Constraints, Gadgets, and Invariants

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Relations and Functions

Let A be a finite set

Relation (k-ary): $R \subseteq A^k$, can be viewed as a function

$$R: A^k \rightarrow \{0,1\}$$

Function (k-ary): $R: A^k \rightarrow \mathbb{R}$ (for optimization)

$R: A^k \rightarrow \mathbb{R}^+$ (for partition functions)

Constraint Problems

Constraint Problems

Instance: $(V;A;\mathcal{C})$ where

CSP(Γ)

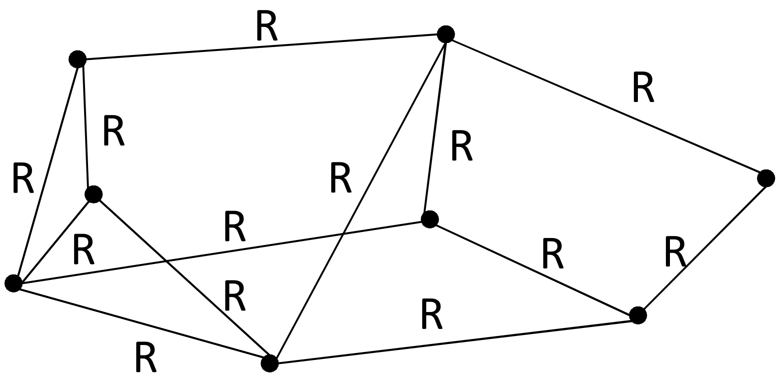
- V is a finite set of variables
- A is a set of values
- $\mathcal{C}$ is a set of constraints $\{R_1(s_1), \ldots, R_q(s_q)\}$

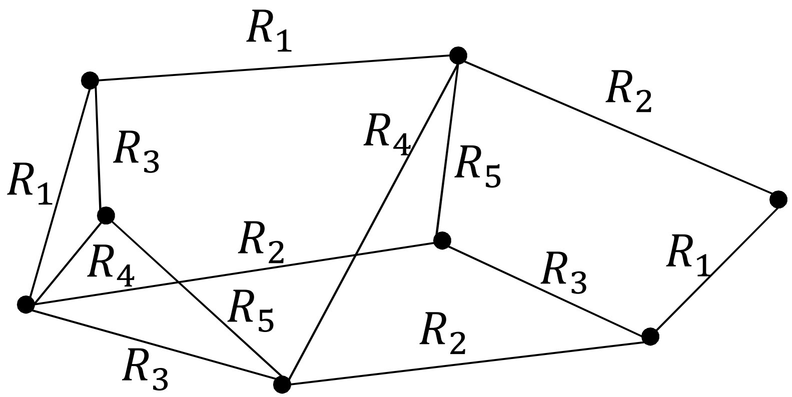
$R_1, \ldots, R_q$ can be relations on A, or

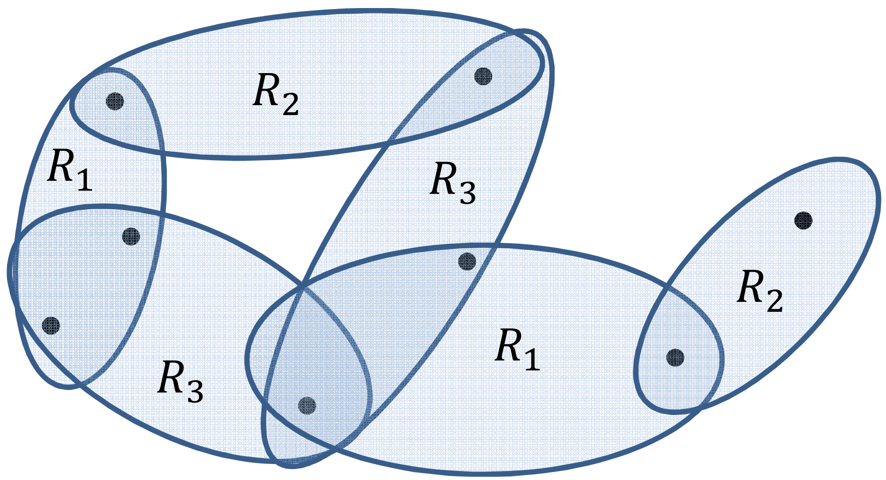
(nonnegative, real/complex) functions on A

Often assumed to be from a fixed set Γ

Constraint Problems II







Constraint Problems III

Instance: $(V;A;\mathcal{C})$ where **CSP(Γ), \$CSP(Γ), #CSP(Γ)**

- V is a finite set of variables
- $\mathcal{C}$ is a set of constraints $\{R_1(s_1), \ldots, R_q(s_q)\}$

