Dynamics for the Random-cluster Model

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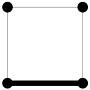
Joint work with Alistair Sinclair (UC Berkeley)

Random-cluster model (Fortuin & Kasteleyn, 1969)

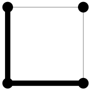
- Probability distribution over the subgraphs of a graph $G = (V, E)$.
- Given parameters $p \in [0, 1]$ and $q > 0$, for each subgraph $(V, A \subseteq E)$:

$$\mu_G(A) \propto p^{|A|} (1 - p)^{|E \setminus A|} q^{c(A)}$$

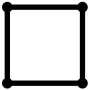
[$c(A)$: # of cmpts in (V, A)]



$p(1-p)^3q^3$



$p^2(1-p)^2q^2$



p^4q

Motivation

$$\mu_G(A) \propto p^{|A|}\,(1-p)^{|E\setminus A|}\,q^{c(A)}$$

Unifying framework for studying several important distributions:

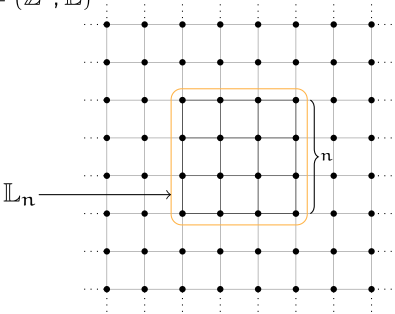
- When $q = 1$, bond percolation model. [$G = K_n$, $G(n,p)$ model]
- For integer $q \geqslant 2$, “dual” to ferromagnetic Ising/Potts model.
- When $q \to 0$, the set of (weak) limits that arises includes:

$$\mu_G \to \mathrm{UST}(G), \quad \mu_G \to \mathrm{USF}(G), \text{ or } \quad \mu_G \to \mathrm{UCS}(G)$$

Random-cluster model in infinite graphs

Infinite measure: If $\{G_n\} \to G$, then $\mu_G := \lim_{n\to\infty} \mu_{G_n}$

Example: $\mathbb{L} = (\mathbb{Z}^2, \mathbb{E})$



Then, $\{\mathbb{L}_n\} \to \mathbb{L}$ and $\mu_{\mathbb{L}} := \lim_{n\to\infty} \mu_{\mathbb{L}_n}$

Phase transition

Phase transition: $\exists\, p_c(q)$ such that w.h.p.,

- $p < p_c(q) \implies$ all components are finite;
- $p > p_c(q) \implies$ there is at least one infinite component.

In $\mathbb{Z}^2$:

$$p_c(q) = \frac{\sqrt{q}}{\sqrt{q}+1}$$

[Beffara, Duminil-Copin 2012]

Finite setting: corresponds to the emergence of a giant component.

Dynamics

Our focus: Markov chains on the random-cluster configurations of a graph G with stationary distribution μ_G.

Mixing time: Number of steps T_{mix} until total variation distance from μ_G is small ($\leqslant 1/4$), starting from any initial configuration.

Motivation:

- Connection between phase transitions and mixing times.
- Algorithms for sampling configurations (MCMC).
- Random-cluster dynamics challenge current techniques.

Heat-bath (HB) dynamics

Given a random-cluster configuration $A \subseteq E$:

1. pick an edge $e \in E$ u.a.r.;
2. replace A by $A \cup \{e\}$ with probability

$$\frac{\mu_G(A \cup \{e\})}{\mu_G(A \cup \{e\}) + \mu_G(A \setminus \{e\})};$$

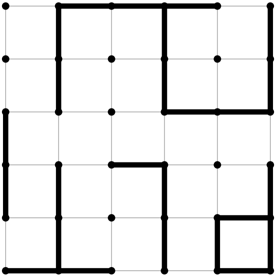
3. else replace A by $A \setminus \{e\}$.

Heat-bath (HB) dynamics (cont.)

$$\frac{\mu_G(A \cup \{e\})}{\mu_G(A \cup \{e\}) + \mu_G(A \setminus \{e\})} = \begin{cases} p & \text{if } e \text{ is not a cut-edge;} \\ \dfrac{p}{p + q(1-p)} & \text{otherwise.} \end{cases}$$

Heat-bath (HB) dynamics (cont.)

$$\frac{\mu_G(A \cup \{e\})}{\mu_G(A \cup \{e\}) + \mu_G(A \setminus \{e\})} = \begin{cases} p & \text{if } e \text{ is not a cut-edge;} \\ \dfrac{p}{p + q(1-p)} & \text{otherwise.} \end{cases}$$



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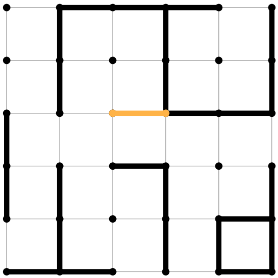

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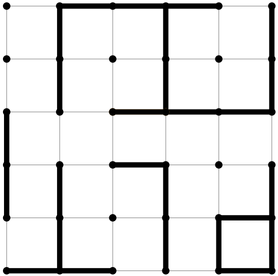

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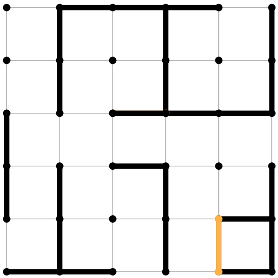

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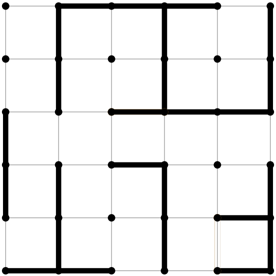


