# The Large Deviations of the Whitening Process in Random Constraint Satisfaction Problems

and of the bootstrap percolation

Guilhem Semerjian

LPT-ENS

05.05.2016 / Simons Institute

based on Braunstein, Dall'Asta, S, Zdeborová

arXiv:1602.01700 and J. Stat. (in press)

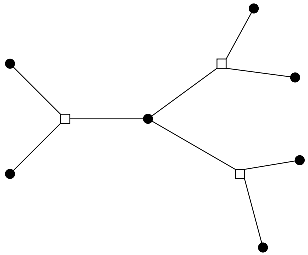
# Outline


1. Hypergraph bicoloring and its phase transitions
2. Rigidity and freezing
3. Main results
4. Minimal contagious sets of random regular graphs
5. Conclusions and perspectives

# An example of CSP

- Hypergraph bicoloring (positive NAE-$k$-SAT) :
  - $N$ variables $\quad \underline{\sigma} = (\sigma_1, \dots, \sigma_N) \in \{-1, 1\}^N$
  - $M$ constraints on the hyperedges of a $k$-uniform hypergraph

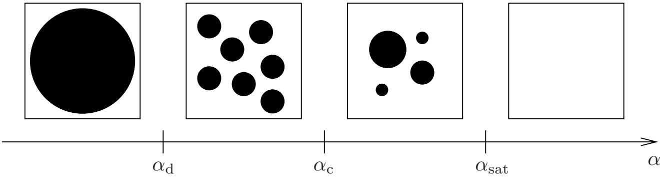
$$\psi_a(\{\sigma_i\}_{i \in \partial a}) = \begin{cases} 1 & \text{at least one +1 and one -1} \\ 0 & \text{all +1 or all -1} \end{cases}$$



solutions : $\quad \mathcal{S} = \{\underline{\sigma} : \psi_a(\underline{\sigma}_{\partial a}) = 1 \ \forall a\}$

# Phase transitions for random CSPs (also $k$-SAT, $q$-COL, ...)

- random hypergraph with $M$ edges (regular or Erdös-Rényi)
  density of constraints $\alpha = M/N$, thermodynamic limit $N, M \to \infty$




- Satisfiability threshold at $\alpha_{\rm sat}(k) \sim 2^{k-1} \ln 2$
- Shattering of solutions in clusters at $\alpha_{\rm d}(k) \sim \alpha_{\rm sat}(k) \frac{\ln k}{k \ln 2}$
  reconstruction threshold on the tree
- Condensation, sub-exponential nb. of clusters at $\alpha_{\rm c}(k) \sim \alpha_{\rm sat}(k)$

# Phase transitions for random CSPs (also *k*-SAT, *q*-COL, ...)

Recent rigorous results on hypergraph bicoloring/random NAESAT :

- satisfiability threshold [Ding, Sly, Sun 13]
- condensation at positive temperature [Bapst, Coja-Oghlan, Rassmann 14]
- typical number of solutions [Sly, Sun, Zhang 16]
- fluctuations of the number of solutions [Rassman 16]
- failure of Survey Propagation for $\alpha > \alpha_{\rm d}$ [Hetterich 16]
- ...

# One more phase transition : rigidity

Coarse-grained description of a cluster : $\underline{\sigma}^* \in \{-1, 1, 0\}^N$

$$\text{with } \sigma_i^* = \begin{cases} 1 & \text{if } \sigma_i = 1 \text{ in all solutions of the cluster} \\ -1 & \text{if } \sigma_i = -1 \text{ in all solutions of the cluster} \\ 0 & \text{otherwise} \end{cases}$$

Frozen variables of a cluster : the ones with $\sigma_i^* = \pm 1$

# One more phase transition : rigidity

- Alternative definition of frozen variables :
  - start with a solution $\underline{\sigma}$
  - a constraint $a$ blocks a variable $\sigma_i = \pm 1$ iff $\sigma_j = -\sigma_i$ for all $j \in \partial a \setminus i$
  - if $i$ is not blocked by any constraint, "whiten" it, $\sigma_i \to 0$
  - repeat until fixed point $\underline{\sigma}^*$ is reached

  Procedure known as whitening, peeling, coarsening...

  Largest subcube containing $\underline{\sigma}$ with no solutions at Hamming distance 1

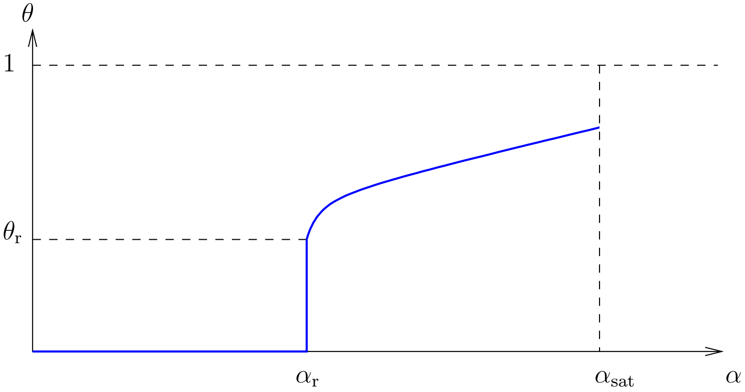
- $\theta$ : fraction of frozen variables ($\sigma_i^* = \pm 1$) in a fixed point