Objective (Decision): whether there is $h: V \to A$ such that,

for any i, $R_i(h(s_i))$ is true

Objective (Optimization): find h that maximizes $\sum_i R_i(h(s_i))$

Objective (Counting): find the number of such solutions h

Objective (Partition function): find the number $\sum_h \prod_i R_i(h(s_i))$

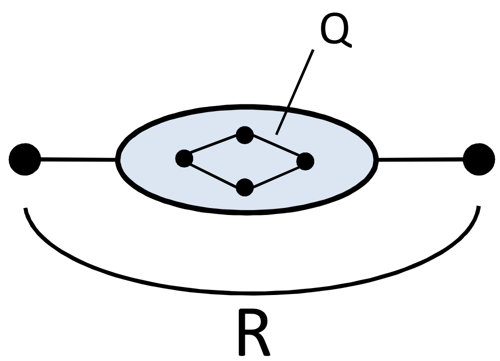
Classification

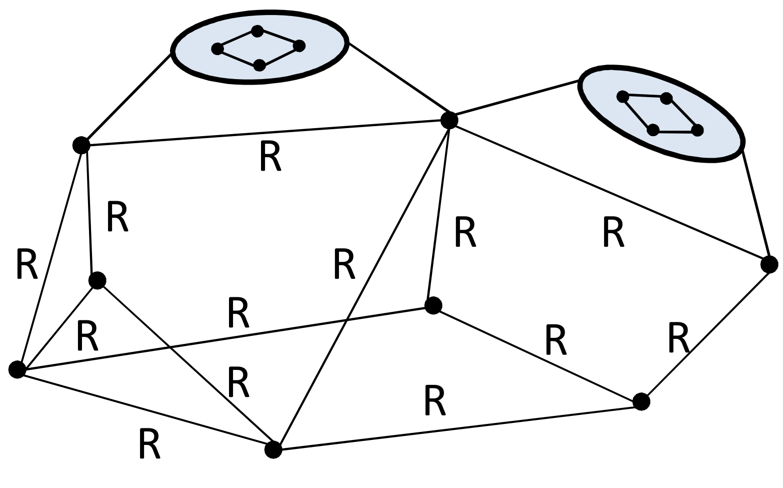
The Classification Problem: Find the complexity of CSP(Γ), \$CSP(Γ), #CSP(Γ) for every constraint language Γ

Gadgets and Reductions

Gadgets and Reductions

`express' R

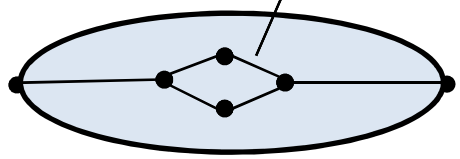




The hope is CSP(R) $\leq$ CSP(Q)

Gadgets and Reductions II

No auxiliary variables



Q is binary
R is 6-ary

Then CSP(R) $\le$ CSP(Q) (in all possible meanings)

More generally, if for every $R \in \Gamma$ there is an instance of CSP(Δ) with relations/functions $Q_1, \ldots, Q_n \in \Delta$ such that

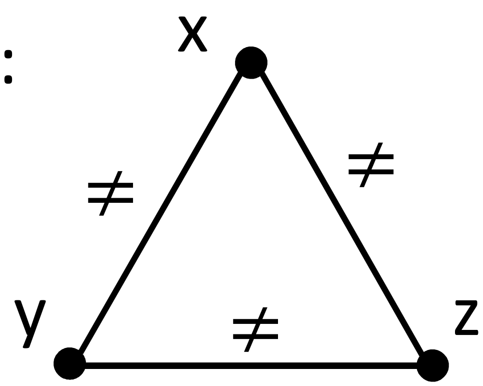
$R(\overline{x}) = Q_1(\overline{x}_1) \wedge \cdots \wedge Q_n(\overline{x}_n)$ then CSP(Γ) $\le$ CSP(Δ)

$R(\overline{x}) = Q_1(\overline{x}_1) + \cdots + Q_n(\overline{x}_n)$ then \$CSP(Γ) $\le$ \$CSP(Δ)

$R(\overline{x}) = Q_1(\overline{x}_1) \times \cdots \times Q_n(\overline{x}_n)$ then #CSP(Γ) $\le$ #CSP(Δ)

Small Example

R:



Define relation R on A

$R = \emptyset$ if $|A| = 2$

R is AllDifferent otherwise

Gadgets and Reductions III

The set of all functions/relations that can be expressed by an instance of CSP(Δ) is called the weak clone generated by Δ, and denoted $\langle\Delta\rangle$

