The Complexity of Approximating Small Degree Boolean #CSP

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Counting CSP

- F (Γ) is a family of functions (constraints)
- $x_1, x_2, \dots, x_n$ are variables taking values from $[q]$ (mainly Boolean domain {0,1} in this talk).
- A function $f \in F$ is applied on $x_{i_1}, x_{i_2}, \dots, x_{i_r}$, where $i_1, i_2, \dots, i_r \in [n]$
- Partition function:

$$\sum_{x_1, x_2, \dots, x_n \in [q]} \prod_{(f, i_1, i_2, \dots, i_r) \in I} f(x_{i_1}, x_{i_2}, \dots, x_{i_r})$$

- #CSP(F) (or #CSP(Γ)) denotes the computational problem

Outline

- Exact counting CSP
- Approximate counting CSP
- Bounded degree CSP

Dichotomies for Boolean #CSP

- [Creignou, Hermann 96] All the functions in F take values in {0,1} (unweighted). Only tractable cases are affine relations.
- [Dyer, Goldberg, Jerrum 07] non-negative values.
- [Bulatov, Dyer, Goldberg, Jalsenius, Richerby 09] real values.
- [Cai, L., Xia 09] complex values.

#CSP over large domain

- [Bulatov 08] Unweighted case
- [Dyer, Richerby 10, 11] Alternative proof and decidable dichotomy
- [Cai, Chen, L. 11] Non-negative weighted functions
- [Cai, Chen 12] Complex weighted

#CSP with degrees at most three

[Cai, L., Xia 09]

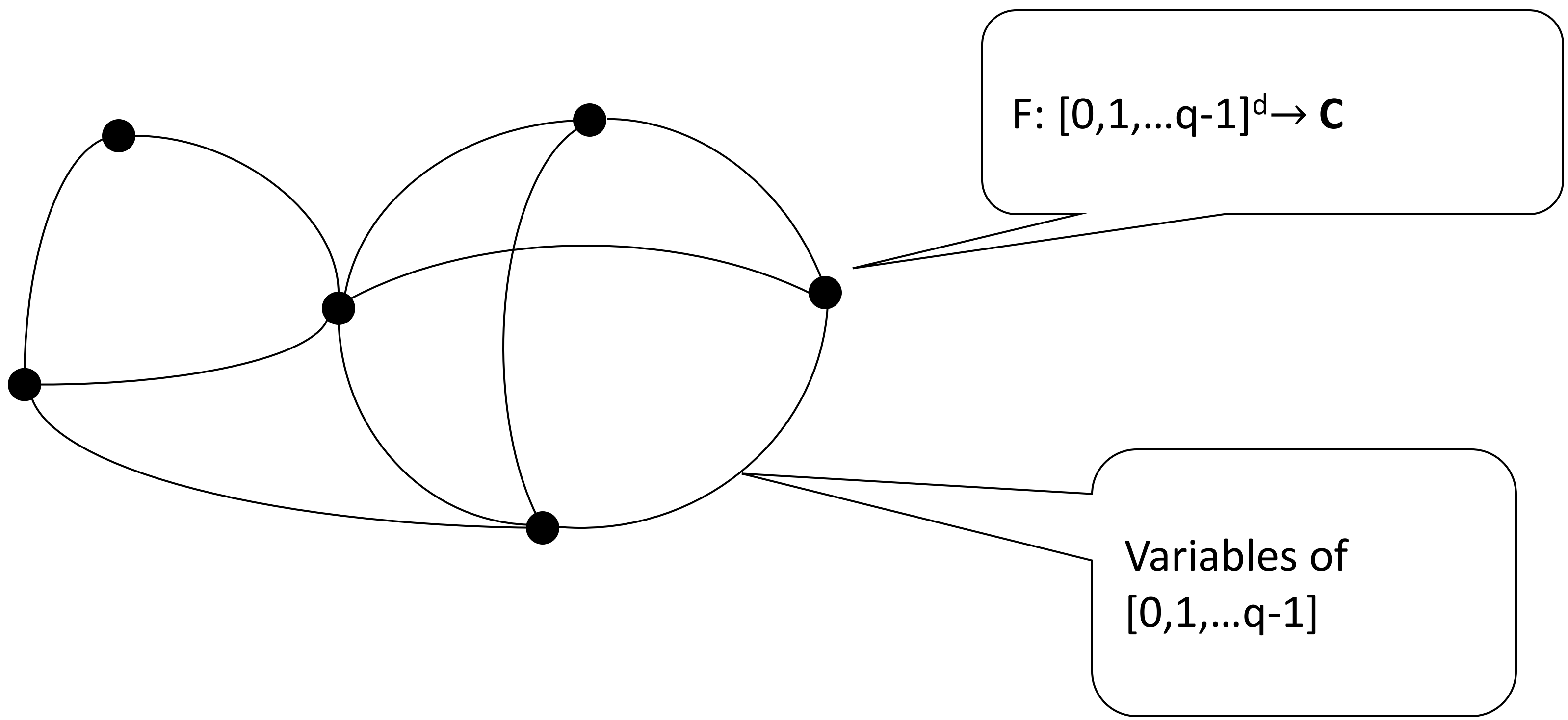
- For any complex value function set F over Boolean domain, $\#CSP_3(F)$ is as hard as $\#CSP(F)$.
- So we have the same dichotomy for $\#CSP_3(F)$
- This is not generally true for #CSP over large domain.

