Simons Workshop on Approximate Counting, Markov Chains and Phase Transitions: Open Problem Session

Scribes: Antonio Blanca, Sarah Cannon, Yumeng Zhang

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1 Yuval Peres: Simple Random Walk on a Transitive Graph

Definition 1.1. A graph is *transitive* if for any two vertices, there is a graph homomorphism mapping one to the other.

Let $G = (V, E)$ be a finite connected transitive graph, and consider a lazy simple random walk on G.

Open Question 1.2. *Does the following inequality holds?*

$$\tau_{\text{mix}}(\varepsilon) \leq C(\varepsilon) \cdot \deg(G) \cdot \text{diam}^2(G)$$

Theorem 1.3 (Babai)**.** *The inequality holds for the relaxation time (i.e., $1/GAP$).*

2 Yuval Peres: Heat-bath dynamics for the Ising model

Consider the ferromagnetic Ising model on a finite graph G with no external field:

$$\mu(\sigma) = \frac{1}{Z} e^{\sum_{u\sim v} J_{uv}\sigma(u)\sigma(v)}$$

Open Question 2.1. *Which starting configurations σ_0, τ_0 maximize $||P^t(\sigma_0, \cdot) - P^t(\tau_0, \cdot)||_{\text{TV}}$?*

Conjecture 2.2. *$\sigma_0 =$ "all plus" and $\tau_0 =$ "all minus" (but this is known not to hold in some other monotone systems, e.g. the noisy voter model on the star graph).*

Open Question 2.3. *Is the relaxation time monotone increasing on any J_{uv}? (Known for cycles, see e.g. Nacu 2003.)*

3 Yuval Peres: Swendsen-Wang dynamics on general graphs

Conjecture 3.1. *The Swendsen-Wang dynamics for the ferromagnetic Ising model with no external field on a graph $G = (V, E)$ mixes in time $O(\text{poly}(n))$ at any temperature, where $n = |V|$.*

Conjecture 3.2. *The Swendsen-Wang dynamics for the ferromagnetic Ising model on a graph $G = (V, E)$ mixes in time $O(n^{1/4})$, where $n = |V|$. (That is, the mean-field case with $p = p_c$ provides the worst case for the mixing time of the chain.)*

4 Heng Guo: Spatial mixing for Ferro 2-spin systems

Consider the two-spin ferromagnetic Ising model on graph $G = (V, E)$ with edge interaction matrix $\begin{bmatrix} \beta & 0 \\ 1 & \gamma \end{bmatrix}$ ($\beta\gamma > 1$, $\beta \leq \gamma$) and non-uniform external fields $\begin{bmatrix} \lambda_v \\ 1 \end{bmatrix}$. The partition function is given by

$$\mathbf{Z} \equiv \sum_{\sigma \in \{+,-\}^{|V|}} \beta^{m_{--}(\sigma)} \gamma^{m_{++}(\sigma)} \prod_{v \in V} \lambda_v^{\mathbf{1}\{\sigma_v = -\}},$$

where $m_{xy} \equiv \#\{e = (u, v) \in E : \sigma_u = x, \sigma_v = y\}, x, y \in \{+, -\}$ is the number of edges between two spins values.

Theorem 4.1 ([Guo,Lu, 2015])**.** *Let $\lambda_c \equiv \left(\frac{\gamma}{\beta}\right)^{\frac{\sqrt{\beta\gamma}}{\sqrt{\beta\gamma}-1}}$. There exists constant $C = C(\beta, \gamma) > 0$ such that for arbitrary tree T (without any assumption on the degrees), if $\lambda_v \leq \lambda_c$ for all $v \in V$, then*

$$|\mathbb{P}(\sigma_{\rho_1} = +) - \mathbb{P}(\sigma_{\rho_2} = + \mid \sigma_L \equiv +)| \leq \exp(-C\ell),$$

where L is the set of leaves at distance l from the root. That is, the difference in the marginal probabilities at the root decays exponentially with l.

Remark 4.2. Their result actually holds for any pair of trees T_1, T_2 that are identical to each other up to level ℓ. But one can not replace the free boundary condition with all $-$ boundary conditions here.