Gadgets and Reductions: Decision

Quantification (Decision)

If for every $R \in \Gamma$ there is an instance of CSP(Δ) with relations $Q_1, \ldots, Q_n \in \Delta$ such that

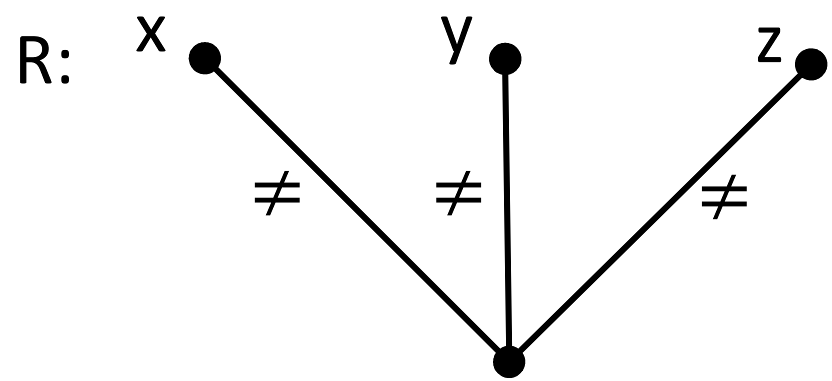
$$R(\overline{x}) = \exists \overline{y}\ Q_1(\overline{x}_1, \overline{y}_1) \wedge \cdots \wedge Q_n(\overline{x}_n, \overline{y}_n)$$

then CSP(Γ) $\leq$ CSP(Δ)

(Jeavons, et al., 1997)

The set of all functions/relations that can be expressed by an instance of CSP(Δ) + existential quantification is called the clone generated by Δ, and denoted $\langle \Delta \rangle_{\exists}$

Small Example II



Define relation R on A = {0,1,2}

R is NotAllDifferent

Gadgets & Reductions: Optimization

Optimization (Maximization):

For a constraint language Δ by $\langle\Delta\rangle_{max}$ we denote the set of functions

$$R(\overline{x}) = \max_{\bar{y}}(Q_1(\overline{x}_1, \overline{y}_1) + \cdots + Q_n(\overline{x}_n, \overline{y}_n)) ,$$

the max-clone.

If $\Gamma \subseteq \langle\Delta\rangle_{max}$, then \$CSP(Γ) $\leq$ \$CSP(Δ).

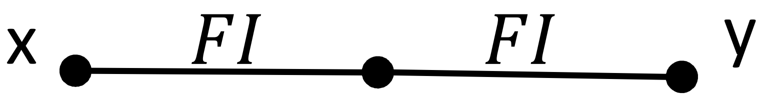
Small Example III

FerroIsing

	0	1
0	λ	1
1	1	λ

$\lambda > 1$

R:



00 $\max\{\lambda+\lambda, 1+1\}$

01 $\max\{1+\lambda, \lambda+1\}$

R:

	0	1
0	2λ	$1+\lambda$
1	$1+\lambda$	2λ

$$1 < \frac{2\lambda}{1+\lambda} < \lambda$$

Gadgets & Reductions: Counting

Counting:

For a constraint language Δ by $\langle\Delta\rangle_\Sigma$ we denote the set of functions

$$R(\overline{x}) = \sum_{\bar{y}} Q_1(\overline{x}_1, \overline{y}_1) \times \cdots \times Q_n(\overline{x}_n, \overline{y}_n),$$

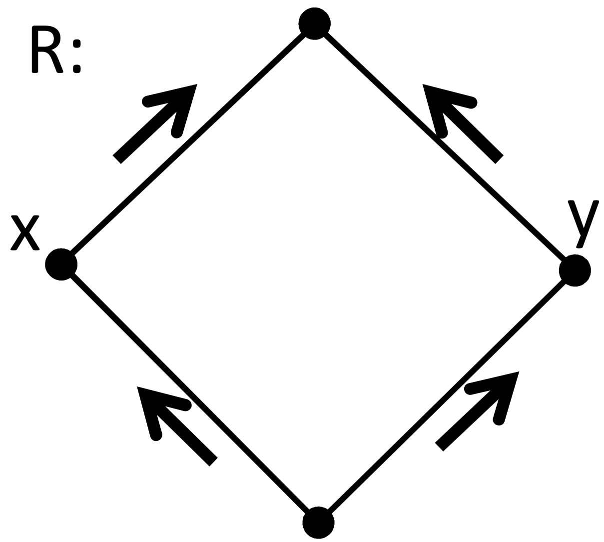
the Σ-clone

If $\Gamma \subseteq \langle\Delta\rangle_\Sigma$, then #CSP($\Gamma$) $\leq$ #CSP(Δ)

