Glauber Dynamics of Lattice Triangulations on Thin Rectangles

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Lattice triangulations: basic facts

Definition

A triangulation of a $m \times n$ rectangle in $\mathbb{Z}^2$ is a *maximal* set of *non-crossing* edges, each of which connects exactly two points of the rectangle and passes through no other point.

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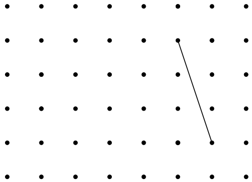

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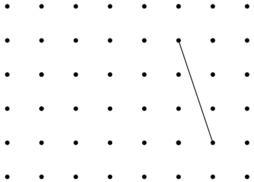

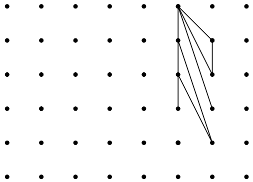

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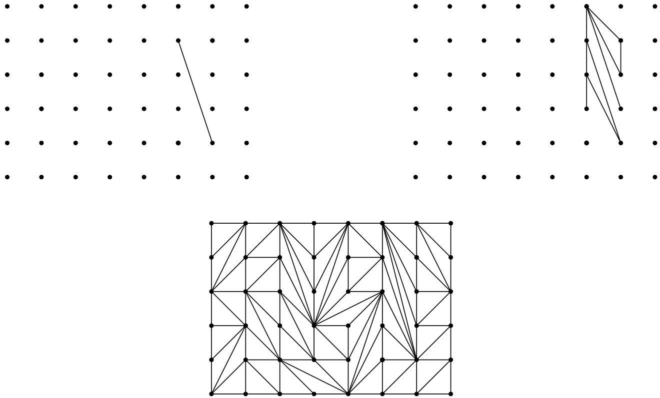
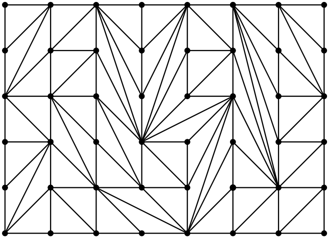


- $\Omega(m, n)$ the set of all triangulations of $R_{m,n}$



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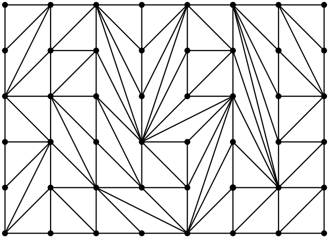
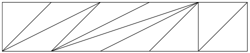
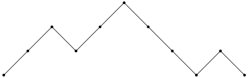


- $m = 1$: $\#\Omega(1,n) = \binom{2n}{n}$. Equivalence with lattice paths



(a) A one dimensional lattice triangulation



(b) The associated lattice path

An important difference w.r.t. spin systems

- The middle point of each (random) edge is a given (deterministic) point in the half-integer lattice;
- Assigning an edge $\sigma_x \Leftrightarrow$ assigning a "spin s_x".
- For a spin system on a graph interaction is local: the law of s_x is determined given the neighbors.
- An edge σ_x has 4 neighboring edges whose midpoints can be, however, very far from x.
- Lack of locality/geometry.

Sampling lattice triangulations

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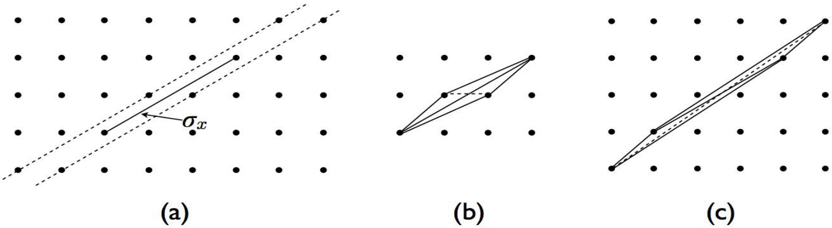

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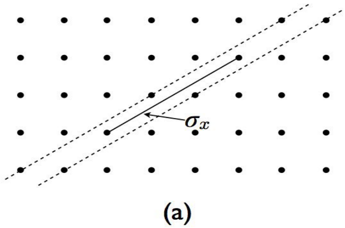

Flip graph on $\Omega(m, n)$ is connected.

Markov chain reversible w.r.t. uniform distribution:

- pick a midpoint x u.a.r.
- flip σ_x with Prob $=1/2$ if flippable.

