The complexity of approximately counting in 2-spin systems on k-uniform bounded-degree hypergraphs

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2-state spin system

Spins: $\{0, 1\}$

Symmetric Interaction matrix: $A = \begin{pmatrix} \beta & 1 \\ 1 & \gamma \end{pmatrix}$

interaction between two spins

$\beta, \gamma \geqslant 0$

$\lambda > 0$

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Instance: $G = (V, E)$

Partition function:

$$w_{A;G}(\sigma) = \prod_{w \in V} \lambda^{|\sigma^{-1}(0)|} \prod_{\{u,v\} \in E} a_{\sigma(u),\sigma(v)}$$

$$Z_{A;G} = \sum_{\sigma : V \to \{0,1\}} w_{A;G}(\sigma)$$

"configuration" σ assigns spins to vertices

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$\beta = 0,\ \gamma = 1$
"hard-core" independent sets

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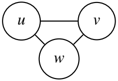
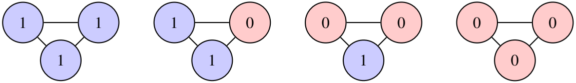
Example: “Hard-core lattice gas” (Independent Sets)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

G





$w_{A;G}(\sigma) = 1$ $\quad$ $w_{A;G}(\sigma) = \lambda$ $\quad$ $w_{A;G}(\sigma) = 0$ $\quad$ $w_{A;G}(\sigma) = 0$

$\times 3$ $\quad$ $\times 3$

$$Z_{A;G} = 1 + 3\lambda$$

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$\beta, \gamma \geqslant 0$

$\beta\gamma < 1$ anti-ferromagnetic

$\lambda > 0$

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underlying phase transition: study of random configs

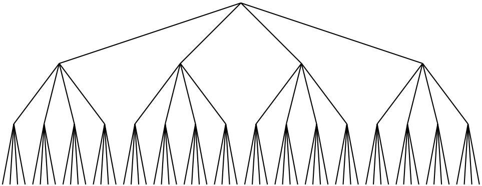
The Gibbs measure $\mu_{A;G}(\sigma) = w_{A;G}(\sigma)/Z_{A;G}$

A Gibbs measure on an infinite graph is a measure such that the induced measure on any finite piece G is given by $\mu_{A;G}(\sigma)$ (conditioned on boundary)

Usually (compactness) there is at least one Gibbs measure, but there can be more than one (or, for some models, infinitely many)

Back

Anti-ferromagnetic 2-spin. $\Delta \geqslant 3$.



Amazing fact: If infinite Δ-regular tree has multiple Gibbs measures (non-uniqueness) $\exists c > 1$ such that it is NP-hard to approximate $Z_{A;G}$ within a factor of c^n on Δ-regular graphs. If $\forall d \leqslant \Delta$ the infinite d-regular tree has a unique Gibbs measure $\exists$ FPTAS for $Z_{A;G}$ on graphs with degree $\leqslant \Delta$.

Sly, Sun 2012 (Sly 2010; Galanis, Štefankovič, Vigoda 2012)

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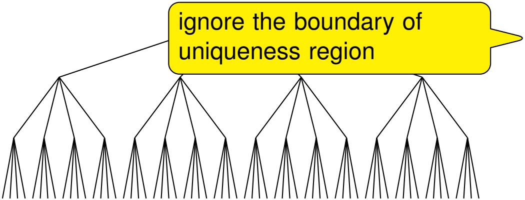

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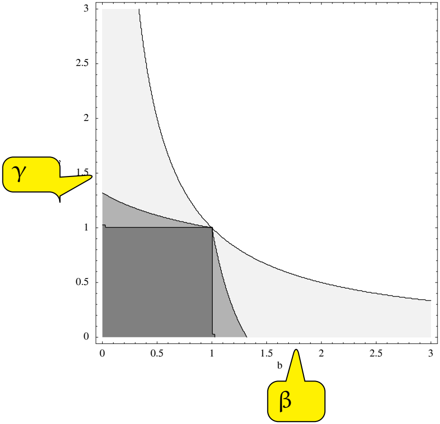
So when are β, γ and λ in the uniqueness regime?



$\lambda = 1$.

• $0 \leqslant \beta < 1$ and $0 < \gamma \leqslant 1$: non-uniqueness on the infinite Δ-regular tree for all sufficiently large Δ.

• $0 \leqslant \beta < 1$ and $\gamma > 1$: uniqueness holds on the infinite Δ-regular tree for all sufficiently large Δ.

the curve for a given Δ sort of as drawn

Easy to tell when parameters are in the uniqueness regime

$$f(x) = \lambda\left(\frac{\beta x+1}{x+\gamma}\right)^{\Delta-1}$$

Uniqueness: $f \circ f$ has unique positive fixed point.

