Golden Gates, Ramanujan Complexes and Ramanujan Digraphs

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Expanders and Extractors, Simons Institute, Berkeley 2017

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- LPS: Ramanujan quotients of T_{p+1}.

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- Think of $PGU_2\left(\mathbb{Z}\left[\frac{1}{5}\right]\right)$ as all $A \in M_2(\mathbb{Z}[i])$ with $A^*A = 5^n I$, $n \in \mathbb{N}$.

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$$S = \left\{ \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}, \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \right\}$$

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- Uses Ramanujan-Petersson conjecture (Eichler/Weyl/Deligne), Functoriality (Jacquet-Langlands).

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$$a^2 + b^2 + c^2 + d^2 = p \qquad (a, b, c, d \in \mathbb{Z})\,.$$

- Write $\alpha = a + bi$, $\beta = c + di$. Each solution gives

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in PGU_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right), \qquad A^* A = \left(|\alpha|^2 + |\beta|^2\right) \cdot I = p \cdot I$$

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- Hardest: Find such a short circuit, given $\{A_i\}, M, \varepsilon$.
- Nice to have: good growth rate. E.g. if $\{A_i\}$ have no relations, there are r^ℓ circuits with ℓ gates.

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- Hard (Ross-Selinger, Sardari): approximate $M \in PU(2)$ by $M' \in \langle S_p \rangle$.

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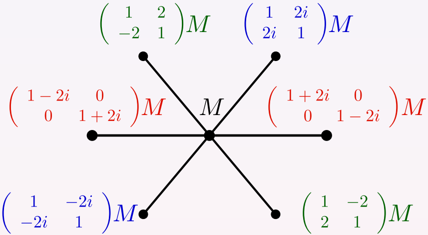

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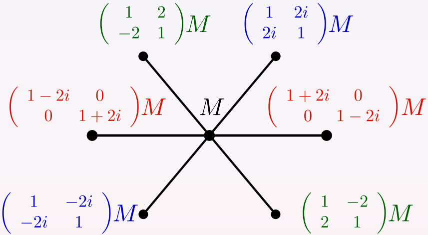

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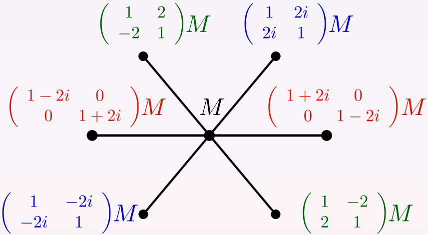

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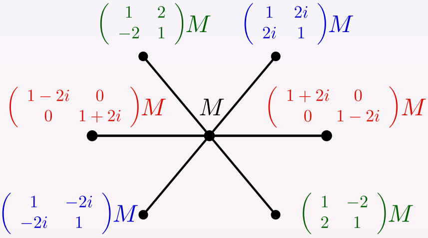

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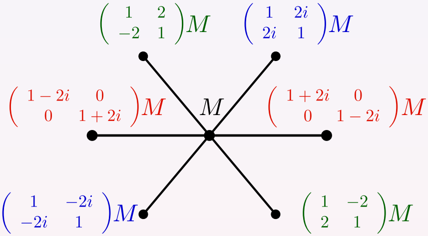

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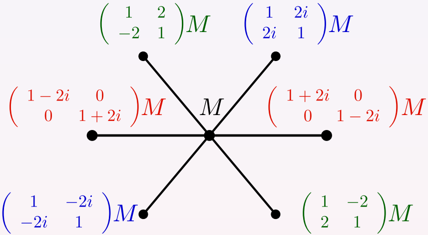

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- Suggests: Expander = small nontrivial spectrum.

Golden Gates

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- Proof uses even more number theory.

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- We find $\Gamma \leq PGU_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$ which acts simply transitively on the *directed edges* of the tree.
- Example:

$$\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \begin{pmatrix} 1-\sqrt{2} & i \\ -i & \sqrt{2}-1 \end{pmatrix} \right\rangle$$

  has a finite index in $\langle C, T \rangle$, and acts simply transitively on the edges of a 3-regular tree.

