Extremal cuts of sparse random graph

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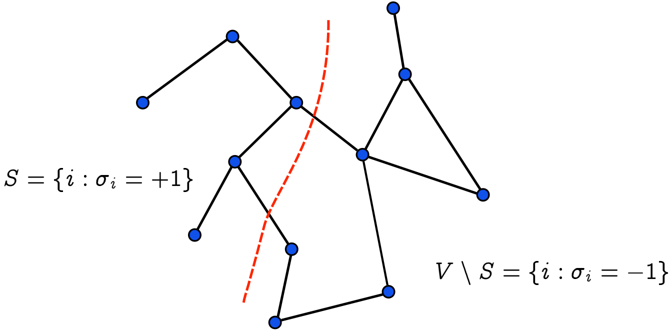
Cuts in graph $G = (V, E)$



$$\text{cut}_G(S) \equiv \Big|\{(i,j) \in E : \; i \in S, \, j \in V \setminus S\}\Big|$$

$$2\text{cut}_G(S) - |E| = \mathcal{H}_G(\sigma) \quad \text{for AF-Ising on } G$$

Extremal cuts

Minimum bisection

$$\mathsf{mcut}(G) = \min\left\{\mathrm{cut}_G(S)\ :\ S \subseteq V\,,\ |S| = |V|/2\right\}$$

Maximum bisection

$$\mathsf{MCUT}(G) = \max\left\{\mathrm{cut}_G(S)\ :\ S \subseteq V\,,\ |S| = |V|/2\right\}$$

Maximum Cut

$$\mathsf{MaxCut}(G) = \max\left\{\mathrm{cut}_G(S)\ :\ S \subseteq V\right\}$$

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Interesting for many reasons

- Clustering similarity matrices
- Graph layout (optimal $V \mapsto \mathbb{N}$)
- Isoperimetric number
- Community structure in social networks
- ...

Complexity: hard to solve (worst case)

$$\left.\begin{array}{r}\text{mcut}\\ \text{MCUT}\\ \text{MaxCut}\end{array}\right\}\quad \text{are NP-hard}$$

(hard to approximate to within $1 + o(1)$ factor:
SDP [Goemans,Williamson '95]; hardness [Trevisan,Sorkin,Sudan,Williamson '00]
mcut [Feige,Krauthgamer '00])
(... very rich theory)

Typical complexity (average graph)?

Random graph models

Erdős-Renyi Random graph $G = (V, E) \sim \mathcal{G}(n, p)$

- $|V| = n$ vertices
- Edges independently present with probability p
  (or uniform among graphs of $m = [\binom{n}{2} p]$ edges)
- Average degree $\gamma = np$

Random regular graph $G = (V, E) \sim \mathcal{G}^{\text{reg}}(n, \gamma)$

- $|V| = n$ vertices
- Uniformly random among graphs with $\deg(i) = \gamma \quad \forall i \in V$.

$p = \gamma/n$, $\gamma = O(1)$, sparse graphs

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Extremal cuts of random graphs: long history

- Bollobas '88 ($\mathcal{G}^{\text{reg}}$, mcut, concentration bound)
- Alon '97 ($\mathcal{G}^{\text{reg}}$, mcut, algorithmic bound)
- Coppersmith, Gamarnik, Hajiaghayi, Sorkin '04 ($\mathcal{G}$, MaxCut transition $\gamma = 0.5$)
- Díaz,Serna,Wormald, '07 ($\mathcal{G}^{\text{reg}}$, MCUT, algorithmic bound)
- Daudé,Martínez,Rasendrahasina,Ravelomanana '12 (MaxCut, scaling-window)
- Gamarnik,Li '14 ($\mathcal{G}$, MaxCut, improved concentration)
- ...