Chayes-Machta (CM) dynamics

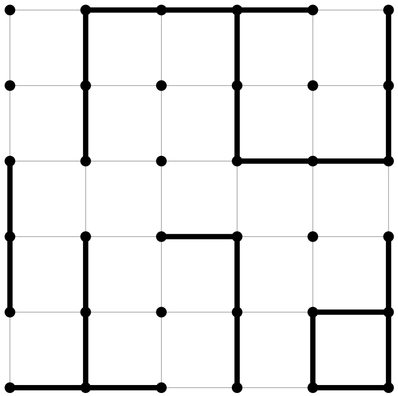
Given a random-cluster configuration $A \subseteq E$:

1. Activate each component of A independently with prob. $1/q$;
2. Add each active edge with prob. p; remove it otherwise.

- Straightforward to check that μ_G is the stationary measure.
- It is well-defined for any $q \geqslant 1$.

CM dynamics: example

$G = \mathbb{L}_6$, $p = 1/2$, $q = 2$



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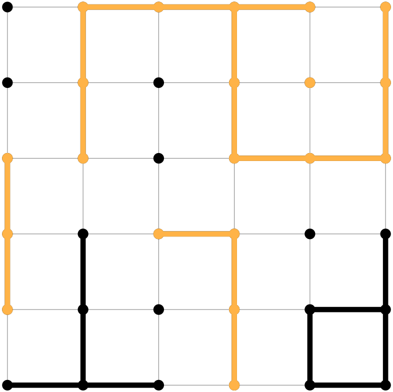

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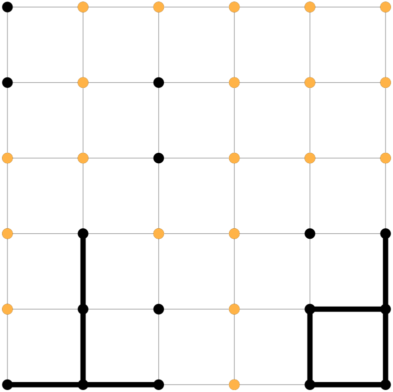

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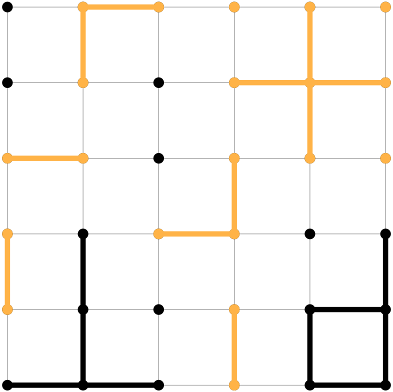

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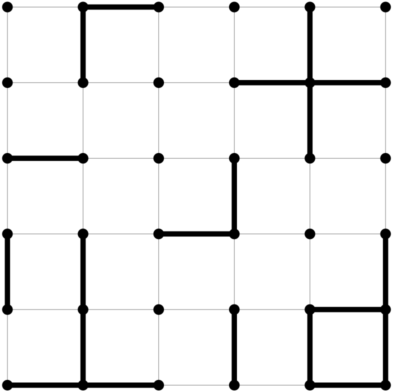


Local and Global dynamics

- The HB dynamics is a local Markov chain, while the CM is global.
- In spin systems, global Markov chains mix fast in regimes where local dynamics are slow.

Theorem 1

$$\frac{T_{\text{mix}}(CM)}{\tilde{O}(|E|^2)} \leqslant T_{\text{mix}}(HB) \leqslant \tilde{O}(|E|^2) \cdot T_{\text{mix}}(CM)$$

Proof idea. Follows by generalizing a technique of [Ullrich 2013].

Part I: Chayes-Machta dynamics in the mean-field $[G = K_n]$

Part II: Heat-bath dynamics in $\mathbb{Z}^2$

Mean-field theory

Mean-field: $G = K_n$ [useful non-trivial starting point]

Phase transition: If $p = \lambda/n$, then $\exists\, \lambda_c(q)$ such that w.h.p.:

- $\lambda < \lambda_c(q) \implies$ all components have size $O(\log n)$.
- $\lambda > \lambda_c(q) \implies$ there is a component of size $\sim \theta_r n$.

Critical value:

$$\lambda_c(q) = \begin{cases} q & \text{if } 0 < q \leqslant 2, \\ 2\left(\frac{q-1}{q-2}\right)\log(q-1) & \text{if } q > 2. \end{cases}$$

[Bollobás, Grimmett, Janson 1996] [Luczak, Łuczak 2006]

Mean-field mixing: Previous work

Previous work on mixing times:

- Most previous results are for Swendsen-Wang (SW) dynamics.
  [similar to CM dynamics, but only for integer q]

- Mixing time of SW dynamics for $q = 2$ fully understood.
  [Cooper,Dyer,Frieze,Rue 2006]
  [Long,Nachmias,Ning,Peres 2011]

- Until recently, only partial results for integer $q \geqslant 3$.
  [Gore,Jerrum 1996], [Huber 2003]

- Independently, mixing time of SW dynamics for integer $q \geqslant 3$ also fully understood. [Galanis, Štefankovič, Vigoda 2015]

Mean-field mixing: Our results

Theorem 2

If $q \in (1, 2]$:

$$T_{\mathsf{mix}}(CM) = \Theta(\log n) \qquad \text{for } \lambda \neq \lambda_c$$