Weighted triangulations and Glauber dynamics

Consider the Gibbs distribution on $\Omega(m, n)$

$$\mu(\sigma) = \frac{\lambda^{|\sigma|}}{Z}, \qquad |\sigma| = \sum_{x \in \Lambda_{m,n}} |\sigma_x|$$

where $|\sigma_x| = \|\sigma_x\|_1$.

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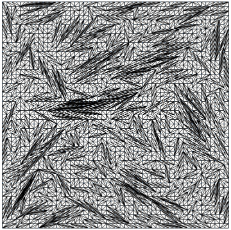
Glauber chain: pick u.a.r. a midpoint $x \in \Lambda_{m,n}$. If the edge σ_x is flippable to edge σ'_x then flip it with probability

$$\frac{\mu(\sigma')}{\mu(\sigma') + \mu(\sigma)} = \frac{\lambda^{|\sigma'_x|}}{\lambda^{|\sigma'_x|} + \lambda^{|\sigma_x|}}.$$

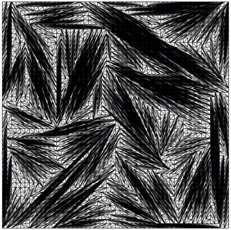
Reversible w.r.t. μ.

Weighted triangulations and Glauber dynamics

Simulations suggest a phase transition (here $n = m = 50$):



$\lambda = 1$



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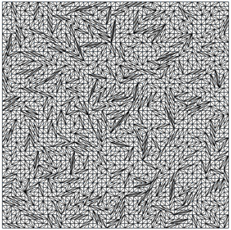

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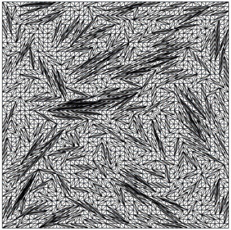

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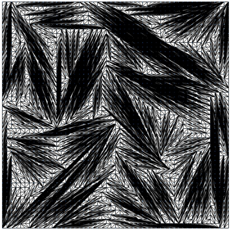

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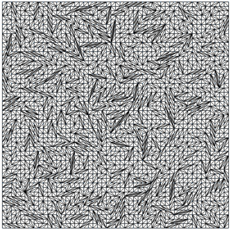

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Conjecture

- $\lambda < 1$: $T_{\mathrm{mix}} = O(mn(n+m))$
- $\lambda > 1$: $T_{\mathrm{mix}} = \exp(\Omega(mn(n+m)))$
- $\lambda = 1$: $T_{\mathrm{mix}} = poly(m, n)$.

Main results for any m, n

Theorem (Rapid mixing for small λ)

There exists $\lambda_0 > 0$ such that, for all $\lambda < \lambda_0$ and any possible set of constraint edges, $T_{\mathrm{mix}} = O(mn(m+n))$.

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Theorem (Slow mixing for $\lambda > 1$)

For all $\lambda > 1$ and without constraint edges
$T_{\mathrm{mix}} \geqslant \exp(c(m+n))$.

Rapid mixing for small λ

Path coupling (Bubley-Dyer 1997) + exponential metric [inspired by S. Greenberg, A. Pascoe, D. Randall '09].

Exponential metric: Fix $\alpha > 1$, and for σ, τ differing only at x set

$$\Delta(\sigma,\tau) = \begin{cases} \alpha^2 - 1 & \text{if } |\sigma_x| = |\tau_x| = 2 \text{ (unit diagonals)} \\ \left|\alpha^{|\sigma_x|} - \alpha^{|\tau_x|}\right| & \text{otherwise.} \end{cases}$$

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Lemma

For $\lambda < \lambda_0 = 1/8$, $\alpha = 8$, there is a coupling such that

$$\mathbb{E}_{\sigma,\tau}[\Delta(\sigma',\tau')] \leqslant \Delta(\sigma,\tau)\left(1 - \tfrac{1}{2|\Lambda_{n,m}|}\right).$$

Torpid mixing for $\lambda > 1$

Definition (Exponential Bottleneck)

A set $A \subset \Omega(m,n)$ such that $\mu(A) \leqslant 1/2$ and

$$\frac{\mu(\partial A)}{\mu(A)} \leqslant e^{-c(m+n)}.$$

Here $\partial A = \{(\sigma,\sigma') : \sigma \in A,\ \sigma' \notin A,\ \sigma \leftrightarrow \sigma'\}$.