For relations: If $\Gamma \subseteq \langle\Delta\rangle_\exists$ then #CSP(Γ) $\leq$ #CSP(Δ)

(B.,Dalmau, 2003)

Small Example IV



$\rightarrow$ is implication

	0	1
0	1	1
1	0	1

Define relation R on A = {0,1}

R is FerroIsing

R:

	0	1
0	2	1
1	1	2

Polymorphisms

Polymorphisms

Operation $f(x_1, \dots, x_n)$ is a polymorphism of relation R if for any $\bar{a}_1, \dots, \bar{a}_n \in R$, it holds $f(\bar{a}_1, \dots, \bar{a}_n) \in R$

Pol(R), Pol(Γ) is the set of all polymorphisms of R, Γ

$R \in \langle \Gamma \rangle_\exists$ if and only if Pol(Γ) $\subseteq$ Pol(R)

If Pol(Γ) $\subseteq$ Pol(Δ) then CSP(Δ) $\le$ CSP(Γ)

#CSP(Δ) $\le$ #CSP(Γ)

Polymorphisms: Examples

Let R = {(0,1),(1,2),(2,0)} on A = {0,1,2} and f(x,y,z) = x – y + z.
f is a polymorphisms of R

$$f\begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; f\begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ldots$$

f(x,y,z) is a majority operation, if f(x,x,y) = f(x,y,x) = f(y,x,x) = x.
If relation R has a majority polymorphism, then $\bar{a} \in R$ if and only if every its binary projection belongs to the corresponding binary projection of R

Polymorphisms: Results

Dichotomy Conjecture for decision CSPs (~): CSP(Γ) is poly time if and only if Γ has a nontrivial polymorphism. Otherwise it is NP-complete

Exact counting: More complicated, but can be described through polymorphisms

Optimization: Multimorphisms

A multimorphism is a collection of operations $m_1, \ldots, m_n$ on A.
$m_1, \ldots, m_n$ is a multimorphism of function R on A if for any $\bar{a}_1, \ldots, \bar{a}_n$

$$R(\bar{a}_1) + \cdots + R(\bar{a}_n) \geq R(m_1(\bar{a}_1, \ldots, \bar{a}_n)) + R(m_n(\bar{a}_1, \ldots, \bar{a}_n))$$

Submodularity: $m_1 = \wedge,\ m_2 = \vee$

$$R(\bar{a}) + R(\bar{b}) \geq R(\bar{a} \wedge \bar{b}) + R(\bar{a} \vee \bar{b})$$

Optimization: Fractional Polymorphisms

Fix a set A and let O^k denote the set of all k-ary operations $m: A^n \to A$. A probability distribution μ on O^k, $\mu: O^k \to [0,1]$ is called a fractional polymorphism of function $\mathrm{R}: A^k \to \mathbb{R}$ if for any $\bar{x}_1, \dots, \bar{x}_n \in A^k$

$$E_{m\sim\mu}\left[R\left(m(\bar{x}_1, \dots, \bar{x}_n)\right)\right] \leq avg\left(R(\bar{x}_1), \dots, R(\bar{x}_n)\right]$$

Submodularity:

k = 2, $\mu(\wedge) = \mu(\vee) = \frac{1}{2}$, that is,

$$\frac{1}{2}\left(R(\bar{x}_1 \wedge \bar{x}_2) + R(\bar{x}_1 \vee \bar{x}_2)\right) \leq \frac{1}{2}(R(\bar{x}_1) + R(\bar{x}_2))$$

Optimization: Results

FPol(R), FPol(Γ) denote the set of all fractional polymorphisms of function R or constraint language Γ

$R \in \langle\Gamma\rangle_{max}$ iff FPol(Γ) $\subseteq$ FPol(R) (Zivny et al. 2009)

$CSP($\Gamma$) is polynomial time iff Γ has a `nontrivial' fractional polymorphism. Otherwise it is NP-hard.