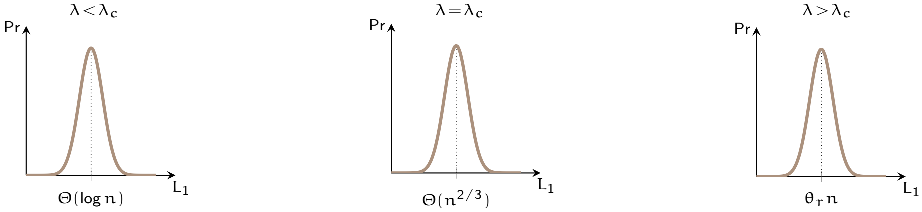
If $q > 2$:

$$T_{\mathsf{mix}}(CM) = \begin{cases} \exp(\Omega(\sqrt{n})) & \text{for } \lambda \in (\lambda_{\mathsf{L}}, \lambda_{\mathsf{R}}) \\ \Theta(\log n) & \text{for } \lambda \notin [\lambda_{\mathsf{L}}, \lambda_{\mathsf{R}}) \\ \Theta(n^{1/3}) & \text{for } \lambda = \lambda_{\mathsf{L}} \end{cases}$$

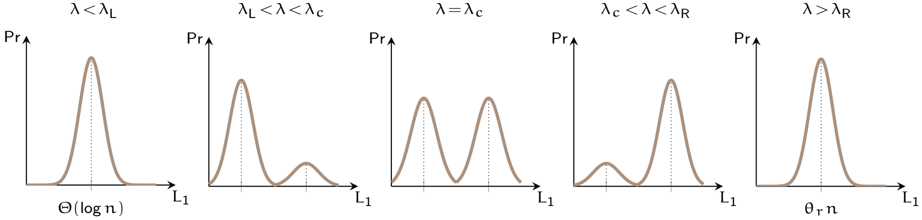
$[\lambda_{\mathsf{L}} < \lambda_c < \lambda_{\mathsf{R}}]$

Mean-field mixing: Interpretation of results

Second order phase transition for $1 < q \leqslant 2$:



First order phase transition for $q > 2$:



Mean-field mixing: Proof ideas

Technique:

- Couple two copies $\{X_t\}$, $\{Y_t\}$ of the CM dynamics, starting from arbitrary initial configurations X_0, Y_0.
- If $\Pr[X_T \neq Y_T] \leqslant 1/4$ then $T_{\mathrm{mix}} \leqslant T$.

Coupling has two main phases:

1. Run $\{X_t\}$ and $\{Y_t\}$ independently until the sizes of largest components are close to expectation. $[\Theta(\log n)$ if $\lambda < \lambda_c$, and $\sim\theta_r n$ if $\lambda > \lambda_c]$
2. Coupled the evolution of $\{X_t\}$, $\{Y_t\}$ until $X_T = Y_T$.

Phase 1: Independent evolution

Observation: If $\mathcal{A}_t$ is the set of active vertices at time t, the configuration in $\mathcal{A}_t$ is replaced by a $G(\mathcal{A}_t, p)$ random graph.

Phase 1a: After $O(\log n)$ steps $\{X_t\}$ has at most one large component.

Phase 1b: Analyze the expected change in size of the largest component.

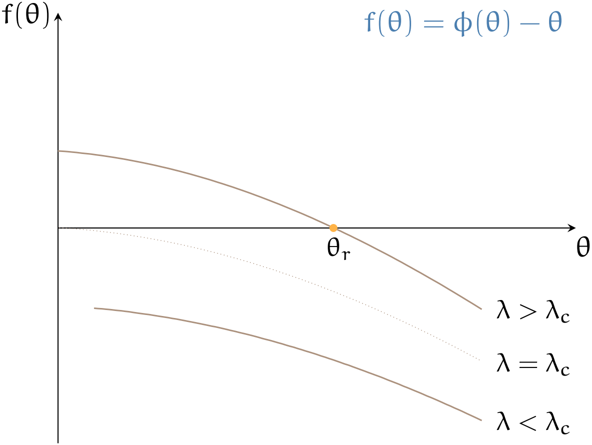
[$\mathcal{L}(X_t)$ largest component of X_t, and $L_1(X_t) =: \theta_t n$]

- If $\mathcal{L}(X_t)$ is inactive: $\mathcal{L}(X_{t+1}) = \mathcal{L}(X_t)$ w.h.p.
- If $\mathcal{L}(X_t)$ is active: # of active vertices $\approx \theta_t n + \frac{(1-\theta_t)n}{q} =: M$

$$E[\, L_1(X_{t+1}) \mid \mathcal{L}(X_t) \text{ active} \,] \;\approx\; L_1(G(M,p)) \;=:\; \phi(\theta_t)n.$$

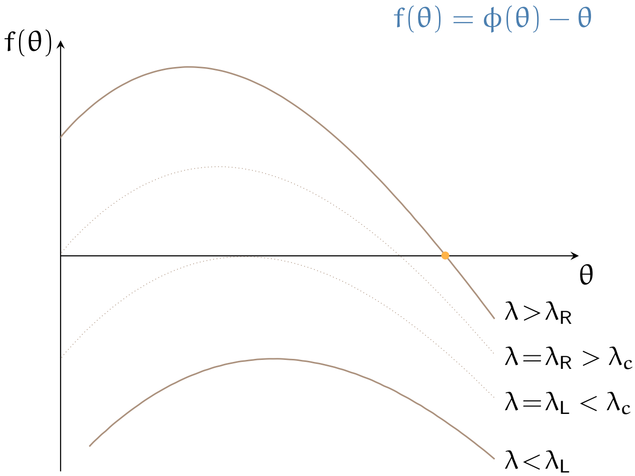
- Drift given by the function $f(\theta) := \phi(\theta) - \theta$.