Holant

- Read twice CSP
- Also known as edge coloring model, tensor network, factor graph…
- More expressive than CSP framework

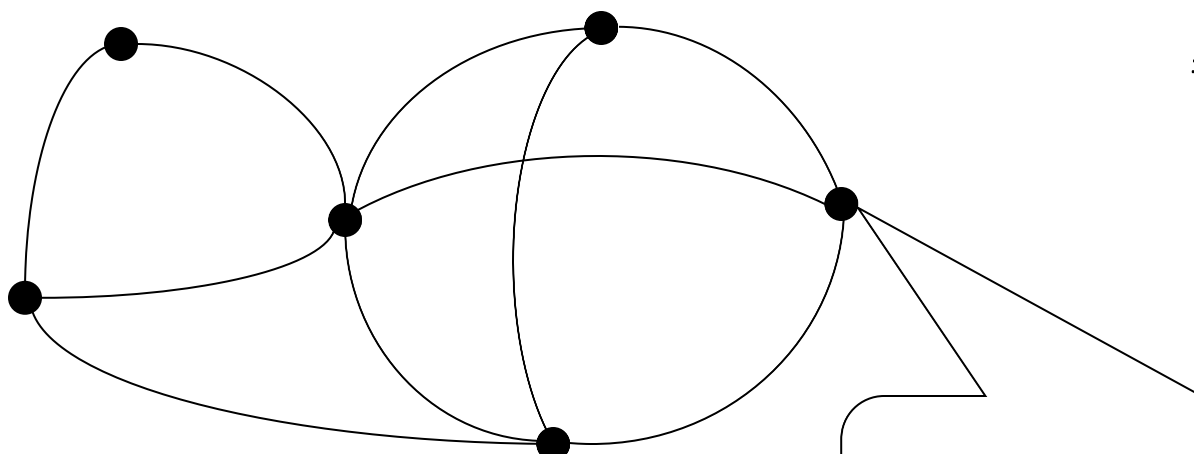
Holant

$$Holant_{\Omega} = \sum_{x_1, x_2, \ldots, x_m \in [q]} \prod_{v \in V} F_v(x\,|_{\mathrm{v}})$$



Examples

$$Holant_{\Omega} = \sum_{x_1, x_2, \ldots, x_m \in \{0,1\}} \prod_{v \in V} F_v(x\,|_{\mathrm{v}})$$



#Perfect Matchings

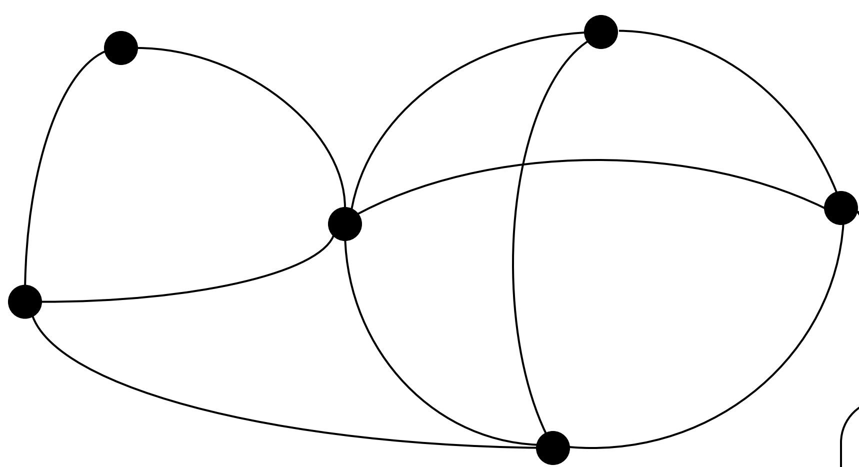
$$F_v(\sigma) = \begin{cases} 1 & wt(\sigma) = 1, \\ 0 & otherwise \end{cases}$$