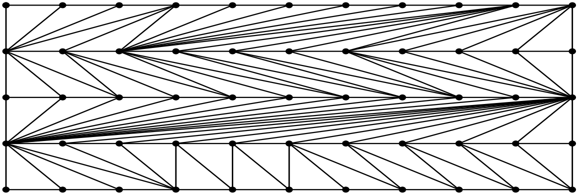
Lemma

Exponential bottleneck $\Rightarrow$

$$T_{\mathrm{mix}} = \Omega\big(\exp[c(n+m)]\big), \qquad c > 0.$$

The Herringbone bottleneck

- A is the set of all Herringbone triangulations.
- Orientation in 1D layers oscillates $+/-$.



- $\sigma \in \partial A$ iff an internal edge is vertical.
- For $\lambda > 1$, $\sigma \in \partial A$ is exponentially unlikely (in $\max(n, m)$) given A.

Optimal bounds on T_{mix} for $m = 1$

Theorem

- $\lambda < 1$: $T_{\text{mix}} = \Theta(n^2)$ *(path coupling + exponential metric)*
- $\lambda > 1$: $T_{\text{mix}} = \exp(\Omega(n^2))$ *(1 layer bottleneck)*
- $\lambda = 1$: $T_{\text{mix}} \sim n^3 \log n$ *(e.g. coupling, D.B. Wilson '01)*

Optimal bounds for thin rectangles ($m = \text{const}, \; n \gg 1$)

Theorem

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- **Lower bound** for $\lambda > 1$: (slightly) improved version of the Herringbone Bottleneck.

- **Upper bound** for $\lambda < 1$:
  - In time $O(n^2)$ the chain enters the set $\widetilde{\Omega}$ of "short" ($O(\log n)$) triangulations. Main tool: Lyapunov function (A. Stauffer '15).
  - Mixing time bounds $O(n^{(1+o(1))}$ of *restricted* chain in $\widetilde{\Omega}$ via Log-Sobolev bounds + improved canonical paths arguments.

Exponential tails of edge length (m fixed).

Lemma (No constraint edges)

Fix $\lambda < 1$. There exist c_1, c_2 such that, for any $t \geq c_1 n^2$ and any $\ell \geq 1$,

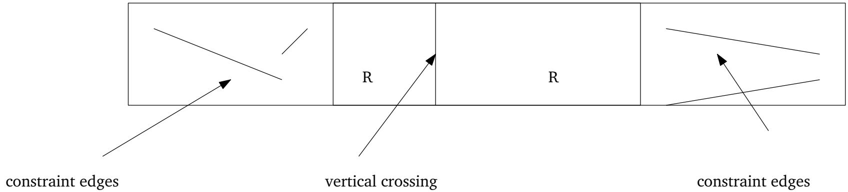
$$\sup_{\sigma} \sup_{x \in \Lambda_{n,m}} \mathbb{P}_\sigma(|\sigma_x(t)| \geq \ell) \leq c_1 \exp(-c_2 \ell)$$

Lemma (Constraint edges τ)

Fix $\lambda < 1$. Let $\bar{\sigma}_x$ the ground state of σ_x in the presence of constraint edges τ. There exist c_1, c_2 such that, for any $t \geq c_1 n^2$, any $\ell \geq 1$ and any x,

$$\sup_{\sigma} \mathbb{P}\left(\cup_y \{\sigma_y(t) \cap \bar{\sigma}_x \neq \emptyset\} \cap \{|\sigma_y(t)| \geq |\bar{\sigma}_x| + \ell\}\right) \leq c_1 \exp(-c_2 \ell)$$

Coupling in presence of constraint edges



- Let R be a $k \times m$ rectangle inside $R_{n,m}$.
- Let τ, τ' be constraint edges *not* intersecting R.

Lemma

Fix $\lambda < 1$ and m. There exists c and k_0 together with a coupling of μ^{τ}, $\mu^{\tau'}$ such that, if $k \geq k_0$, with probability at least $1 - \exp[-ck]$ there exist ϵk common vertical crossings of unit edges in R.

Back to thin rectangles: $T_{\text{mix}} = O(n^2)$ for any $\lambda < 1$

Step 1: Burn-in phase.