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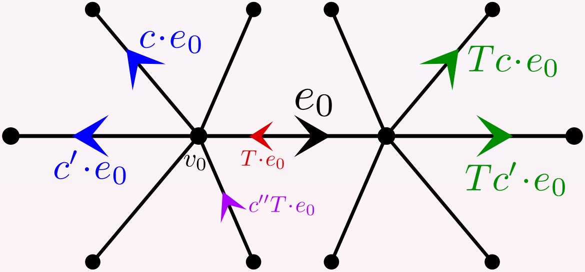

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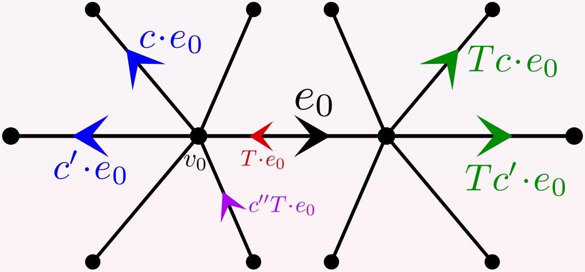

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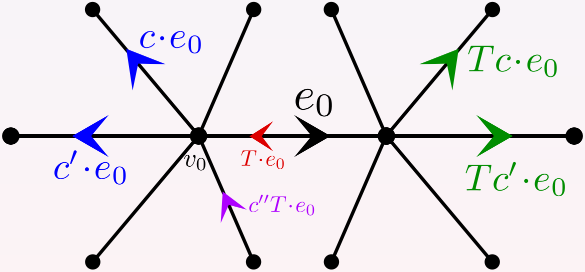

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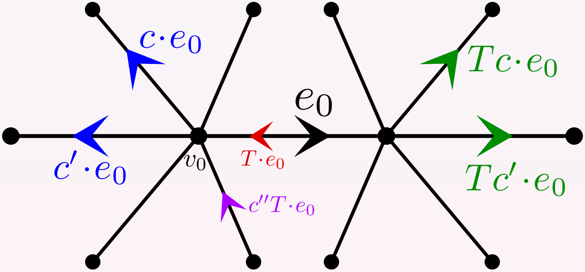

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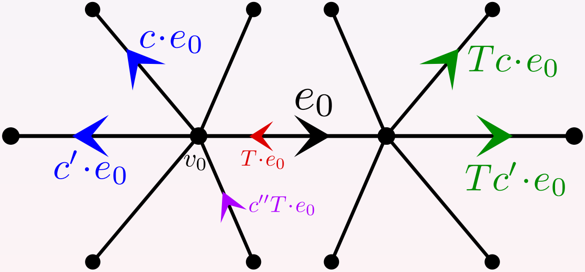

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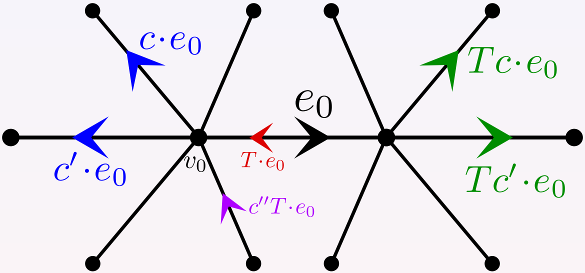
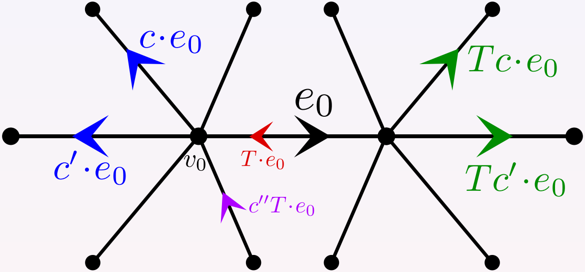

$c \cdot e_0$
$Tc \cdot e_0$
e_0
$c' \cdot e_0$
v_0
$T \cdot e_0$
$Tc' \cdot e_0$
$c''T \cdot e_0$



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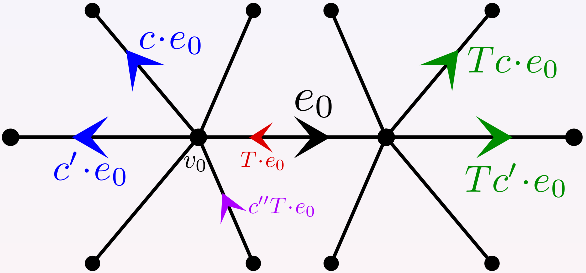

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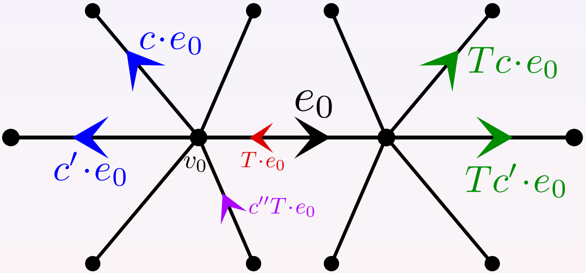