Typical result (sparse case):

If $G \sim \mathcal{G}(n, \gamma/n)$ then, with high probability

$$\frac{n\gamma}{4} + C_1 n\sqrt{\gamma} \leq \mathsf{MCUT}(G) \leq \frac{n\gamma}{4} + C_2 n\sqrt{\gamma}$$

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A classical argument (Bollobas '88)

$|V| = n$, $|E| = n\gamma/2$. Fix $S \subseteq V$, $|S| = n/2$

- Each edge is cut with probability 1/2

$$\mathbb{E}[\mathrm{cut}(S)] = \frac{|E|}{2} = \frac{n\gamma}{4}.$$

(also factor 0.5 approximation of MaxCut)

- Azuma-Höeffding argument

$$\mathbb{P}\Big\{|\mathrm{cut}(S) - \mathbb{E}\mathrm{cut}(S)| \geq \Delta\Big\} \leq 2\exp\Big(-\frac{\Delta^2}{4n\gamma}\Big)$$

(simpler: Binomial($\frac{n^2}{4}, \frac{\gamma}{n}$) in $\mathcal{G}(n, \gamma/n)$)

- Union bound

$$\mathbb{P}\Big\{\max_{S,|S|=n/2}\Big|\mathrm{cut}(S) - \frac{n\gamma}{4}\Big| \geq \delta n\sqrt{\gamma}\Big\} \leq 2^{n+1}\, e^{-n\delta^2/4}.$$

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So, w.h.p.

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Extremal cuts and Ising spins

Ising Hamiltonian: $$\mathcal{H}_G(\sigma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E} \sigma_i \sigma_j \ : \{-1,+1\}^n \mapsto \mathbb{R}\,.$$

$$\mathsf{MaxCut}(G) = \max_{\sigma} \sum_{(i,j)\in E} \left(\frac{1-\sigma_i\sigma_j}{2}\right)$$
$$= \frac{1}{2}|E| - \frac{1}{2}\min_{\sigma} \sum_{(i,j)\in E} \sigma_i\sigma_j = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}}\min_{\sigma}\{\mathcal{H}_G(\sigma)\}$$

$(S = \{i : \sigma_i = +1\}, \quad |E| = \frac{n\gamma}{2})$.

$$|S| = \frac{n}{2} \iff \Omega_n = \left\{\sigma \in \{-1,+1\}^n : \sum_{i=1}^{n} \sigma_i = 0\right\}$$

$$\mathsf{MCUT}(G) = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}}\min_{\sigma\in\Omega_n}\{\mathcal{H}_G(\sigma)\}\,,$$
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$$\begin{aligned}
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&= \frac{1}{2}|E| - \frac{1}{2} \min_{\sigma} \sum_{(i,j)\in E} \sigma_i \sigma_j = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}} \min_{\sigma} \{\mathcal{H}_G(\sigma)\}
\end{aligned}$$

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Statistical physics: Ising measures

Ising measures (on balanced configurations):

$$\mathcal{H}_G(\sigma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E} \sigma_i \sigma_j \qquad (\sigma \in \Omega_n)\,,$$

$$\mu_{n,\beta}(\sigma) \equiv \frac{1}{Z_n(\beta)}\, e^{\beta \mathcal{H}_G(\sigma)}\,, \qquad Z_n(\beta) \equiv \sum_{\tilde\sigma \in \Omega_n} e^{\beta \mathcal{H}_G(\tilde\sigma)}\,.$$

(ferromagnetic iff $\beta \geq 0$).

Ground state energy:

$$\lim_{\beta\to+\infty} \frac{1}{\beta} \log Z_n(\beta) = \max_{\sigma\in\Omega_n} \{ \mathcal{H}_G(\sigma) \} \qquad \Longrightarrow \qquad \mathsf{mcut}(G)\,,$$

$$\lim_{\beta\to-\infty} \frac{1}{\beta} \log Z_n(\beta) = \min_{\sigma\in\Omega_n} \{ \mathcal{H}_G(\sigma) \} \qquad \Longrightarrow \qquad \mathsf{MCUT}(G)\,.$$

Insights from statistical physics

Conjectures

- Fu,Anderson, '86
- Mézard,Parisi, '01
- [Zdéborova,Boettcher '10] For $G_n \sim \mathcal{G}^{\text{reg}}(n,\gamma)$, w.h.p.

$$\mathsf{MCUT}(G_n) = \mathsf{MaxCut}(G_n) + o(n) = |E_n| - \mathsf{mcut}(G_n) + o(n)\,.$$

Theorem (Bayati,Gamarnik,Tetali, '09)

For $G_n \sim \mathcal{G}(n,\gamma/n)$, or $G_n \in \mathcal{G}^{\text{reg}}(n,\gamma)$ w.h.p.

$$\liminf_{n\to\infty} \frac{1}{n}\mathsf{MaxCut}(G_n) = \limsup_{n\to\infty} \frac{1}{n}\mathsf{MaxCut}(G_n)$$

Sub-additivity: no limit value.