  Either $\theta = 0$ or $\theta \geq \theta_{\rm min} > 0$ [Maneva, Mossel, Wainwright 07]

  unfrozen / frozen solutions

# One more phase transition : rigidity

Typical fraction of frozen variables (solution chosen u.a.r.) :




$\alpha_d(k) \leq \alpha_r(k)$ : stronger form of correlation (naive reconstruction)

At large $k$, $\alpha_r(k) \sim \alpha_d(k)$

# Frozen variables and algorithmic difficulty

- Frozen solutions should be hard to find :
  need to set collectively order $N$ variables

- Indeed heuristic algorithms output unfrozen solutions

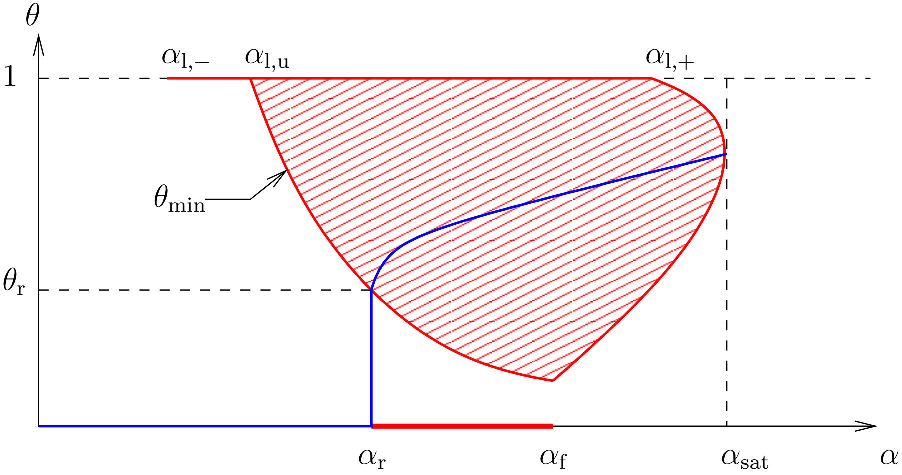
- Algorithmic barrier : no known algorithm finds solutions in polynomial time for

  $\alpha > \alpha_{\rm d}(k) \sim \alpha_{\rm r}(k)$ (at large $k$)

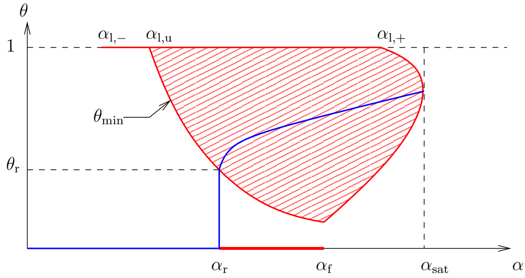
- Up to which densities do (atypical) unfrozen solutions exist ?

  Called freezing transition, $\alpha_{\rm f}(k)$

# Main results (I)



# Main results (I)

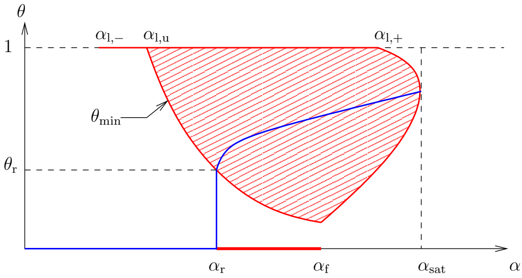



- Unfrozen solutions exist up to $\alpha_{\mathrm{f}}(k) \sim \frac{1}{2}\,\alpha_{\mathrm{sat}}(k)$

  previously, $\alpha_{\mathrm{f}}(k) \leq \frac{4}{5}\,\alpha_{\mathrm{sat}}(k)$ [Achlioptas, Ricci-Tersenghi 06]

Recall $\alpha_{\mathrm{r}}(k) \sim \alpha_{\mathrm{d}}(k) \sim \frac{\ln k}{k \ln 2}\,\alpha_{\mathrm{sat}}(k)$

# Main results (I)




- Locked solutions ($\theta = 1$, all variables frozen, sol. = whitening f.p.)
  - appear at $\alpha_{l,-}(k) \sim \frac{1}{k}\,\alpha_{sat}(k)$
  - disappear at $\alpha_{l,+}(k) \sim \alpha_{sat}(k)$
  - are the only frozen solutions up to $\alpha_{l,u}(k) \sim \alpha_d(k)$