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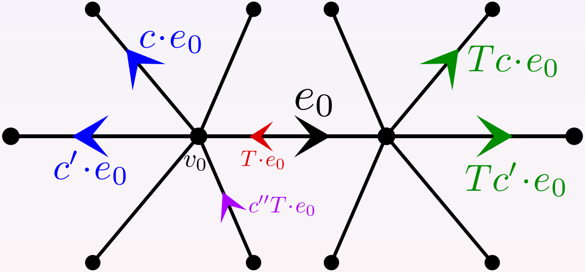

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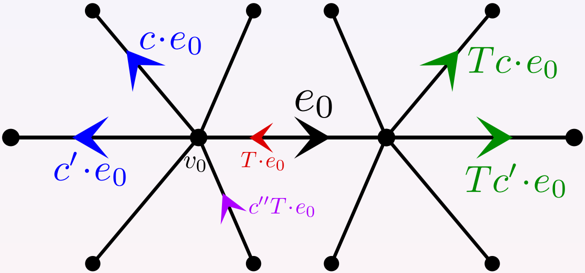

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- $\sqrt{k-1}$: spectrum of NBRW on the k-regular tree.

Icosahedral gates

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- $\Gamma = \langle C, T \rangle$ is the full $\{7+5\varphi\}$-arithmetic group in the Icosian ring:

$$\mathbb{I} = \left\{ \frac{1}{2} \begin{pmatrix} (a+b\varphi) + (c+d\varphi)\, i \\ + (e+f\varphi)\, j + (g+h\varphi)\, k \end{pmatrix} \,\middle|\, \begin{matrix} a,b,c,d,e,f,g,h \in \mathbb{Z} \\ a+c+e+g \equiv b+d+f+h \equiv 0 \pmod 2 \\ (c,e,a) \equiv (b,d,f) \text{ or } \equiv (1,1,1)+(b,d,f) \pmod 2 \end{matrix} \right\} \subseteq \mathbb{H}.$$

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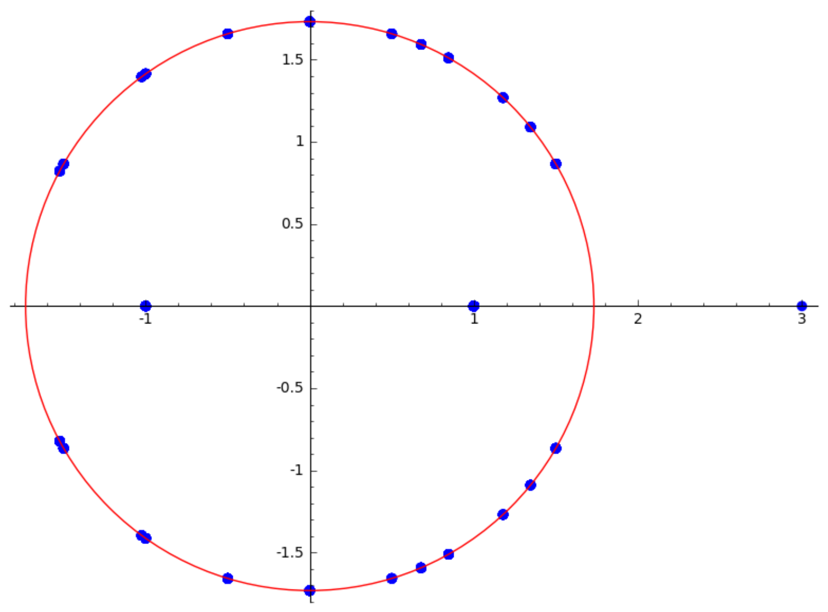
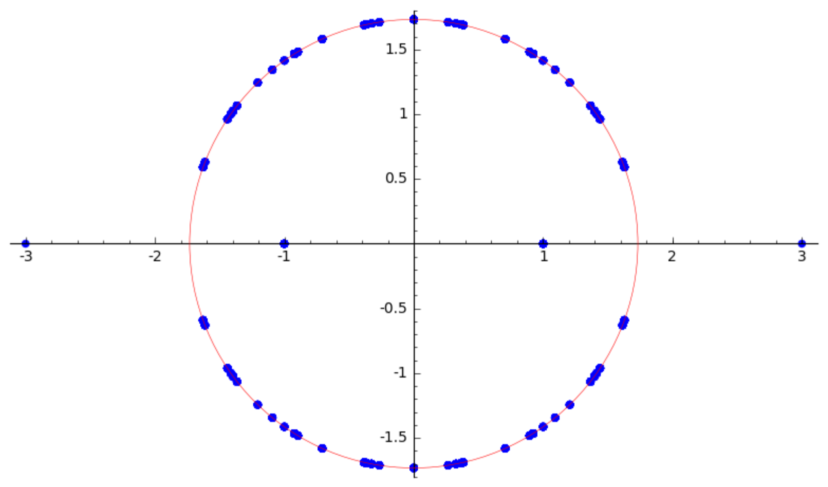
- For arithmetic quotients $\Gamma_q \backslash \Gamma$, we obtain Cayley Ramanujan digraphs.