(Thapper, Zivny, 2013, Kolmogorov et al. 2015)

Approximation: Approximation Polymorphisms

Fix a set A and let O^k denote the set of all k-ary operations $m: A^k \to A$.
A probability distribution μ on O^k, $\mu: O^k \to [0,1]$ is called an **α-approximation polymorphism** of function R: $A^k \to \mathbb{R}$ if for any $\bar{x}_1, \ldots, \bar{x}_n \in A^k$

$$\alpha \cdot E_{m \sim \mu}\left[R\big(m(\bar{x}_1, \ldots, \bar{x}_n)\big)\right] \geq avg(R(\bar{x}_1), \ldots, R(\bar{x}_k))$$

Let α_Γ be the greatest constant such that there is a \`nontrivial' α_Γ-approximation polymorphism of Γ. Then (assuming the Unique Games Conjecture) α_Γ is the approximation threshold for \$CSP(Γ). (Raghavendra, 2008)

Approximate Counting

Approximate Counting: Clones

Clones for approximate counting are $\langle\Gamma\rangle_\Sigma$ + limits = $\langle\Gamma\rangle_\omega$, that is, $R \in \langle\Gamma\rangle_\omega$ iff there are $R_1, R_2, \ldots \in \langle\Gamma\rangle_\Sigma$ such that $\lim R_k = R$

If $\Gamma \subseteq \langle\Delta\rangle_\omega$ then #CSP(Γ)$\leq_{AP}$#CSP(Δ)

Any `morphisms' for approximate counting?

Morphisms for Approximate Counting

Observation: For any constraint language Γ of rational-valued functions there is a constraint language Δ of relations (possibly on a different set) such that #CSP(Γ) $\approx$ #CSP(Δ)

Partial operation $f(x_1, \ldots, x_n)$ is a partial polymorphism of relation R if for any $\bar{a}_1, \ldots, \bar{a}_n \in R$, it holds f($\bar{a}_1, \ldots, \bar{a}_n$) belongs to R or does not exist

PPol(R), PPol(Γ) is the set of all partial polymorphisms of R, Γ

$R \in \langle\Gamma\rangle$ if and only if PPol(Γ) $\subseteq$ PPol(R)

Can We Do Better?

We need to find some sort of `morphisms' for $\langle \Gamma \rangle_{\Sigma}$ or/and $\langle \Gamma \rangle_{\omega}$

Nothing known yet, but there are options …

CWDB? I

Option 1. Does one of the existing types of `morphisms' work?

Function $f:\{0,1\}^k \to \mathbb{R}^+$ is Log-Super-Modular (LSM) if for any $\bar{x}_1, \bar{x}_2 \in \{0,1\}^k$

$$f(\bar{x}_1)f(\bar{x}_2) \leq f(\bar{x}_1 \wedge \bar{x}_2)f(\bar{x}_1 \vee \bar{x}_2)$$

FerroIsing $\in$ LSM, AntiFerroIsing $\notin$ LSM

LSM is closed under $\langle\cdot\rangle_\Sigma$ and $\langle\Gamma\rangle_\omega$

Not clear if it is true for other multimorphisms

Conservative Case

Set of operations (constraint language) Γ on A is conservative if it contains all the unary operations on A

Almost complete complexity classification of conservative constraint languages

(many people in different combinations, 2014, 2015)

CWDB? II

Option 2. Properties of Fourier coefficients?

Let $f : \{0,1\}^n \to \mathbb{R}^+$ be a function and $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$
Fourier coefficient $\hat{f}(S)$ is given by

$$\hat{f}(S) = \frac{1}{2^n} \sum_{x_1,\ldots,x_n \in \{0,1\}^n} f(x_1, \ldots, x_n)(-1)^{x_{i_1} + \cdots + x_{i_k}}$$

Let PF denote the set of functions f such that $\hat{f}(S) \geq 0$ for all S. PF is closed under $\langle \cdot \rangle_\Sigma$ and $\langle \cdot \rangle_\omega$

Some interesting constraint languages from PF and LSM

CWDB? III

Option 3. Looking for \`morphisms' w.r.t. $\langle\cdot\rangle_\omega$ is wrong.

We may want to relax the closure operator

A probability distribution μ on O^k, $\mu\colon O^k \to [0,1]$ is called a log-approximation polymorphism of function R: $A^k \to \mathbb{R}^+$ if it is a 1-approximation polymorphism of log R, that is,

$$E_{m\sim\mu}\left[\log R\big(m(x_1, \ldots, x_k)\big)\right] \geq avg(\log R(x_1), \ldots, \log R(x_k))$$

Log-Approximation Polymorphisms

If μ is a approximation polymorphism of Γ, it is a log-approximation polymorphism of any $R \in \langle\Gamma\rangle$

For any Γ $\langle\Gamma\rangle \subseteq \langle\Gamma\rangle_\omega$

Thus #CSP(Γ) $\leq_{AP}$ #CSP(Δ) whenever $\langle\Gamma\rangle \subseteq \langle\Delta\rangle$

Thank You!