$wt(\sigma) = number\ of\ \ 1s\ in\ \sigma$

Examples

$$Holant_\Omega = \sum_{x_1, x_2, \ldots, x_m \in \{0,1\}} \prod_{v \in V} F_v(x\,|_{\mathrm{v}})$$

#Matchings

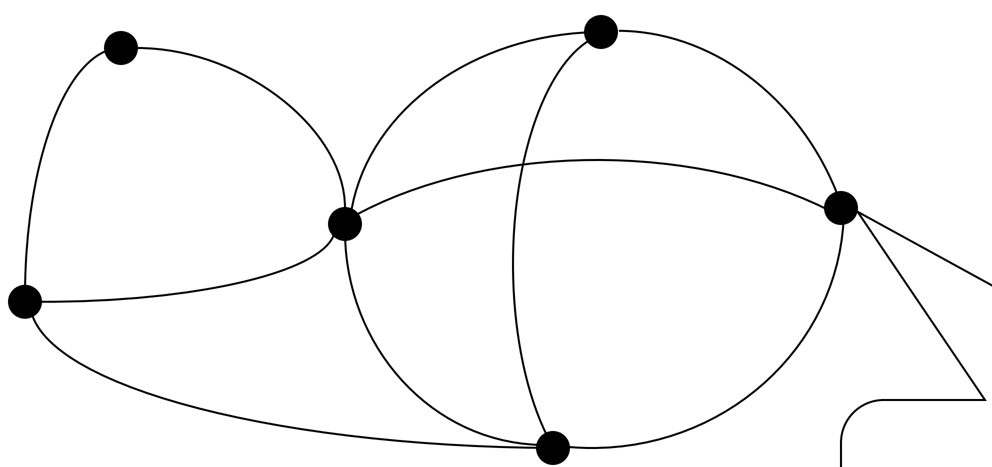


$wt(\sigma) = number\ of\ \ 1s\ in\ \sigma$

$$F_v(\sigma) = \begin{cases} 1 & wt(\sigma) \leq 1, \\ 0 & otherwise \end{cases}$$

Examples

$$Holant_{\Omega} = \sum_{x_1, x_2, \dots, x_m \in \{0,1\}} \prod_{v \in V} F_v(x\,|_{\mathrm{v}})$$



#Edge Covers

$$F_v(\sigma) = \begin{cases} 1 & wt(\sigma) \geq 1, \\ 0 & otherwise \end{cases}$$

$wt(\sigma) = number\ of\ \ 1s\ in\ \sigma$

New interesting tractable problems: an example

- NTW_3 is the Not-Two function of arity 3:

$$NTW_3(\sigma) = \begin{cases} 0 & wt(\sigma) = 2, \\ 1 & otherwise \end{cases}$$

- $\#\mathrm{CSP}(\mathrm{NTW}_3)$ is #P-complete

- Counting $\mathrm{Holant}(\mathrm{NTW}_3)$ is in P.
  (why? An exercise.)

Dichotomies for Holant

- Symmetric Complex Holant* [Cai, L., Xia 09]
- Symmetric Real Holant^c [Cai, L., Xia 09]
- Symmetric Complex Holant^c [Cai, Huang, L. 10]
- Complex Holant* [Cai, L., Xia 11]
- Symmetric Real Holant [Huang, L. 12]
- Symmetric Complex Holant [Cai, Guo, Williams 13]

Holant*: all the unary functions are available

Holant^c : two constant unary functions are available

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Approximate Counting

- Let $\epsilon > 0$ be an approximation parameter and Z be the correct counting number of the instance, the algorithm returns a number Z' such that $(1-\epsilon)Z \leq Z' \leq (1+\epsilon)Z$, in time ploy($n$,1/$\epsilon$).
- Fully polynomial-time approximation scheme (FPTAS).
- Fully polynomial-time randomized approximation scheme (FPRAS) is its randomized version.