Phase 1: Drift function ($1 < q \leqslant 2$)



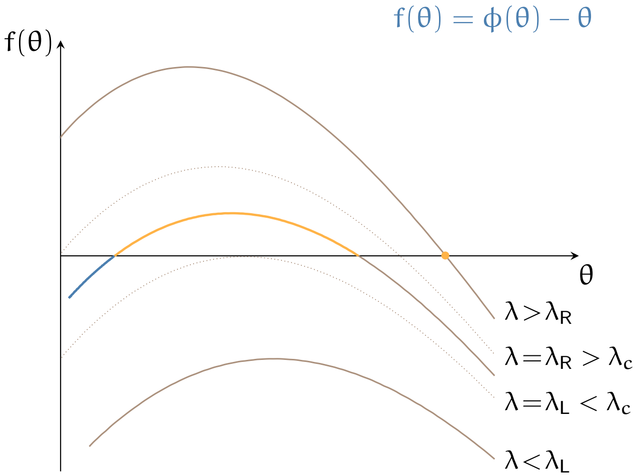
The drift function f has the desired sign.

Phase 1: Drift function ($q > 2$)



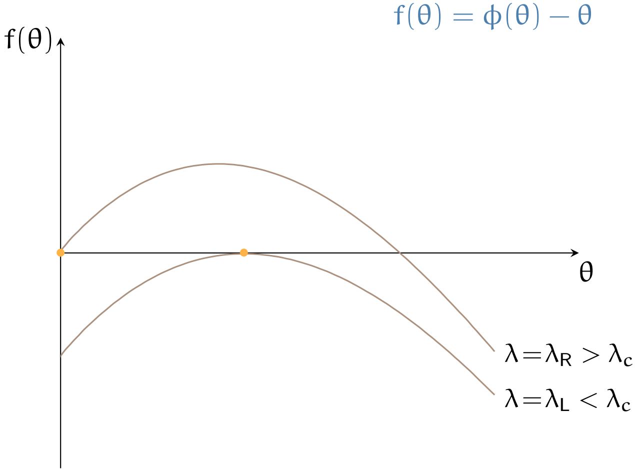
When $\lambda < \lambda_L$ and $\lambda > \lambda_R$ drift always has desired sign.

Phase 1: Drift function ($q > 2$)



When $\lambda_L < \lambda < \lambda_R$, drift does not always have desired sign.

Phase 1: Drift function ($q > 2$)



When $\lambda = \lambda_L$ or $\lambda = \lambda_R$, drift can be 0.

Proofs: Phase 2 - Coupled evolution

Phase 2a: Coupling to achieve the same component structure.
[Assuming $L_1(X_0) \approx L_1(Y_0)$]

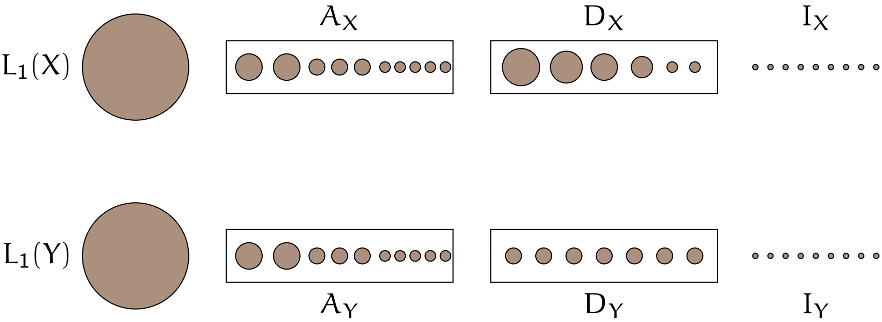
1. Couple the activation of components in a way such that both active subgraphs have the same size w.h.p.
2. Couple edge resampling using arbitrary bijection between active edges.

Observations:

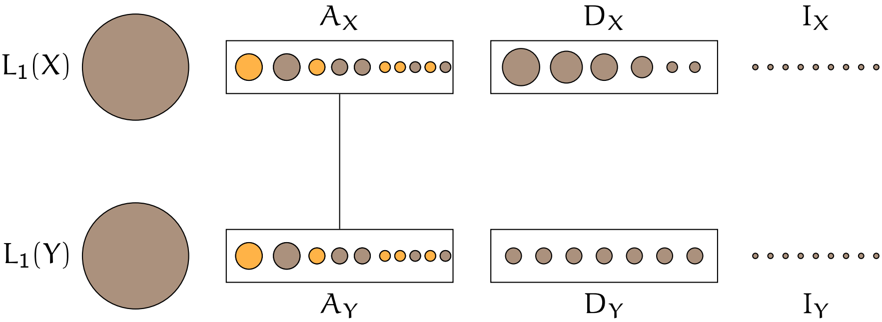
- If activation coupling succeeds, the activated subgraphs will have the same component structure after step 2.
- After $T = O(\log n)$ consecutive successes of the activation coupling X_T and Y_T have the same component structure w.h.p.

Phase 2b: If X_0 and Y_0 have the same component structure, $\{X_t\}$, $\{Y_t\}$ can be coupled such that $X_T = Y_T$ for some $T = O(\log n)$ w.h.p.

