Optimal Online Algorithms via Linear Scaling

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Multiple-Choice Secretary Problem

[Hajiaghayi, Kleinberg, Parkes, EC 2004], [Kleinberg, SODA 2005], . . .

- n items arrive online over time, each with a weight
- Weights are adversarial
  Arrival order is uniformly random
- Upon arrival of an item:
  Immediate decision whether to accept or reject
- Constraint: Accept at most k items
- Objective: Maximize sum of weights of accepts items

Theorem (Kleinberg, SODA 2005)

There is a $1 - O(1/\sqrt{k})$*-competitive algorithm, but no* $1 - o(1/\sqrt{k})$*-competitive algorithm.*

α-competitive if $\mathbf{E}[\mathrm{ALG}] \geq \alpha \cdot \mathrm{OPT}$

Algorithm

In round ℓ, upon arrival of item j

- Let $S^{(\ell)}$ be $\lfloor \frac{\ell}{n} k \rfloor$ items of highest value that arrived so far
- If $j \in S^{(\ell)}$ and $|\text{Accepted}| < k$
  Set $\text{Accepted} := \text{Accepted} \cup \{j\}$

Value of tentative selection

Let $p := 9\sqrt{\frac{1}{k}}$. For $pn \leq \ell \leq (1 - p)n$, we have

- $E[v(S^{(\ell)})] \geq \frac{\ell}{n} OPT \left(1 - 9\sqrt{\frac{1}{\frac{\ell}{n}k}}\right)$
- $E[v_j C_\ell] = \frac{1}{\ell} E[v(S^{(\ell)})] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1}{\frac{\ell}{n}k}}\right)$

$$C_\ell = \begin{cases} 1 & \text{if } j \in S^{(\ell)} \\ 0 & \text{otherwise} \end{cases}$$

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Conflict probability

Let $p := 9\sqrt{\frac{1}{k}}$. For $pn \leq \ell \leq (1-p)n$, we have

- $E\left[|\text{Accepted}|\right] \leq \sum_{\ell'=1}^{\ell-1} E[C_{\ell'}] \leq \frac{\ell}{n} k$
- $Pr\left[|\text{Accepted}| \geq k\right] \leq \exp\left(-\frac{n-\ell}{n}\sqrt{k}\right)$

Putting the pieces together

For $pn \leq \ell \leq (1-p)n$ we have:

- $E[v_j C_\ell] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1}{\frac{\ell}{n}k}}\right)$
- $Pr\left[|\text{Accepted}| \geq k\right] \leq \exp\left(-\frac{n-\ell}{n}\sqrt{k}\right)$

Despite dependencies:

$$E[\text{value from round } \ell] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1}{\frac{\ell}{n}k}}\right)\left(1 - \exp\left(-\frac{n-\ell}{n}\sqrt{k}\right)\right)$$

Adding up:

$$E[ALG] \geq \sum_{\ell=pn}^{(1-p)n} \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1}{\frac{\ell}{n}k}}\right)\left(1 - \exp\left(-\frac{n-\ell}{n}\sqrt{k}\right)\right)$$
$$= \left(1 - O\left(\sqrt{\frac{1}{k}}\right)\right) OPT \qquad \square$$

Outline

Online Packing Linear Programs

$\max c^T x$

s.t.

m constraints

$$A x \le b$$

$\le d$ non-zero

for all j: $\quad 0 \le x_j \le 1$

$$B = \min_{i \in [m]} \frac{b_i}{\max_{j \in [n]} a_{i,j}}$$

Results for Online Packing LPs

$(1-\epsilon)$-competitive algorithms if B is large enough

Feldman, Henzinger, Korula,
Mirrokni, Stein (ESA 2010): $B = \Omega\left(\frac{m \log(nK)}{\epsilon^3}\right)$

Agrawal, Wang, Ye (2009): $B = \Omega\left(\frac{m \log(nK/\epsilon)}{\epsilon^2}\right)$

Molinaro and Ravi (ICALP 2012): $B = \Omega\left(\frac{m^2 \log(m/\epsilon)}{\epsilon^2}\right)$

K., Radke, Tönnis,
Vöcking (STOC 2014): $B = \Omega\left(\frac{\log d}{\epsilon^2}\right)$

matches lower bound
by Agrawal et al., 2009

Algorithm

In round ℓ, upon arrival of variable x_j

- $\tilde{x}^{(\ell)} :=$ optimal solution to "known" LP with scaled capacities $\frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Theorem

- $E[ALG] = \left(1 - O\left(\sqrt{\frac{\log d}{B}}\right)\right) OPT$
- $E[ALG] = \Omega\left(d^{-\frac{2}{B-1}}\right) OPT$

Analysis

For $pn \leq \ell \leq (1-p)n$ we have:

- $E[c^T x^{(\ell)}] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1+\ln d}{\frac{\ell}{n} B}}\right)$
- $Pr\left[\left(\sum_{\ell' \leq \ell} A x^{(\ell')}\right)_i > b_i\right] \leq \frac{1}{d} \exp\left(-\frac{n-\ell}{n}\sqrt{B}\right)$

Despite dependencies:

$$E[c^T y^{(\ell)}] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1+\ln d}{\frac{\ell}{n} B}}\right) \left(1 - \exp\left(-\frac{n-\ell}{n}\sqrt{B}\right)\right)$$

Adding up:

$$E[ALG] \geq \sum_{\ell=pn}^{(1-p)n} \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1+\ln d}{\frac{\ell}{n} B}}\right) \left(1 - \exp\left(-\frac{n-\ell}{n}\sqrt{B}\right)\right)$$

$$= \left(1 - O\left(\sqrt{\frac{\log d}{B}}\right)\right) OPT \qquad \square$$

Algorithm

In round ℓ, upon arrival of variable x_j

- $\tilde{x}^{(\ell)} :=$ optimal solution to “known” LP with scaled capacities $\frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Theorem

- $E[ALG] = \left(1 - O\left(\sqrt{\frac{\log d}{B}}\right)\right) OPT$
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- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Value of tentative selection

Let $p := 9\sqrt{\frac{1+\ln d}{B}}$. For $pn \leq \ell \leq (1-p)n$, we have

- $E[c^T \tilde{x}^{(\ell)}] \geq \frac{\ell}{n} OPT \left(1 - 9\sqrt{\frac{1+\ln d}{\frac{\ell}{n} B}}\right)$
- $E[c^T x^{(\ell)}] = \frac{1}{\ell} E[c^T \tilde{x}^{(\ell)}] \geq \frac{1}{n} OPT \left(1 - 9\sqrt{\frac{1+\ln d}{\frac{\ell}{n} B}}\right)$

Algorithm

In round ℓ, upon arrival of variable x_j

- $\tilde{x}^{(\ell)} :=$ optimal solution to "known" LP with scaled capacities $\frac{\ell}{n} b$
- $x_j^{(\ell)} := 1$ with probability $\tilde{x}_j^{(\ell)}$, all other entries 0
- If $A(y + x^{(\ell)}) \leq b$, set $y := y + x^{(\ell)}$

Conflict probability

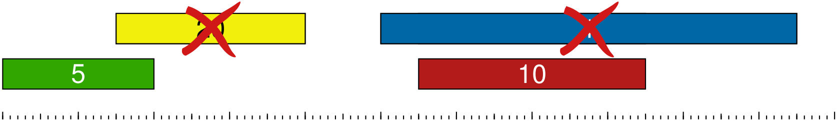
Let $p := 9\sqrt{\frac{1+\ln d}{B}}$. For $pn \leq \ell \leq (1-p)n$, we have

$$Pr\left[\left(\sum_{\ell' \leq \ell} Ax^{(\ell')}\right)_i > b_i\right] \leq \frac{1}{d} \exp\left(-\frac{n-\ell}{n}\sqrt{B}\right)$$

Temp Secretary Problem

[Fiat, Gorelik, Kaplan, Novgorodov, ESA 2015]

- n items arrive online over time, each with a weight and a length
- Weights are adversarial
  Arrival times are drawn i.i.d. from a distribution (e.g. $U[0, 1]$)
- Upon arrival of an item:
  Immediate decision whether to accept or reject
- Constraint: Accept at most k items simultaneously
- Objective: Maximize sum of weights of accepts items



Algorithm for Identical Lengths

[K., Tönnis, ESA 2016]

Important Observation:
If all items have length γ, we can accept at most $k\lceil 1/\gamma \rceil$ arriving in $[0, 1]$

for *every arriving item j* **do**
    Set $t := \tau_j$;
    Let $S^{(t)}$ be the $\lfloor tk/\gamma \rfloor$ highest-valued items i with $\tau_i \leq t$;
    if $j \in S^{(t)}$ **then**
        if *Accepted* $\cup \{j\}$ *is a feasible schedule* **then**
            Set Accepted := Accepted $\cup \{j\}$;

Theorem

If all lengths are γ, the algorithm is

- $1/2 - O(\sqrt{\gamma})$*-competitive and*
- $1 - O(1/\sqrt{k}) - O(\sqrt{\gamma})$*-competitive*

Summary

- General simple template for online algorithms:
  Pretend you could start from scratch now, but leave some space!

- Optimal guarantees for multiple-choice secretary problem and online packing LPs

- Simple and better algorithm for temp secretary problem

- Approach also works for submodular objective functions

  [K., Tönnis, unpublished]

Outlook

- What about cost-minimization problems, e.g., scheduling?

  [Göbel, K., Tönnis, ESA 2015]

- What about non-uniformly drawn permutations?

  [K., Kleinberg, Niazadeh, STOC 2015]

- Are there algorithms that are good in multiple models?

Thank you!

Questions?