Complexity of Approximate Counting

- As hard as NP problem rather than #P
- Approximation Preserving (AP) Reduction
- NP-hardness (#SAT-equivalent)
- #BIS (independent sets for bipartite graphs)
  - Conjectured to be of intermediate complexity.
  - Plays a similar role as the Unique Game for optimization problems.
  - A large number of other problems are proved to have the same complexity as #BIS (#BIS-equivalent) or at least as hard as #BIS (#BIS-hard)

Trichotomy

[Dyer, Goldberg, Jerrum 2010]

For relations over Boolean domain, #CSP(Γ) is divided into three classes:

- #CSP(Γ) in FP (if every relation in Γ is affine)
- #CSP(Γ) $=_{AP}$ #BIS
- #CSP(Γ) $=_{AP}$ #SAT

No non-trivial FPTAS/FPRAS (No life below #BIS)

A non-example from weighted version

- For a single binary function $\begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$
- $\beta\gamma > 1$: FPRAS [Jerrum, Sinclair 93] [Goldberg, Jerrum, Paterson 03]
- $\beta\gamma < 1$:
  - FPTAS in uniqueness range [Li, L., Yin 12,13]
  - NP-hard in non-uniqueness range [Sly, Sun 12]

- Asymmetric binary function is open

Weighted Version

[Bulatov, Dyer, Goldberg, Jerrum, McQuillan 12]

[Chen, Dyer, Goldberg, Jerrum, L., McQuillan, Richerby 13]

Assuming all unary functions, $\#CSP^*(F)$ is divided into three classes:

- FP even for exact counting
- #BIS-hard (LSM family)
- #SAT-equivalent

Outline

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Trichotomy ($d \geq 6$)

[Dyer, Goldberg, Jalsenius, Richerby 2011]

Let Γ be a Boolean constraint language and let $d \geq 6$. Then $\#CSP_d^c(\Gamma)$ is divided into three classes:

- $\#CSP_d^c(\Gamma)$ in FP (if every relation in Γ is affine)
- $\#CSP_d^c(\Gamma) =_{AP}$ #BIS
- $\#CSP_d^c(\Gamma) =_{AP}$ #SAT

$\#CSP_d^c(\Gamma)$ is $\#CSP(\Gamma \cap \{\mathbf{0}, \mathbf{1}\})$ for instances with a maximum degree of d.

A non-example without Pinning

- For the relation $(X \vee Y \vee Z)$, there is an FPTAS for $\mathrm{CSP}_6(X \vee Y \vee Z)$ [Bezakova, Galanis, Goldberg, Guo, Stefankovic 16]

- For the relation mon-k-CNF: $X_1 \vee X_2 \vee \cdots \vee X_k$, there is an FPTAS/FPRAS for $\mathrm{CSP}_\mathrm{d}(\mathrm{mon{-}k{-}CNF})$ for large degree d (= $\frac{c2^{\frac{k}{2}}}{k^2}$) [Sly et. al. 2016]

Single symmetric relation

[Galanis, Goldberg 16]

- For any non-affine symmetric relation f over Boolean domain, there exists a constant Δ such that $CSP_d(f)$ is NP-hard for any $d \geq \Delta$

- To identify the threshold degree for a given relation?

- Asymmetric case? Weighted case?

Partial Classification ($d \geq 3$)

[Dyer, Goldberg, Jalsenius, Richerby 2011]

Let Γ be a Boolean constraint language and $d \geq 3$. Then $\#CSP_d^c(\Gamma)$ is divided into four classes:

- $\#CSP_d^c(\Gamma)$ in FP (if every relation in Γ is affine)
- $\#CSP_d^c(\Gamma) =_{AP}$ #BIS
- $\#CSP_d^c(\Gamma) =_{AP}$ #SAT
- Γ is a set of monotone relations

Monotone Relations

- Any monotone relation can be written as monotone CNFs (fox example $(X \vee Y) \wedge (X \vee Z)$)
- After suitable pinning, we can realize the relation $X \vee Y$
- $CSP_6(X \vee Y)$ is NP-hard [Sly 10]. This leads to the trichotomy for $d \geq 6$

d=5

- $CSP_5(X \vee Y)$ is FPTASable [Weitz 06]
- $CSP_5(X_1 \vee X_2 \vee \cdots \vee X_k)$ is FPTASable for any k [Liu, L. 15]
- Dis-Mon-CNF: Mon-CNFs where the variables in different clauses are disjoint

  for example: $(X \vee Y) \wedge (Z \vee W)$
- CSP_5[Dis-Mon-CNF] is FPTASable

d=5 (a conjecture)