Proofs: Phase 2 - Activation coupling



Proofs: Phase 2 - Activation coupling



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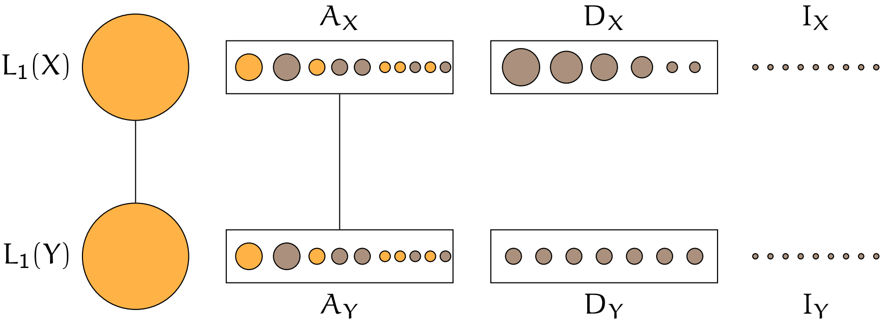

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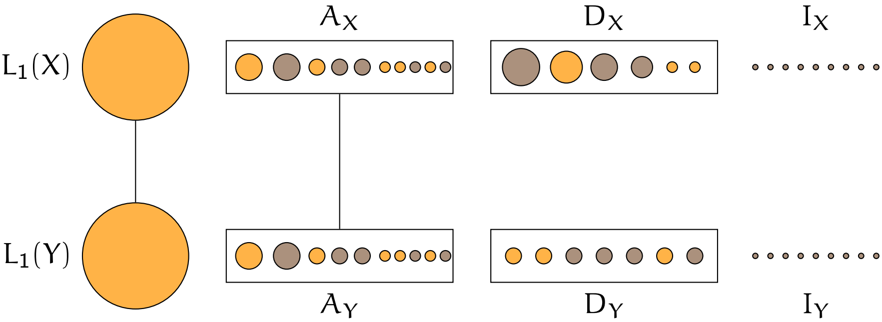

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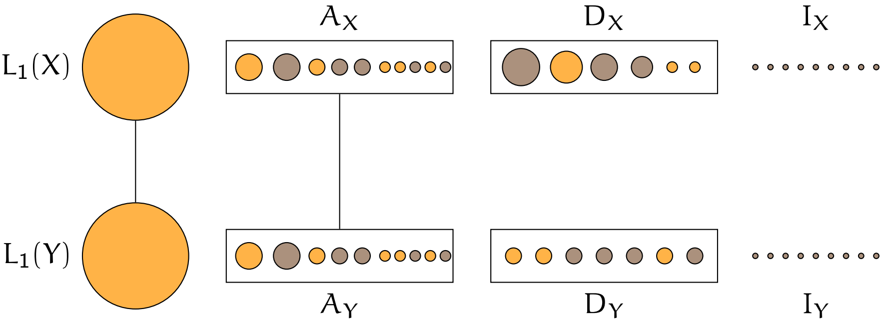

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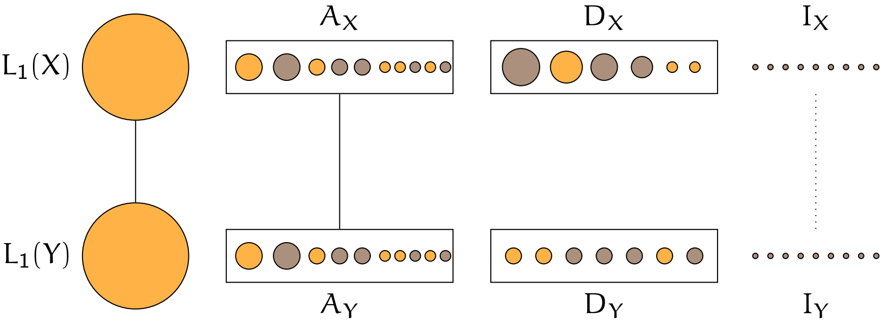
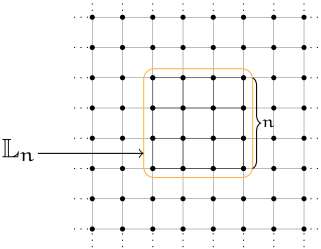


- First part of activation creates a discrepancy $\mathcal{D} = O(\sqrt{n})$.
- Correct $\mathcal{D}$ using coupling of binomial distributions on I_X and I_Y.

Part II: Random-cluster dynamics in $\mathbb{Z}^2$

Heat-bath dynamics in $\mathbb{L}_n$



Given a random-cluster configuration $A \subseteq E_n$ of $\mathbb{L}_n$:

1. pick an edge $e \in E$ u.a.r.;
2. replace A by $A \cup \{e\}$ with probability $\dfrac{\mu_G(A \cup \{e\})}{\mu_G(A \cup \{e\}) + \mu_G(A \setminus \{e\})}$;
3. else replace A by $A \setminus \{e\}$.

Heat-bath dynamics in $\mathbb{L}_n$: Mixing times

Previous work: [Ullrich 2014]

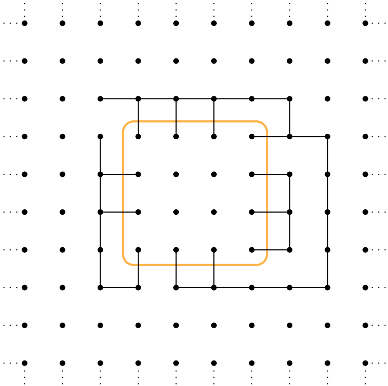
$T_{\text{mix}}(\mathrm{HB}) = O(n^6 \log^2 n)$, for all integer $q \geqslant 1$ and all $p \neq p_c(q)$.