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```
In[1]:= EQS = {y == λ ((β x + 1) / (x + γ)) ^ (Δ - 1),
              x == λ (( β y + 1) / (y + γ)) ^ (Δ - 1), x > 0, y > 0};

        NSolve[ EQS /. {β → 0, γ → 1, λ → 1, Δ → 3} , {x, y}, Reals]
        NSolve[ EQS /. {β → 0, γ → 1, λ → 1, Δ → 4} , {x, y}, Reals]
        NSolve[ EQS /. {β → 0, γ → 1, λ → 1, Δ → 5} , {x, y}, Reals]
        NSolve[ EQS /. {β → 0, γ → 1, λ → 1, Δ → 6} , {x, y}, Reals]

Out[2]= {{x → 0.465571, y → 0.465571}}

Out[3]= {{x → 0.380278, y → 0.380278}}

Out[4]= {{x → 0.324718, y → 0.324718}}

Out[5]= {{x → 0.06377, y → 0.73411}, {x → 0.285199, y → 0.285199}}
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```

ind set on 3-regular tree: Nodes “in” with probability $x/(1+x) \sim 0.32$.

6-regular: Nodes “in” with probability 0.06 and 0.42 alternate layers

What about increased arity?

Recall: 2-state spin system (without external field)

Spins: $\{0, 1\}$

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Instance: $G = (V, E)$

Partition function:

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What about increased arity?

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Symmetric arity-k Boolean function $f : \{0,1\}^k \to \mathbb{R}_{\geqslant 0}$

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k-uniform hypergraph $H = (V, \mathcal{F})$ with max degree $\leqslant \Delta$ (each vertex in $\leqslant \Delta$ hyperedges)

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$$w_{f;H}(\sigma) = \prod_{\{v_1,\ldots,v_k\} \in \mathcal{F}} f(\sigma(v_1), \ldots, \sigma(v_k))$$

$$Z_{A;G} = \sum_{\sigma : V \to \{0,1\}} w_{A;G}(\sigma)$$

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What about increased arity?

Often $f : \{0,1\}^k \to \{0,1\}$

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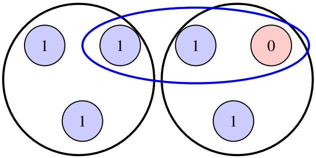
Complications with larger arity!

There may be no computational threshold, or if there is, it might not coincide with the uniqueness threshold

Example: strong independent sets

(Liu, Lin 2015, Yin, Zhao 2015)

$f(s_1,\ldots,s_k)=1$ iff at most one of $s_1,\ldots,s_k$ is 0.



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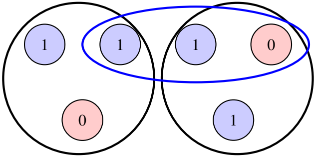
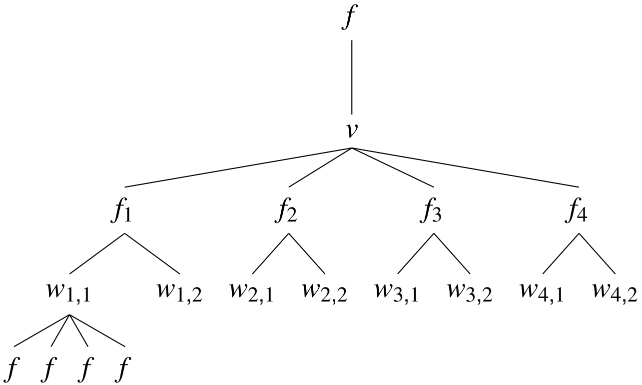


Strong Independent Set. $k = 3$. $\Delta = 5$.



Strong Independent Set. $k = 3$.

Uniqueness only for $\Delta \leqslant 3$

```
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        NSolve[EQS /. {β → 0, γ → 1, λ → 1, Δ → 3}, {x, y}, Reals]
        NSolve[EQS /. {β → 0, γ → 1, λ → 1, Δ → 4}, {x, y}, Reals]


Out[3]= {{x → 0.34781, y → 0.34781}}

Out[4]= {{x → 0.584659, y → 0.0979558}, {x → 0.0979558, y → 0.584659}, {x → 0.27
```

Uniqueness on the Δ-uniform hypertree iff $\Delta \leqslant 3$

$\Delta \leqslant 3$: (Liu, Lin 2015, Yin, Zhao 2015) (implicitly) establish strong spatial mixing which leads to approximation scheme

$\Delta = 4, 5$: Strong spatial mixing fails (due to non-uniqueness)

$\Delta \geqslant 6$: Non-uniqueness leads to intractability

Yin, Zhao natural gadgets cannot be used to show hardness for $4, 5$ so these cases remain open

For “natural” functions f

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FPRAS should exist up to SSM threshold, which is (in general) below the uniqueness threshold

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Above uniqueness threshold, approximation may be hard

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Not clear in general whether there exists a computational threshold or, if this exists, whether it coincides with the uniqueness threshold

Our result

Definition. For $k \geqslant 2$, let EASY(k) be the set containing the following seven functions.