Adjacency spectrum of $PSL_2(\mathbb{F}_{13})$ with respect to $\begin{pmatrix} 12 & 9 \\ 7 & 12 \end{pmatrix}, \begin{pmatrix} 6 & 8 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 4 & 12 \\ 1 & 7 \end{pmatrix}$



Adjacency spectrum of $PGL_2(\mathbb{F}_{17})$ with respect to $\left(\begin{smallmatrix}16 & 14\\ 12 & 16\end{smallmatrix}\right), \left(\begin{smallmatrix}5 & 13\\ 13 & 14\end{smallmatrix}\right), \left(\begin{smallmatrix}3 & 16\\ 1 & 12\end{smallmatrix}\right)$

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- Approximation is much harder.

PU(3)

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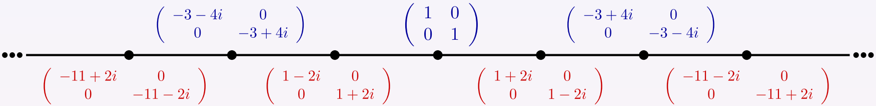

$\begin{pmatrix} -3-4i & 0 \\ 0 & -3+4i \end{pmatrix}$
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$\begin{pmatrix} -3+4i & 0 \\ 0 & -3-4i \end{pmatrix}$
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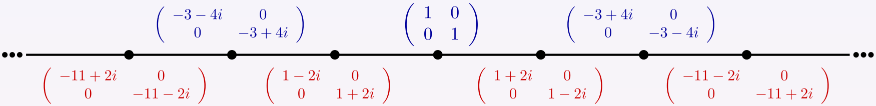
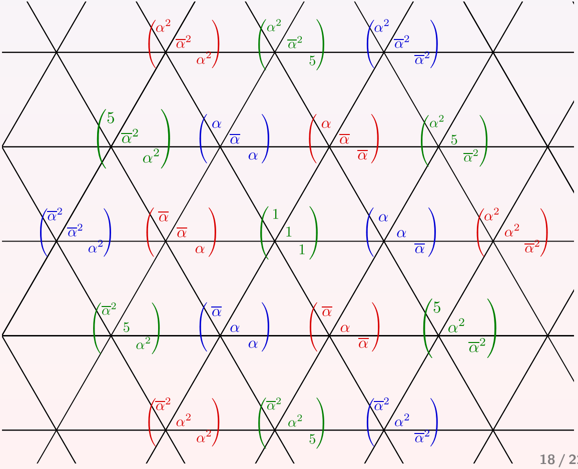


Is replaced in PGU_3
by a Euclidean plane:
$(\alpha = 1 + 2i)$



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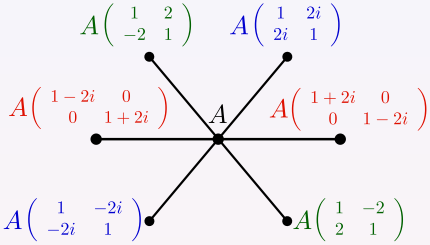

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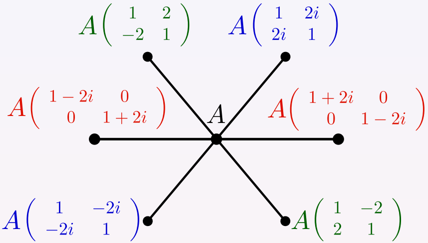

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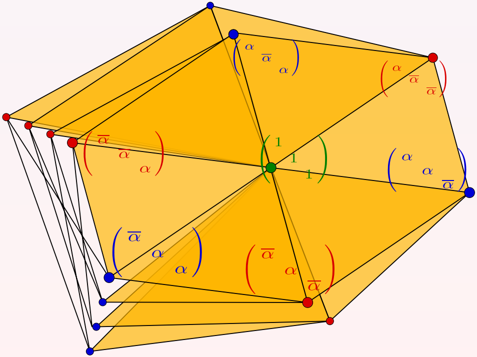

- Jacobi's six-square theorem: for $p \equiv 1 \pmod 4$, there are $12\left(p^2+1\right)$ solutions to

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- Siegel's Mass formula allows us to count the solutions: count solutions in $PGU_2(\mathbb{Q}_p)$ for all p, including $\mathbb{Q}_\infty = \mathbb{R}$.

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- Golden qutrits?
- Similar results on $PU(4)$ - not as nice. Work in progress!

Thank You!