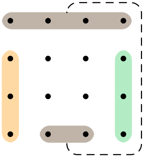
Holds only for integer q, produces weak bounds, and the proof is indirect.

Theorem 3

$T_{\text{mix}}(\mathrm{HB}) = \Theta(n^2 \log n)$, for all $q \geqslant 1$ and all $p \neq p_c(q)$.

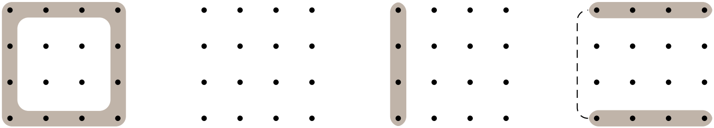
Boundary conditions





Boundary condition: A partition of the vertices in $\partial\mathbb{L}_n$ that encodes the connectivities from $\mathbb{L}_n^c$.

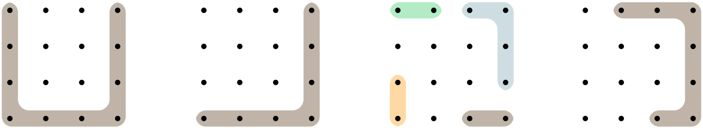
Boundary conditions: examples



wired free

A boundary condition η is side-homogeneous if:

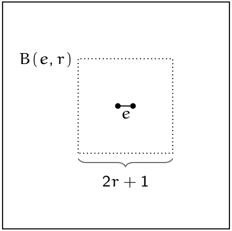
1. all wired vertices in η belong to the same component of $\mathbb{L}_n^c$; and
2. η is either free or wired along each side of $\mathbb{L}_n$



not side-homogeneous!

Spatial Mixing (SM)

Let $B(e, r)$ be a square box of radius r around e:



$\mathbb{L}_n$

SM holds if for all e and all pairs of configurations A_1^c, A_2^c on $B^c(e, r)$:

$$| \, \mu^{\eta}_{\mathbb{L}_n}(\, e = 1 \mid A_1^c \,) \; - \; \mu^{\eta}_{\mathbb{L}_n}(\, e = 1 \mid A_2^c \,) \,| \; \leqslant \; \mathrm{e}^{-\Omega(r)}$$

Spatial Mixing (cont.)

Exponential decay of connectivities (EDC): [Beffara, Duminil-Copin 2012]

For $p < p_c(q)$, $q \geqslant 1$ and all $u, v \in \mathbb{Z}^2$,

$$\mu_{\mathbb{L}}(u \leftrightarrow v) \leqslant e^{-d(u,v)}$$

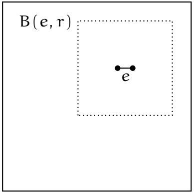
Previous work on SM: [Alexander 2004]

- EDC holds in finite volumes with arbitrary boundary conditions.
- SM holds for certain restricted class of boundary conditions, for all $p < p_c(q)$ and all integer $q \geqslant 1$.

Lemma 1. SM holds for side-homogeneous boundary conditions, for all $p < p_c(q)$ and all $q \geqslant 1$.

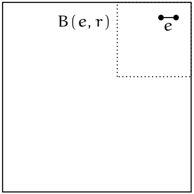
Proof of Lemma 1 (SM)

Case 1: $\partial \mathbb{L}_n \cap B(e, r) = \emptyset$



- Influence on e from $B^c(e, r)$ iff there are paths from ∂B to e.
- EDC ensures that influence decays exponentially with r.

Case 2: $\partial \mathbb{L}_n \cap B(e, r) \neq \emptyset$



- Influence on e also from the boundary condition on $\mathbb{L}_n$.

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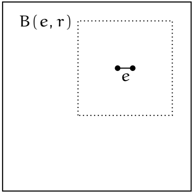

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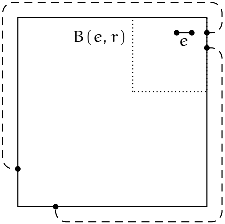


- Influence on e also from the boundary condition on $\mathbb{L}_n$.
- Far regions could affect the state of e.
- Side-homogeneous boundaries avoid this!

Outline of proof of Theorem 3

Theorem 3

$T_{mix}(HB) = \Theta(n^2 \log n)$, for all $p \neq p_c$ and all $q \geqslant 1$.

Proof Sketch.

Lemma 2. SM implies that $T_{mix} = O(n^2 \log n\, (\log \log n)^2)$.

Lemma 3. If $T_{mix} = O\left(n^{2+\varepsilon}\right)$, then $T_{mix} = O(n^2 \log n)$.

Therefore, $T_{mix} = O(n^2 \log n)$ for all $p < p_c$ and all $q \geqslant 1$.

Lemma 4. If $T_{mix} \leqslant M$ for $p < p_c$, then $T_{mix} = O(M)$ for $p > p_c$.

Lemma 5. $T_{mix} = \Omega(n^2 \log n)$.

Proof of Lemma 2: SM implies fast mixing

On spin systems: [Martinelli, Olivieri, Schonmann 1994]
[Dyer, Sinclair, Vigoda, Weitz 2004]
[Mossel, Sly 2013]

Proof idea.

- Couple two copies $\{X_t\}$, $\{Y_t\}$ of the heat-bath Markov chain, starting from arbitrary initial configurations.
- If $\Pr[X_T \neq Y_T] \leqslant 1/4$ then $T_{\text{mix}} \leqslant T$.

Identity Coupling: Use same random edge e and same uniform random number to decide if add/remove e.