Recall $\alpha_r(k) \sim \alpha_d(k) \sim \alpha_{sat}(k)\frac{\ln k}{k \ln 2}$

# The idea of the computation

- Parallel version of the whitening process :
  - initial condition $\underline{\sigma}^0 = \underline{\sigma}$ a solution
  - discrete time parallel evolution :

$$\sigma_i^{t+1} = \begin{cases} \sigma_i & \text{iff } \exists a \in \partial i \, , \ \forall j \in \partial a \setminus i \, , \quad \sigma_j^t = -\sigma_i \\ 0 & \text{otherwise} \end{cases}$$

- Monotonous evolution, fixed-points obtained as $\underline{\sigma}^* = \lim_{t \to \infty} \underline{\sigma}^t$
- For a finite time horizon $T$, biased measure over solutions :

$$\mu(\underline{\sigma}, T, \epsilon) = \frac{1}{Z(T,\epsilon)} \mathbb{I}(\underline{\sigma} \in \mathcal{S}) e^{\epsilon \sum_i |\sigma_i^T|}$$

- $Z(T, \epsilon)$ : generating function of the number of solutions classified by the number of white variables after $T$ steps

# The idea of the computation

- $\sigma_i^T$ depends on $\underline{\sigma}$ through variables at distance $\leq T$ from $i$
- $\mu(\underline{\sigma}, T, \epsilon)$ has interactions at distance $T$
- they can be made local with additional variables (whitening times)
- then graphical model on a sparse random factor graph
  $\Rightarrow$ "routine" cavity method computation
- Large $T$ limit can be taken analytically to get the fixed points

Very similar to previous works on minimal contagious sets for bootstrap percolation [Altarelli, Braunstein, Dall'Asta, Zecchina 13] [Guggiola, S. 15]

# Main results (II)

For each $T$, threshold $\alpha_T(k)$ such that for $\alpha < \alpha_T(k)$,

typical configurations of $\mu(\underline{\sigma}, T, \epsilon)$ are unfrozen (for a well-chosen $\epsilon$)

- $\alpha_T(k)$ grows with $T$, $\alpha_{\rm f}(k)$ obtained as $\lim_{T\to\infty} \alpha_T(k)$
- For fixed $T$, at large $k$ :
  - $\alpha_1(k) \sim \frac{\alpha_{\rm sat}(k)}{\ln k}$ recall $\alpha_{\rm d}(k) \sim \alpha_{\rm sat}(k) \frac{\ln k}{k \ln 2}$
  - $\alpha_2(k) \sim \frac{\alpha_{\rm sat}(k)}{\ln \ln k}$
  - in general $\alpha_T(k) \sim \frac{\alpha_{\rm sat}(k)}{\ln^{\circ T} k}$ $T$-times iterated logarithm

# Minimal contagious sets

- bootstrap percolation dynamics : inactive vertices become active if they have $\geq l$ active neighbors
- $\theta_{\min}(k, l)$ : minimal fraction of active vertices in order to activate completely a $k + 1$ regular random graph
- for $l = k$, corresponds to the decycling number (Feedback Vertex Set)
- for $l = k - 1$, corresponds to the de-3-coring number

Analytic results for (lowerbounds on) $\theta_{\min}(k, l)$ (RS and 1RSB)
[Guggiola, S. 15]

# Minimal contagious sets

Special cases :

- decycling of 3- and 4-regular graphs :

  $\theta_{\min}(2,2) = \frac{1}{4}$ , $\theta_{\min}(3,3) = \frac{1}{3}$

  First (second) one proven (conjectured) [Bau, Wormald, Zhou 02]

- de-3-coring of 5- and 6-regular graphs :

  $\theta_{\min}(4,3) = \frac{1}{6}$ , $\theta_{\min}(5,4) = \frac{1}{4}$

Conjecture : these 4 cases are the only ones that saturate the lowerbound :

for all $k, l$, $\theta_{\min}(k,l) \geq \frac{2l-k-1}{2l}$ [Dreyer, Roberts 09]

Conjecture for the decycling number at large degree :
$\theta_{\min}(k,k) = 1 - \frac{2\ln k}{k} - \frac{2}{k} + O\left(\frac{1}{k \ln k}\right)$
ok with rigorous bound [Haxell, Pikhurko, Thomason 08]

# Minimal contagious sets

Definition as a problem about processes on infinite trees :

- $\mathcal{C}_\theta$ = probability measures $\mu$ on $\{0,1\}^{\mathbb{T}_{k+1}}$ that are translationally invariant (ergodic), with $\mu[\sigma_0 = 1] = \theta$
- $\max\{\theta : \exists\, \mu \in \mathcal{C}_\theta$ with $\mu[0 \leftrightarrow \infty] = 0\}$ ?

# Conclusions and perspectives

- Freezing transition rather close to the satisfiability
- Done on the regular hypergraph bicoloring,
  should generalize to other CSPs
- RS computation, RSB effects should not spoil large $k$ asymptotics
- Biasing the measure, with interactions between variables at finite distance, can turn atypical properties into typical ones, in a large density range

  Could it help to break the algorithmic barrier ?