(interpolation as in [Franz,Leone,Toninelli, '03]).

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A new result

Theorem (Dembo,Montanari,Sen '15)

For $G_n \sim \mathcal{G}^{\text{reg}}(n,\gamma)$, $G_n \sim \mathcal{G}(n,\gamma/n)$ *w.h.p.*

$$\frac{1}{n}\mathsf{mcut}(G_n) = \frac{\gamma}{4} - \mathsf{P}_*\sqrt{\frac{\gamma}{4}} - o_\gamma(\sqrt{\gamma}) + o_n(1)\,,$$
$$\frac{1}{n}\mathsf{MCUT}(G_n) = \frac{\gamma}{4} + \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o_\gamma(\sqrt{\gamma}) + o_n(1)\,,$$
$$\frac{1}{n}\mathsf{MaxCut}(G_n) = \frac{\gamma}{4} + \mathsf{P}_*\sqrt{\frac{\gamma}{4}} + o_\gamma(\sqrt{\gamma}) + o_n(1)\,,$$

where $\mathsf{P}_* = \ldots$ *(wait a minute).*

Remarks

- The $\sqrt{\gamma}$ term has a well defined coefficient.
- The coefficient has an 'explicit' formula.
- Same for 3 problems and 2 graph models.
- Partial confirmation of [Zdéborova,Boettcher '10] conjecture.
- $\mathsf{P}_* \approx 0.7632$ (numerically; so $\mathsf{P}_*/\sqrt{2} \approx 0.5397$).

What is P_*?

GOE random matrix:

$$J \in \mathbb{R}^{n\times n}\,, \quad J = J^{\mathsf{T}}\,, \quad (J_{ij})_{i<j} \sim \mathrm{N}(0, n^{-1}), \quad J_{ii} \sim \mathrm{N}(0, 2n^{-1})\,.$$

Sherrington-Kirkpatrick spin-glass model

$$\mathcal{H}_{\rm SK}(\sigma) \equiv \frac{1}{2}\,\sigma^{\mathsf{T}} J\sigma\,.$$

Finally

$$\mathsf{P}_* \equiv \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big[\max_{\sigma\in\{-1,+1\}^n}\{\mathcal{H}_{\rm SK}(\sigma)\}\Big]\,.$$

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- The limit exists (consequence of [Guerra,Toninelli '02])
- Given by *'Parisi's formula'* ([Talagrand '06])
- Clarifies why 'standard' combinatorial methods were unsuccessful.
- Set of near-extremal cuts predicted to have ∞-RSB structure.

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Proof strategy

1. Proof for mcut, MCUT with $G \sim \mathcal{G}(n, \gamma/n)$:

   Concentration, interpolation,'smooth max'

2. Extend to $G' \sim \mathcal{G}^{\text{reg}}(n, \gamma)$: Couple G' and $G \sim \mathcal{G}(n, \gamma'/n)$
   (Tricky: need $\gamma - \gamma' \gg \sqrt{\gamma}$ so $|E_G \Delta E_{G'}| \gg n\sqrt{\gamma}$)

3. Prove that $\mathrm{MaxCut}(G) - \mathrm{MCUT}(G) = o(n\sqrt{\gamma})$
   (Tricky part: $\frac{n}{2} - |S| \leq n\gamma^{-\delta}$, $\delta < \frac{1}{4}$)

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mcut for $\mathcal{G}(n, \gamma/n)$:

Recall (our claim)

$$\mathsf{mcut}(G) = \frac{n\gamma}{4} - \sqrt{\frac{\gamma}{4}} \max_{\sigma \in \Omega_n} \{\mathcal{H}_G(\sigma)\} \,,$$

By concentration, suffices to show

$$\frac{1}{n}\mathbb{E}\Big[\max_{\sigma \in \Omega_n} \{\mathcal{H}_G(\sigma)\}\Big] = \mathsf{P}_* + o_\gamma(1) + o_n(1) \,.$$

We know

$$\frac{1}{n}\mathbb{E}\Big[\max_{\sigma \in \{-1,+1\}^n} \{\mathcal{H}_{\rm SK}(\sigma)\}\Big] = \mathsf{P}_* + o_n(1) \,.$$

Idea: Interpolation ($\sim$ smart path, $\sim$ Lindeberg method, ...)