Monotonicity: If $Y_t \subseteq X_t$, then $Y_{t+1} \subseteq X_{t+1}$.

Therefore, it is sufficient to consider the starting states $Y_0 = \emptyset$, $X_0 = E_n$.

Proof of Lemma 2: SM implies fast mixing (cont.)

- A union bound implies:

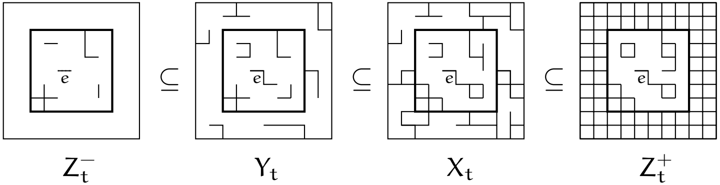
$$\Pr[X_t \neq Y_t] \leqslant \sum_{e \in E} \Pr[X_t(e) \neq Y_t(e)]$$

- Consider $\{Z_t^-\}$, $\{Z_t^+\}$ such that:

$$\emptyset = Z_0^- = Y_0 \subseteq X_0 = Z_0^+ = E_n$$

- $\{X_t\}$, $\{Y_t\}$, $\{Z_t^-\}$, $\{Z_t^+\}$ are coupled via the identity coupling.

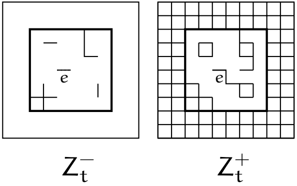
- $\{Z_t^-\}$, $\{Z_t^+\}$ ignore updates in $B^c = B^c(e, r)$ for a suitable r.



Proof of Lemma 2: SM implies fast mixing (cont.)

$$\Pr[X_t(e) \neq Y_t(e)] \;\leqslant\; \Pr[Z_t^-(e) \neq Z_t^+(e)]$$

Stationary measures:



$$\{Z_t^-\} \;\to\; \mu^{\eta}_{\mathbb{L}_n}(\,\cdot\,|\,B^c = 0\,)$$
$$\{Z_t^+\} \;\to\; \mu^{\eta}_{\mathbb{L}_n}(\,\cdot\,|\,B^c = 1\,)$$

For sufficiently large T:

$$Z_T^-(\cdot) \;\approx\; \mu^{\eta}_{\mathbb{L}_n}(\,\cdot\,|\,B^c = 0\,)$$
$$Z_T^+(\cdot) \;\approx\; \mu^{\eta}_{\mathbb{L}_n}(\,\cdot\,|\,B^c = 1\,)$$

SM with $r = O(\log n)$:

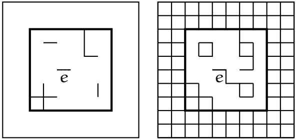
$$\left|\mu^{\eta}_{\mathbb{L}_n}(\,e=1\,|\,B^c = 1\,) - \mu^{\eta}_{\mathbb{L}_n}(\,e=1\,|\,B^c = 0\,)\right| \leqslant e^{-\Omega(r)} = O(n^{-2})$$

Proof of Lemma 2: SM implies fast mixing (cont.)

$$\Pr[X_t(e) \neq Y_t(e)] \leqslant \Pr[Z_t^-(e) \neq Z_t^+(e)]$$

Stationary measures:

$$\{Z_t^-\} \to \mu^{\eta}_{\mathbb{L}_n}(\cdot \mid B^c = 0)$$
$$\{Z_t^+\} \to \mu^{\eta}_{\mathbb{L}_n}(\cdot \mid B^c = 1)$$



Z_t^- Z_t^+

For sufficiently large T:

$$Z_T^-(\cdot) \approx \mu^{\eta}_{\mathbb{L}_n}(\cdot \mid B^c = 0)$$
$$Z_T^+(\cdot) \approx \mu^{\eta}_{\mathbb{L}_n}(\cdot \mid B^c = 1)$$

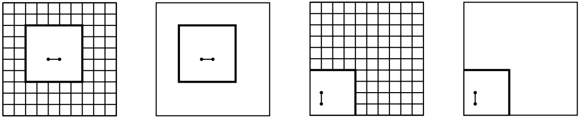
T should be large enough s.t. $\{Z_t^-\}$, $\{Z_t^+\}$ are well mixed.

SM with $r = O(\log n)$:

$$\left| \mu^{\eta}_{\mathbb{L}_n}(e = 1 \mid B^c = 1) - \mu^{\eta}_{\mathbb{L}_n}(e = 1 \mid B^c = 0) \right| \leqslant e^{-\Omega(r)} = O(n^{-2})$$

Proof of Lemma 2: SM implies fast mixing (cont.)

- $\{Z_t^-\}$ and $\{Z_t^+\}$ are "lazy" heat-bath dynamics in $B(e, r)$ with side-homogeneous boundary conditions:



- If $T_{mix}(\mathbb{L}_n) \leqslant F_0(n)$ for all side-homogeneous boundaries, then:

$$T \;=\; F_0(\log n)\log n \cdot \frac{n^2}{\log^2 n} = F_1(n)$$

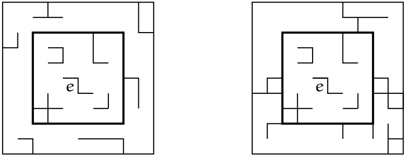
- If $F_0(n) \leqslant e^{n^2}$ [crude bound], then: $F_4(n) = O(n^2 \log n (\log\log n)^2)$

Proof of Lemma 3

Lemma 3. If $T_{\text{mix}} = O\left(n^{2+\varepsilon}\right)$, then $T_{\text{mix}} = O(n^2 \log n)$.