Open Question 4.3. *Does the same correlation decay result holds on general graphs instead of trees? Can this be used to derive a FPTAS for counting Ferro-Ising configurations?*

5 Alexander Holroyd: Restless Markov Chain

Open Question 5.1. *Does there exist a Markov chain (X_n) on a compact metric space such that for some $k \geq 1$ and $\epsilon > 0$, (X_n) satisfies:*

1. *Perfect mixing in k steps: $\forall x, P_x(X_k \in \cdot) = \pi(\cdot)$ (where π is the stationary distribution).*

2. *ϵ-restless: under the metric d of prescribed state space, $\forall x$, $\mathbb{P}_x(d(x, X_1) > \epsilon) = 1$.*

A couple of examples that *do not* work:

1. State space $\Omega = [0,1]^2$. In one step, $(x, y) \to (y, \text{Unif}[0,1])$. This Markov chain satisfies perfect mixing with $k = 2$, but does not satisfy restless for any ϵ. (Consider the diagonal.)

2. State space $\Omega = S^n$, the n-dimensional sphere. In each step, $x \to \text{Unif}\{y : \langle x, y\rangle = 0\}$, i.e. the chain takes the current location as the north pole and jumps to a uniform point on the equator. This chain doesn't mix perfectly.

3. No chain on finite state space and no reversible Markov chain can satisfy both conditions. However, there are examples defined on non-compact metric space, the trivial example being an uncountable set with discrete metric ($d(x, y) \equiv \delta\{x = y\}$). There is also an example with a non-compact countable state space.

6 James Martin: Maximal Independent Set

For any graph G, consider the set of "maximal independent sets" on G, which are configurations that are valid independent sets themselves and one can not change any vertex from 0 to 1 without violating any constraints. Equivalently, a maximal independent set is a configuration where every 1 must be adjacent to all 0's and every 0 must be adjacent to at least one 1. The Gibbs measure can be defined as

$$\mu(\sigma) = \frac{1}{Z}\lambda^{\sum_{v\in V}\mathbf{1}\{\sigma_v=1\}}\mathbf{1}\{\sigma \text{ is an max ind. set}\}.$$

Open Question 6.1. *What can we say about this model? Possible topics include correlation decay, uniqueness, phase diagrams, etc.*

For instance, it appears for small λ there are ten possible ground states. For large λ, we might expect the same behavior as for independent sets without the maximality condition.

7 Alexander Barvinok: Permanent Approximation

Theorem 7.1. *Let $Z = (z_{i,j})$ be an $n \times n$ complex matrix such that $\delta \le Re(z_{i,j}) \le 1$ and $|Im(z_{i,j})| \le \frac{1}{2}\delta^3$. Then $Per(Z) \neq 0$.*

Corollary 7.2. *Let $A = (a_{i,j})$ be a positive matrix, where $\delta \le a_{i,j} \le 1$. The permanent of A can be approximated in deterministic time $n^{O(\log n - \log(\varepsilon))}$.*

Open Question 7.3. *Is there a $\gamma > 0$ (absolute) such that if A is doubly stochastic and $B = (b_{i,j})$ satisfies $|b_{i,j}| \le \frac{\gamma}{n}$ for all i, j, then $Per(A + iB) \neq 0$?*

If this question is answered affirmatively, it would imply that the permanent of a nonnegative matrix can be approximated in deterministic time $n^{\log n - \log\varepsilon}$.

8 James Lee: Markov chains supported on vertex-expanders

Definition 8.1. Graph $G = (V, E)$ is a *vertex-expander* if for all $S \subseteq V$ with $|S| \leq \frac{1}{2}|V|$, then $|N(S)| \geq 1.1 \cdot |S|$, where $N(S)$ is all vertices at distance at most one from S, and 1.1 can be replaced with any constant larger than 1.

Conjecture 8.2. *For any graph G that is a vertex-expander, there exists an assignment of conductances (symmetric weights) to the edges of G such that the l^2 mixing time is $O(\log n)$ to the uniform distribution. (Stated alternately, there exists a reversible Markov chain supported on G that mixes in time $O(\log n)$).*

Theorem 8.3 (L. O. S.)**.** *It is possible to achieve $O((\log n)^{3/2})$.*

9 Raimundo Briceño: Shift of finite type and strong spatial mixing

[Notes by Raimundo Briceño]