Let Γ be a Boolean constraint language. Then $\#CSP_5^c(\Gamma)$ is divided into four classes:

- $\#CSP_5^c(\Gamma)$ in FP (every relation in Γ is affine)
- FPTAS for $\#CSP_5^c(\Gamma)$ ($\Gamma \subset$Dis-Mon-CNF)
- $\#CSP_d^c(\Gamma) =_{AP}$ #BIS
- $\#CSP_d^c(\Gamma) =_{AP}$ #SAT

d=5 (an attempted proof)

- If a monotone relation is not from Dis-Mon-CNF, after suitable pinning, we can realize one of the following two relations:
  - $S_2 = (X \vee Y) \wedge (X \vee Z)$
  - $K_3 = (X \vee Y) \wedge (X \vee Z) \wedge (Y \vee Z)$

- $\#CSP_5^c(S_2)$ is NP-hard [Liu, L. 2015]

- The complexity of $\#CSP_5^c(K_3)$ is open.
- A proof of its NP-hardness will lead to the conjectured classification for $\#CSP_5^c$

d=4

- Both $\#CSP_4^c(K_3)$ and $\#CSP_4^c(S_2)$ are open
- No new FPTASable cases are known
- The same conjectures
  - $\#CSP_5^c(S_2)$ is NP-hard
  - $\#CSP_4^c(S_2)$ is NP-hard
  - The same classification as d=5

d=3

- New tractability: FPTAS for $\#CSP_3^c(K_4)$ (and $\#CSP_3^c(K_3)$) [Liu, L. 15]

- $\#CSP_3^c(S_2)$ is open

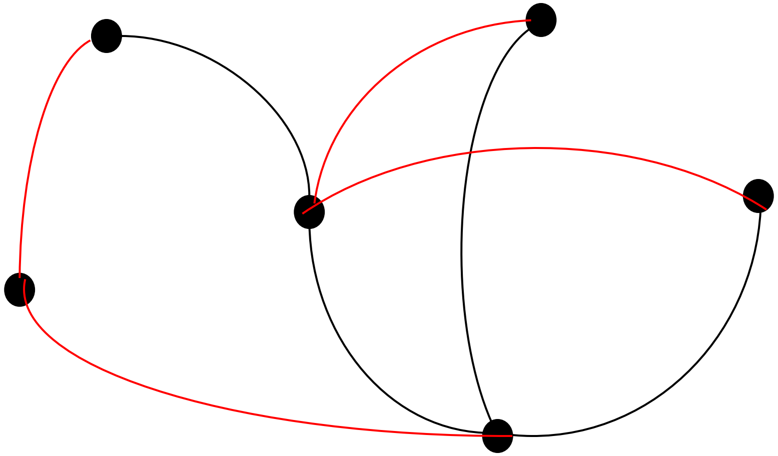
- The picture is much more complicate and also much more interesting

Holant Problems (d=2)

- Much more FPTASable (FPRASable) problems
  - Matching
  - Edge cover
  - B-matching and b-edge-cover
  - Not-all-equal
  - Fibonacci gate problems
- Any hardness result? Perfect matching?

Counting Edge Covers

- A set of edges such that every vertex has at least one adjacent edge in it



Counting Edge Covers

- A set of edges such that every vertex has at least one adjacent edge in it
- FPRAS for 3-regular graphs based on Markov Chain Monte Carlo[Bezakova, Rummler 2009].
- FPTAS for general graphs based on correlation decay approach. [Lin, Liu, L. 2014]

b-matching and b-edge-cover

- b-matching: $F_v(\sigma) = \begin{cases} 1 & wt(\sigma) \leq b, \\ 0 & otherwise \end{cases}$
- b-edge-cover: $F_v(\sigma) = \begin{cases} 1 & wt(\sigma) \geq b, \\ 0 & otherwise \end{cases}$

- FPRAS for counting b-matching with $b \leq 7$ and b-edge-cover with $b \leq 2$. [Huang, L., Zhang 16] (next talk)

Taking Home Messages

- Many problems for approximating Boolean #CSP remain open especially when there are degree bounds and/or weights.
- Many recent progresses in this field make the complete classification within reach.
- Some concrete (open) problems are more important as they play crucial roles in the classification.

A list of Problems

- Asymmetric 2-spin systems
- $\#CSP_d^c(K_3), \#CSP_d^c(S_2)$ with d=3,4,5
- #b-matchings with d>7

Thank You！