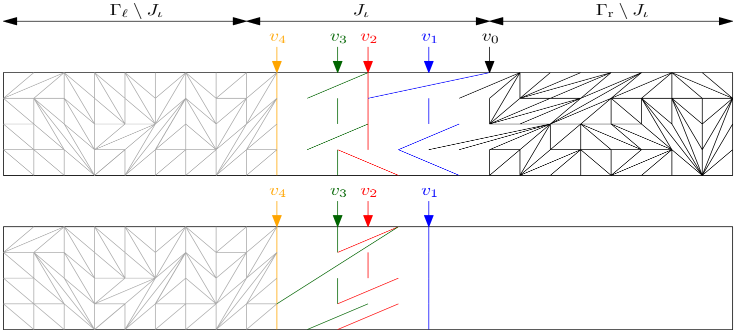
For some $T = c(\lambda)n^2$, uniformly in the initial condition and w.h.p.

$$\sigma(t) \in \widetilde{\Omega}\,, \qquad t \in [T, T + n^{10}].$$

$\widetilde{\Omega}$ is the set of triangulations with at most $O(\log n)$ edges.

- The restricted chain to $\widetilde{\Omega}$ is irreducible with reversible measure $\widetilde{\mu} := \mu(\cdot \mid \widetilde{\Omega})$.
- Because of structural properties $\widetilde{\mu}, \mu$ well coupled.
- Sufficient to prove $\widetilde{T}_{\text{mix}} = o(n^2)$.

Step 2: spatial mixing in $\widetilde{\Omega}$



Lemma (Spatial mixing)

The relative density of the marginals on the left block (light gray) of $\tilde{\mu}$ conditioned on two arbitrary (short) triangulations in the right block (dark gray) is exponentially (in $|J_c|$) close to one if $|J_c| = \Omega(\mathrm{polylog}(n))$.

Step 3: Log-Sobolev constant in $\widetilde{\Omega}$

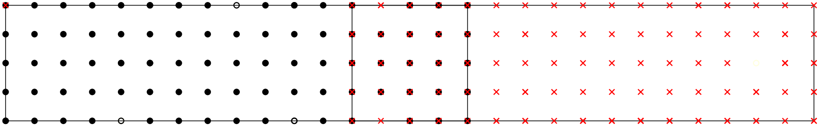


Figure: The rectangle Λ decomposed into two almost-halves Λ_1, Λ_2 with $\Lambda_1 \cap \Lambda_2 \equiv \Omega(\log n) \times m$ rectangle.

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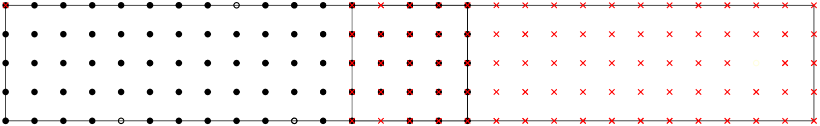

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Spatial mixing implies quasi-factorization of the entropy:

$$\mathrm{Ent}_\Lambda(f^2) \leqslant (1 + n^{-\varepsilon})\tilde{\mu}\left[\mathrm{Ent}_{\Lambda_1}(f^2 | \Lambda \setminus \Lambda_1) + \mathrm{Ent}_{\Lambda_2}(f^2 | \Lambda \setminus \Lambda_2)\right].$$

$$\Downarrow$$

Multiscale analysis of the Log-Sobolev constant.

Notation

- Dirichlet form:

$$\mathcal{E}(f,f) = \frac{1}{2n} \sum_{\sigma,\sigma' \in \widetilde{\Omega}} \tilde{\mu}(\sigma) p(\sigma,\sigma')(f(\sigma) - f(\sigma'))^2.$$

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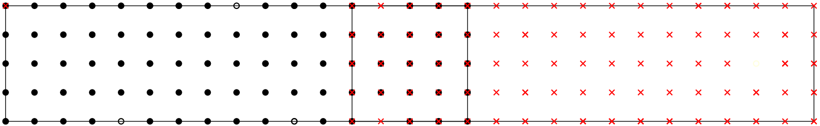
Theorem

$$c_S(n) \leqslant n^{1+o(1)} \Rightarrow \widetilde{T}_{\mathrm{mix}} = O(n^{1+o(1)}).$$

High level overview



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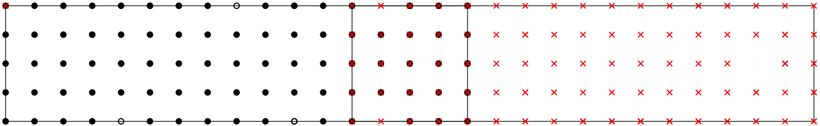

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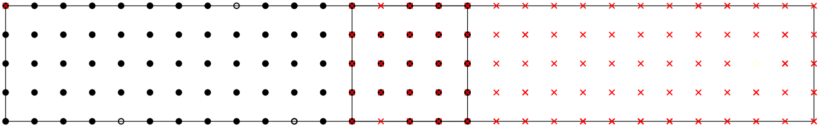

