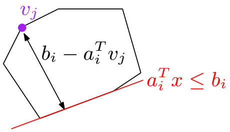


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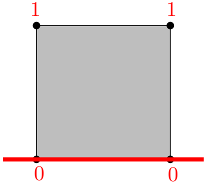


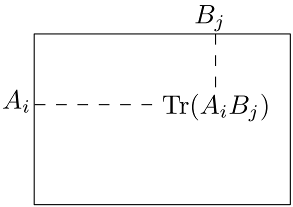
# Positive semidefinite rank

$M \in \mathbb{R}^{p \times q}$ with nonnegative entries

- Positive semidefinite factorization:

$$M_{ij} = \langle A_i, B_j \rangle \quad \text{where} \quad A_i, B_j \in \mathbf{S}_+^d$$

- $\text{rank}_{\text{psd}}(M)$ = size of smallest psd factorization



# Example of positive semidefinite factorization

Consider $M_{ij} = (i-j)^2$ for $1 \leq i, j \leq n$:

$$M = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 \\ 1 & 0 & 1 & 4 & 9 \\ 4 & 1 & 0 & 1 & 4 \\ 9 & 4 & 1 & 0 & 1 \\ 16 & 9 & 4 & 1 & 0 \end{bmatrix}$$

- $\text{rank}_{\text{psd}}(M) = 2$ (independent of $n$): Let

$$A_i = \begin{bmatrix} 1 & i \\ i & i^2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix}^T \text{ and } B_j = \begin{bmatrix} j^2 & -j \\ -j & 1 \end{bmatrix} = \begin{bmatrix} -j \\ 1 \end{bmatrix} \begin{bmatrix} -j \\ 1 \end{bmatrix}^T.$$

  One can verify that $M_{ij} = \text{Tr}(A_j B_j)$.

# SDP lifts and PSD rank

**Theorem**

*Let $P$ be a polytope with slack matrix $M$. Then* $xc_{SDP}(P) = \mathrm{rank}_{\mathrm{psd}}(M)$.

# Connection with sums of squares

## Theorem

*Let* $P = \text{conv}(X)$ *be a polytope.*

- *If* $P$ *has a SDP lift of size* $d$*, then there exists a subspace* $\mathcal{V}$ *of* $\mathbb{R}^X$ *such that the following holds:*
  (i) $\dim \mathcal{V} \leq d^2$
  (ii) *Any facet* $b - a^T x \geq 0$ *of* $P$ *has a s.o.s. certificate from* $\mathcal{V}$ *i.e., there exist* $h_\alpha \in V$ *s.t.*

$$b - a^T x = \sum_\alpha h_\alpha(x)^2 \quad \forall x \in X$$

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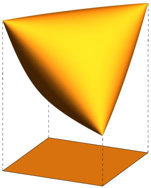
Lasserre hierarchy: $\mathcal{V} =$ subspace of polynomials of degree at most $k$

# Connection with sums of squares: example 1

- $X = \{-1, 1\}^2$, $\text{conv}(X) = [-1, 1]^2$.
- Four facet inequalities:

$$1 - x_1 \geq 0, \quad 1 + x_1 \geq 0, \quad 1 - x_2 \geq 0, \quad 1 + x_2 \geq 0$$



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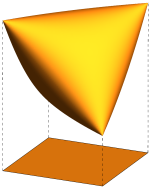

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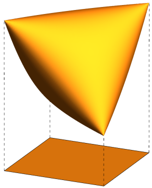


- Similar certificate holds for the other facets
- Subspace

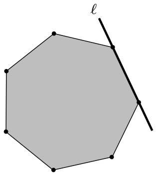
$$\mathcal{V} = \text{span}(1, x_1, x_2)$$

has dimension 3.
This yields an SDP lift of $[-1, 1]^2$ of size 3.

# Example 2: polygons

- $P = \text{conv}(X)$ with $X = N$ roots of unity
- Facet inequality

$$\ell(x, y) = \cos(\pi/N) - \cos(\pi/N)x - \sin(\pi/N)y \geq 0$$



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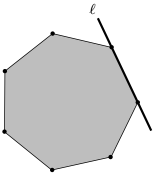


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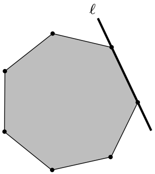


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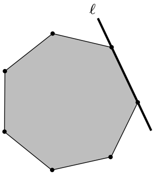


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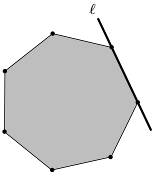


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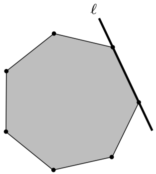


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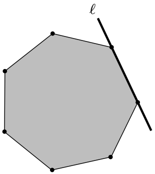


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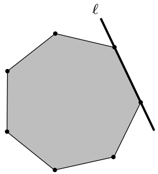


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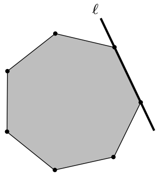


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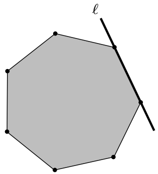
- $P = \text{conv}(X)$ with $X = N$ roots of unity
- Facet inequality

$$\ell(x,y) = \cos(\pi/N) - \cos(\pi/N)x - \sin(\pi/N)y \geq 0$$




- Any s.o.s. certificate of $\ell$ must have deg $\geq N/4$

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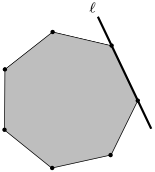


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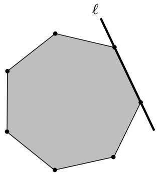


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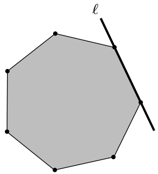


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Subspace of dimension $\sim \log N$

# Hierarchies and extended formulations

$P = \text{conv}(X)$. Certify nonnegativity of facets $\ell$ of $P$

| | **Hierarchies** | **Extended formulations** |
|---|---|---|
| **LP** | $X = \{0,1\}^n \cap \{g_i(x) \geq 0\}$ Handelman/Sherali-Adams<br><br>$\ell =$ nonneg. comb. of $x^\alpha (1-x)^\beta \prod_i g_i(x)^{\gamma_i}$ | $\ell =$ nonneg. combinations of some well-chosen $a_i : X \to \mathbb{R}_+$ |
| **SDP** | $\ell =$ s.o.s. of degree $\leq k$ | $\ell =$ s.o.s. from a well-chosen subspace $\mathcal{V}$ |