Proof Sketch. Establish recurrence for $\max_{e \in E} \Pr[X_t(e) \neq Y_t(e)]$

Main new ingredient: Bound the speed of propagation of disagreements.

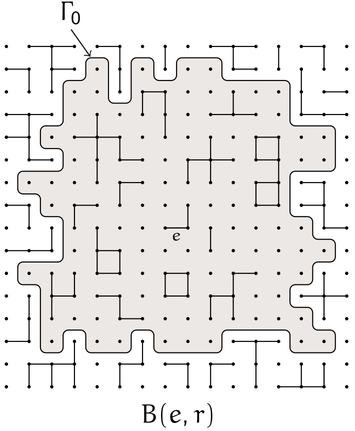


X_t Y_t

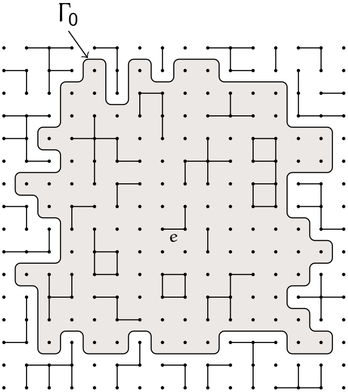
If $X_t(B(e,r)) = Y_t(B(e,r))$, how many steps until $X_T(e) \neq Y_T(e)$?

Lemma 6. If $X_0(B(e,r)) = Y_0(B(e,r))$ and $Y_0 \sim \mu^{\eta}_{\mathbb{L}_n}$, then:

$$\Pr[X_{kn^2}(e) \neq Y_{kn^2}(e)] \leqslant e^{-\Omega(r^{1/4})} \quad [\text{for } k < r^{1/4}]$$


Γ_0
e
$B(e, r)$

Speed of propagation of disagreements: Proof

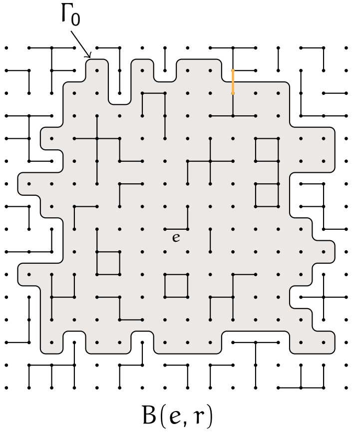


$B(e, r)$

1. If $e_t \notin \partial\Gamma_t$, then $\Gamma_{t+1} = \Gamma_t$;
2. o.w., $\Gamma_{t+1} = \Gamma_t \setminus c_t$.

Key observation: $X_t(\Gamma_t) = Y_t(\Gamma_t)$

Speed of propagation of disagreements: Proof



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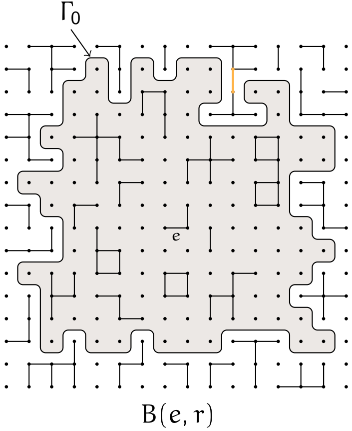

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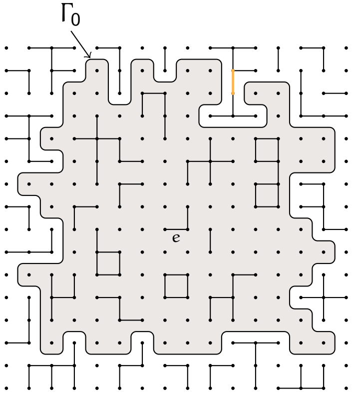
Speed of propagation of disagreements: Proof



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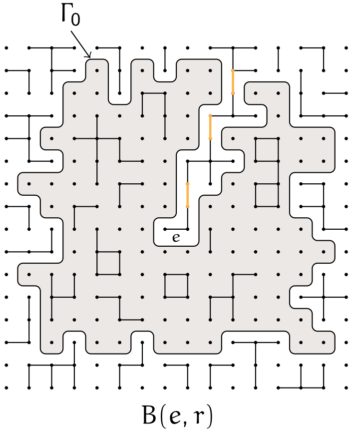
- Since $Y_t \sim \mu^{\eta}_{\mathbb{L}_n}$, EDC only small clusters in $B(e, r)$.

Speed of propagation of disagreements: Proof



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Key observation: $X_t(\Gamma_t) = Y_t(\Gamma_t)$

- Since $Y_t \sim \mu^{\eta}_{\mathbb{L}_n}$, EDC only small clusters in $B(e, r)$.
- Hence, many updates to $\partial\Gamma_t$ are required to reach e.

Open problems

Mean-field:

- Better upper bounds for the mixing time of local dynamics.
  [Should be $\tilde{O}(n^2)$, instead of $\tilde{O}(n^4)$]

$\mathbb{Z}^2$:

- Mixing time of the heat-bath at the critical point $\lambda = \lambda_c(q)$.
  [Conjecture: polynomial for $q \leqslant 4$, exponential for $q > 4$]
  [Duminil-Copin, Sidoravicius, Tassion 2015]
  [Laanait, Messager, Miracle-Solé, Ruiz, Shlosman 1991]

General Graphs:

- Dynamics for $q \in (0, 1)$?

- Analysis of dynamics in other graphs.

Thanks!