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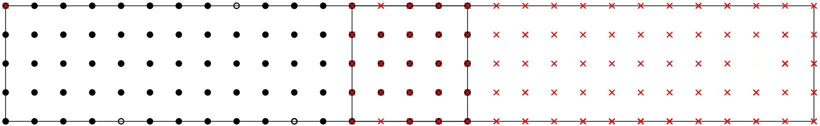

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$$\Rightarrow c_S(n) \leq \text{const} \times c_S(polylog(n)).$$

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  not feasible.
- Need e.g. a $O(poly(L_n))$ bound on $c_S(L_n)$ by different means.

A $poly(L_n)$ upper bound on $c_S(L_n)$

- $c_S(L_n) = O(n \times T_{\rm rel}(L_n))$
- $T_{\rm rel}(L_n) \leq \mathcal{C}$ (congestion rate)

$$\mathcal{C} := \max_{\eta\sim\eta'} \sum_{\substack{\sigma,\sigma': \\ \Gamma_{\sigma,\sigma'}\ni(\eta,\eta')}} \frac{\mu(\sigma)\mu(\sigma')}{\mu(\eta)p(\eta,\eta')}|\Gamma_{\sigma,\sigma'}|$$

where, for any $\sigma, \sigma' \in \widetilde{\Omega}$, $\Gamma_{\sigma,\sigma'}$ is a path in $\widetilde{\Omega}$ from σ to σ'.

- Typically $\mathcal{C} = O(\exp(cL_n))$. We need $O(poly(L_n))$.

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$$\rho = \min_{x \in \mathcal{X}} \mathbb{P}_x(X_T \in \mathcal{X}')$$

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- $\mathcal{X}' \subset \mathcal{X}$ be such that for any pair $x, y \in \mathcal{X}'$ it is possible to define a canonical path $\Gamma_{x,y}$ entirely contained in $\mathcal{X}'$. Let $\mathcal{C}(\mathcal{X}')$ be the associated congestion rate.
- Fix time T and let

$$\rho = \min_{x \in \mathcal{X}} \mathbb{P}_x(X_T \in \mathcal{X}')$$

Lemma (Canonical paths with burn-in time)

$$T_{\text{rel}} \leq \frac{6T^2}{\rho} + \frac{3\mathcal{C}(\mathcal{X}')}{\rho^2}$$

Back to thin rectangles

Theorem

Consider the original triangulation chain on $n \times m$ rectangle with (possibly) boundary edges sticking in but not longer than $n/4$. Then

$$T_{\mathrm{rel}} = O(poly(n)).$$

Corollary (The needed $poly(Ln)$ bound)

For the restricted chain on $\widetilde{\Omega}$ on $L_n \times m$ rectangle
$T_{\mathrm{rel}}(L_n) = O(poly(L_n)) = O(polylog(n))$.

Sketch of proof

Define $\Omega' \subset \Omega$ as follows:

- any edge does not exceed by more than $O(\log n)$ its minimal allowed (by the boundary edges) length.
- for any $x \neq y$, if σ_y crosses the ground state edge $\bar{\sigma}_x$ at x then $|\sigma_y| \leq |\bar{\sigma}_x| + O(\log n)$.

Lemma

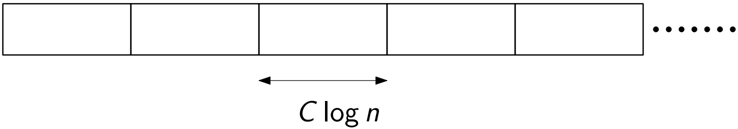
Fix $T = cn^2 m$. Then Ω' satisfies the hypotheses of the "canonical paths with burn-in lemma" with

$$\rho = \min_{\sigma} \mathbb{P}(\sigma(T) \in \Omega') \geq 1/2$$

and congestion rate $\mathcal{C}' = O(poly(n))$ for a suitable choice of canonical paths in Ω'.

Key feature of the set Ω'

- Pb: Given σ, η construct path between them.
- In principle, to flip σ_x to η_x one may need to reshuffle edges in σ with midpoints very far from x.
- If $\sigma, \eta \in \Omega'$ "very far" is not more than $O(\log n)$.



- It is possible to construct the path by processing the slabs left-to-right without never changing more than 2 slabs at a time.