Interpolation: Two steps

Step 1: For $t \in [0,1]$, $G(t) = (V, E(t)) \sim \mathcal{G}(n, \gamma(1-t)/n)$

$$\mathcal{H}_t(\sigma;\gamma) = \frac{1}{\sqrt{\gamma}} \sum_{(i,j)\in E(t)} \sigma_i\sigma_j + \frac{\sqrt{t}}{2}\sum_{i,j=1}^{n} J_{ij}\sigma_i\sigma_j$$

$\mathcal{H}_0 = \mathcal{H}_G \qquad \mathcal{H}_1 = \mathcal{H}_{\text{SK}} \qquad\qquad (\sigma \in \Omega_n).$

$G(t)$ multi-graph of i.i.d. Poisson$(\gamma(1-t)/n)$ multi-edges.

Step 2: From $\max_{\sigma\in\Omega_n}\{\mathcal{H}_{\text{SK}}(\sigma)\}$ to $\max_{\sigma\in\{-1,+1\}^n}\{\mathcal{H}_{\text{SK}}(\sigma)\}$.

Step 1: 'smooth max'

Free energy density (balanced configurations):

$$\phi_n(\beta; t, \gamma) \equiv \frac{1}{n}\mathbb{E}\Big[\log\Big\{\sum_{\sigma\in\Omega_n} e^{\beta\mathcal{H}_t(\sigma;\gamma)}\Big\}\Big]$$

$$\Big|\frac{1}{\beta}\phi_n(\beta; t, \gamma) - \frac{1}{n}\mathbb{E}\big[\max_{\sigma\in\Omega_n}\{\mathcal{H}_t(\sigma;\gamma)\}\big]\Big| \le \frac{\log 2}{\beta}\,.$$

$\Longrightarrow$ just find $\beta(\gamma) \to \infty$ with

$$\sup_{n,t} \frac{1}{\beta}\Big|\frac{\partial\phi_n}{\partial t}(\beta; t, \gamma)\Big| \to 0\,.$$

Lemma

$$\Big|\frac{\partial\phi_n}{\partial t}(\beta; t, \gamma)\Big| \le C\left[\frac{\beta^3}{\gamma^{1/2}} + \frac{\beta^4}{\gamma}\right].$$

Proof: Poisson & Gaussian calculus.

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$$\phi_n(\beta;t,\gamma) \equiv \frac{1}{n}\mathbb{E}\Big[\log\Big\{\sum_{\sigma\in\Omega_n} e^{\beta\mathcal{H}_t(\sigma;\gamma)}\Big\}\Big]$$

$$\Big|\frac{1}{\beta}\phi_n(\beta;t,\gamma) - \frac{1}{n}\mathbb{E}\big[\max_{\sigma\in\Omega_n}\{\mathcal{H}_t(\sigma;\gamma)\}\big]\Big| \le \frac{\log 2}{\beta}\,.$$

$\Longrightarrow$ just find $\beta(\gamma)\to\infty$ with

$$\sup_{n,t}\frac{1}{\beta}\Big|\frac{\partial\phi_n}{\partial t}(\beta;t,\gamma)\Big| \to 0\,.$$

Lemma

$$\Big|\frac{\partial\phi_n}{\partial t}(\beta;t,\gamma)\Big| \le C\left[\frac{\beta^3}{\gamma^{1/2}} + \frac{\beta^4}{\gamma}\right].$$

Proof: Poisson & Gaussian calculus.

Extensions [Subhabrata Sen, in progress]

- Max q-cut in sparse random graphs

  $\Longrightarrow$ Potts spin-glass.

- Max-SAT for $m = (1 + \delta)n$ random linear Eqn. over $\{0, 1\}$
  (in UNSAT phase, p variables per Eqn.)

  $\Longrightarrow$ p-spin model.

Conclusion

- Extremal cuts in random graphs

  $\Longrightarrow$ balanced Ising spins $\sigma_i \in \{-1, +1\}$

- Same for many other models, when $\gamma \to \infty$.

- Quite helpful, but much remains to understand!

  [Zdéborova,Boettcher '10] For $G_n \sim \mathcal{G}^{\text{reg}}(n, \gamma)$, fixed $\gamma \geq 3$, w.h.p.

$$\mathsf{MCUT}(G_n) = \mathsf{MaxCut}(G_n) + o(n) = |E_n| - \mathsf{mcut}(G_n) + o(n)\,.$$

Thanks!