Let $\mathcal{A}$ be a finite set and consider an infinite graph $\mathcal{G}$. For definiteness, fix $\mathcal{G} = \mathbb{Z}^2$. Now, let $\Omega \subseteq \mathcal{A}^{\mathbb{Z}^2}$ be a *shift of finite type (SFT)*, i.e. Ω can be described by a finite family of forbidden configurations $\mathcal{F} \subseteq \bigcup_{\Lambda \Subset \mathbb{Z}^2} \mathcal{A}^\Lambda$ such that

$$\Omega = \left\{ \omega \in \mathcal{A}^{\mathbb{Z}^2} : \forall \Lambda \Subset \mathbb{Z}^2, \omega|_\Lambda \notin \mathcal{F} \text{ up to translation} \right\}. \tag{1}$$

Many supports of relevant models (e.g. Potts model, hard-core model, q-colourings, Widom-Rowlinson model, etc.) fit in this framework. For example, the support of the hard-core model (i.e. independent sets of $\mathbb{Z}^2$) corresponds to $\Omega \subseteq \{0,1\}^{\mathbb{Z}^2}$ with $\mathcal{F} = \left\{ 11, \begin{array}{c} 1 \\ 1 \end{array} \right\}$.

Given an SFT Ω, a set $U \subseteq \mathbb{Z}^2$ and $u \in \mathcal{A}^U$, we denote by $[u]$ the cylinder set $\{\omega \in \Omega : \omega|_U = u\}$.

Definition 9.1. We will say that Ω satisfies the *TSSM property* if there exists $g \in \mathbb{N}$ such that for all $U, V, S \Subset \mathbb{Z}^2$ with $\text{dist}(U, V) \geq g$, and all $\alpha \in \mathcal{A}^U$, $\beta \in \mathcal{A}^V$, $\sigma \in \mathcal{A}^S$, we have

$$[\alpha] \cap [\sigma] \neq \emptyset \wedge [\sigma] \cap [\beta] \neq \emptyset \implies [\alpha] \cap [\sigma] \cap [\beta] \neq \emptyset, \tag{2}$$

where $\text{dist}(U, V)$ denotes the graph distance between the sets U and V.

Let Φ be a nearest-neighbour interaction (i.e. Φ gives "weights" to configurations on vertices and edges) and let μ be a Gibbs measure for Φ fully supported on Ω.

- Suppose that μ satisfies *strong spatial mixing* (for all shapes!). Does Ω necessarily satisfies the TSSM property?
- And conversely, given Ω that satisfies the TSSM property. Can we always find Φ and a corresponding μ that satisfies strong spatial mixing?

Note 9.2. As far as I know, all the models with strong spatial mixing measures that are supported on an SFT Ω are such that Ω satisfies the TSSM property. If $\Omega = \mathcal{A}^{\mathbb{Z}^2}$, a Bernoulli process satisfies strong spatial mixing trivially. If $\Omega = \{\text{independent sets}\}$ we know that the hard-core model satisfies strong spatial mixing if the activity parameter λ is small enough. If $\Omega = \{q\text{-colourings}\}$, we know that the uniform Gibbs measure satisfies strong spatial mixing if $q \geq 6$.

When $\Omega = \{4\text{-colourings}\}$, Ω does not satisfy this property. In fact, this can be used to prove there is no Gibbs measure supported on Ω that satisfies strong spatial mixing.

We know that if we have a strong spatial mixing measure μ supported on an SFT Ω in $\mathbb{Z}^2$ with exponential decay $f(n) = Ce^{-\gamma n}$ and $\gamma > 4 \log |\mathcal{A}|$, the support of μ must satisfy the TSSM property (see [1]).

References

[1] R. Briceño, *The topological strong spatial mixing property and new conditions for pressure approximation*, 2014. Version 1. Nov. 9, 2014. arXiv: 1411.2289.

10 Cris Moore: Irreducible Markov Chains with Large Spectral Gap

Open Question 10.1. *Let G be any graph. For each vertex of G, choose transition probabilities to its neighboring vertices such that the resulting (not necessarily reversible) Markov chain has the uniform stationary distribution. What choice of transition probabilities yields the fastest mixing time?*

For a more concrete question, let G be the n-dimensional hypercube. There is known to be a reversible Markov chain supported on the n-dimensional hypercube that reaches uniformity in time $O(n \log n)$.

Open Question 10.2. *Is there a (non-reversible) Markov chain supported on the n-dimensional hypercube that reaches uniformity in time $O(n)$? (Said another way, we're interested in Markov chains whose transitions are between hypercube vertices at Hamming distance 1. If you allow occasional transitions between vertices at Hamming distance 2, it's known how to get $O(n)$ mixing time to uniformity [Dani, Hayes, Moore; unpublished]).*