$$f^{(k)}_{\text{zero}}(x_1,\ldots,x_k) = 0, \quad f^{(k)}_{\text{one}}(x_1,\ldots,x_k) = 1, \quad f^{(k)}_{\text{allzero}}(x_1,\ldots,x_k) = \mathbf{1}\{x_1 = \ldots = x_k = 0\},$$
$$f^{(k)}_{\text{allone}}(x_1,\ldots,x_k) = \mathbf{1}\{x_1 = \ldots = x_k = 1\}, \quad f^{(k)}_{\text{EQ}}(x_1,\ldots,x_k) = \mathbf{1}\{x_1 = \ldots = x_k\},$$
$$f^{(k)}_{\text{even}}(x_1,\ldots,x_k) = \mathbf{1}\{x_1 \oplus \cdots \oplus x_k = 0\}, \quad f^{(k)}_{\text{odd}}(x_1,\ldots,x_k) = \mathbf{1}\{x_1 \oplus \cdots \oplus x_k = 1\}.$$

Observation. If $f \in$ EASY(k). Then it is easy to compute $Z_{f;H}$.

Theorem. For any other symmetric Boolean function $f : \{0,1\}^k \to \{0,1\}$, $\exists \Delta_0$ such that $\forall \Delta \geqslant \Delta_0$, $\exists c > 1$ such that it is NP-hard to approximate $Z_{f;H}$ within a factor of c^n on k-uniform hypergraphs with degree $\leqslant \Delta$.

Connection to counting Constraint Satisfaction Problems (#CSPs)

Γ: Set of Boolean functions (constraint langauage)
Each arity k function in Γ is of the form $f : \{0,1\}^k \to \{0,1\}$.

CSP instance I: Set V of variables. Each constraint $f(v_1, \ldots, v_k)$ applies a k-ary function $f \in \Gamma$ to a tuple of (not necessarily distinct) variables.

Name #CSP$_{\Delta,c}(\Gamma)$.
Instance n-variable instance I of a CSP(Γ). Each variable is used at most Δ times.
Output number $\widehat{Z}$ such that $c^{-n} Z_{\Gamma;I} \leqslant \widehat{Z} \leqslant c^n Z_{\Gamma;I}$,
$Z_{\Gamma,I}$: number of satisfying assignments of I.

Counting CSP Corollary

Corollary. *Let $k \geqslant 2$ and let $f : \{0,1\}^k \to \{0,1\}$ be a symmetric Boolean function such that $f \notin \mathsf{EASY}(k)$. Then, there exists Δ_0 such that for all $\Delta \geqslant \Delta_0$, there exists $c > 1$ such that $\#\mathsf{CSP}_{\Delta,c}(\{f\})$ is* NP*-hard.*

What is known about bounded-degree Boolean #CSPs

• Adding a degree bound $\Delta = 3$ makes no difference to the difficulty of exact counting CSPs (Creignou and Hermann 1996, Cai, Lu, Xia 2009).

If Γ is affine then #CSP(Γ) is in FP. Otherwise #CSP$_3(\Gamma)$ is #P-complete.

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Γ is affine if every function is $f^{(k)}_{\text{even}}$ or $f^{(k)}_{\text{odd}}$ for some k.

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• This restriction also leaves the complexity unchanged for decision CSPs (Dalmau and Ford 2003 in the special case where Γ includes the two unary pinning functions.

• $\delta_0(0) = 1$ and $\delta_0(1) = 0$.

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bounded-degree decision has not been considered without pinning

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$\Delta = 2$ is holant. Not fully classified for counting or decision. Decision is as hard as the general case if the relation is not a "Delta-matroid". Feder 2001

Approximate counting

Dyer, Goldberg, Jalsenius, Richerby 2012 For every $\Delta \geqslant 6$ and $k \geqslant 3$ and every symmetric k-ary Boolean function $f \notin$ EASY(k), there is no FPRAS for #CSP$(\{f, \delta_0, \delta_1\})$ unless NP $=$ RP.

Not true for our setting!

Example: weak independent sets

$f(s_1, \ldots, s_k) = 1$ iff at least one of $s_1, \ldots, s_k$ is 1.

Not in EASY(k) for any $k \geqslant 2$.

Bordewich, Dyer, Karpinski 2008: For every $\Delta \leqslant (k-1)/2$, there is an FPRAS for the partition function $Z_{f;H}$ on the class of k-uniform hypergraphs H with maximum degree at most Δ. (so not hard for every $\Delta \geqslant 6$ as above)

Back to the result

Name #Hyper2Spin(f, Δ, c).

Instance An n-vertex k-uniform hypergraph H with maximum degree at most Δ.

Output A number $\widehat{Z}$ such that $c^{-n} Z_{f;H} \leqslant \widehat{Z} \leqslant c^n Z_{f;H}$.

Theorem. Let $k \geqslant 2$ and let $f : \{0, 1\}^k \to \{0, 1\}$ be a symmetric Boolean function such that $f \notin$ EASY(k). Then there exists Δ_0 such that for all $\Delta \geqslant \Delta_0$, there exists $c > 1$ such that #Hyper2Spin(f, Δ, c) is NP-hard.