Subquadratic Algorithms for Succinct Stable Matching

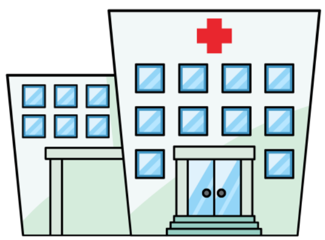
Daniel Moeller

University of California, San Diego

Joint work with Mohan Paturi and Stefan Schneider

The Stable Matching Setting

- Many situations involve matching members of two disjoint sets

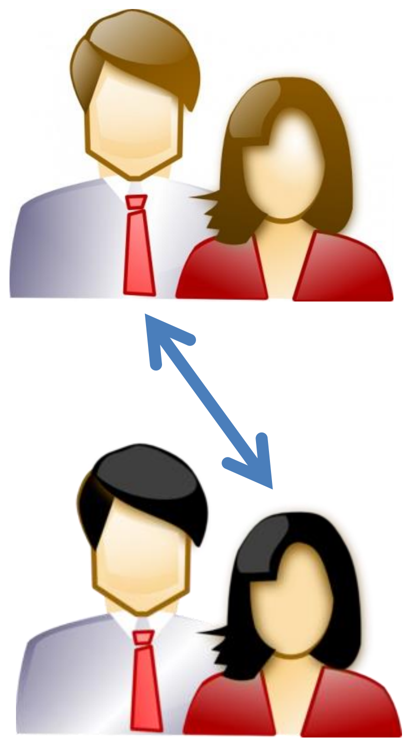




- These situations can often be in flux

The Stable Marriage Problem

- Given:
  - Two disjoint sets of participants (men/women)
  - A preference list for each participant
- Find a matching with no *blocking pairs*



Gale – Shapley

- *Deferred Acceptance Algorithm*
  - Men make proposals in order of their preference list until they are accepted
  - Quadratic time

Hardness

- Input size is quadratic
- Quadratic lower bound for finding and verifying a stable matching
  - [Ng, Hirschberg ‘90]
  - [Segal ‘07]
  - [Gonczarowski, Nisan, Ostrovsky, Rosenbaum ‘15]

Succinct Preferences

- Preferences can have structure
- It may be infeasible to list all participants
- d-list and d-attribute model
  - [Bhatnagar, Greenberg, Randall ’08]

d-Attribute Preferences

- Sometimes participants can be ranked by several attributes
  - online dating (income, height, sense of humor)
  - universities (academics, social life, sports)
- There are d fixed attributes
- Each participant's preferences are determined by the weight they place on each attribute

3-attribute Preferences Example

- m's weight vector

  $$\left(0, \frac{1}{2}, \frac{1}{2}\right)$$

- m's preference list

  $$w_1, w_3, w_4, w_2$$

Woman	Attribute Values
w_1	$(1,4,8)$
w_2	$(5,2,1)$
w_3	$(3,6,2)$
w_4	$(7,2,4)$

d-List Preferences

- Groups might share the same preferences
  - student athletes
  - sorority members
  - engineers
- d lists
  - Each participant uses one of them
  - Special case of the d-attribute setting

2-list Preferences Example

- Preference Lists

σ_1	σ_2	π_1	π_2
m_1	m_3	w_1	w_4
m_2	m_1	w_2	w_3
m_3	m_4	w_3	w_2
m_4	m_2	w_4	w_1

Man	List
m_1	π_2
m_2	π_1
m_3	π_2
m_4	π_1

Woman	List
w_1	σ_2
w_2	σ_1
w_3	σ_2
w_4	σ_2

Questions?

- Subquadratic algorithms?
- Finding a stable matching
  - Arbitrary attributes and weights
  - Small integers/Boolean
  - d-list preferences
- Verifying a stable matching
  - Arbitrary attributes and weights
  - Small integers/Boolean
  - d-list preferences

Algorithmic Results

- Finding a stable matching
  - $\tilde{O}(n^{2-1/\lfloor d/2 \rfloor})$ algorithm for the one-sided, d-attribute model
    - Strongly subquadratic for constant d
  - $O(C^{2d} n(d + \log n))$ algorithm when weights and attributes are integers from a set of size C
    - Strongly subquadratic for $d < \frac{1}{2}\log_C n$

Algorithmic Results

- Verifying a stable matching
  - $\tilde{O}(n^{2-1/2d})$ algorithm for the d-attribute model
    - Strongly subquadratic for constant d
  - $O(dn)$ algorithm for the d-list model
    - Subquadratic for $d = \mathrm{o(n)}$
  - $\tilde{O}(n^{2-1/O(c \log^2 c)})$ randomized algorithm for Boolean attributes and weights
    - Where $d = c \log n$
    - [Alman, Williams ‘15]

Hardness Result

- No strongly subquadratic algorithm for $d = \omega(\log n)$
  - Assuming the Strong Exponential Time Hypothesis (SETH)
  - Reduction from Maximum Inner Product

Hardness Sketch

(Boolean) Maximum Inner Product

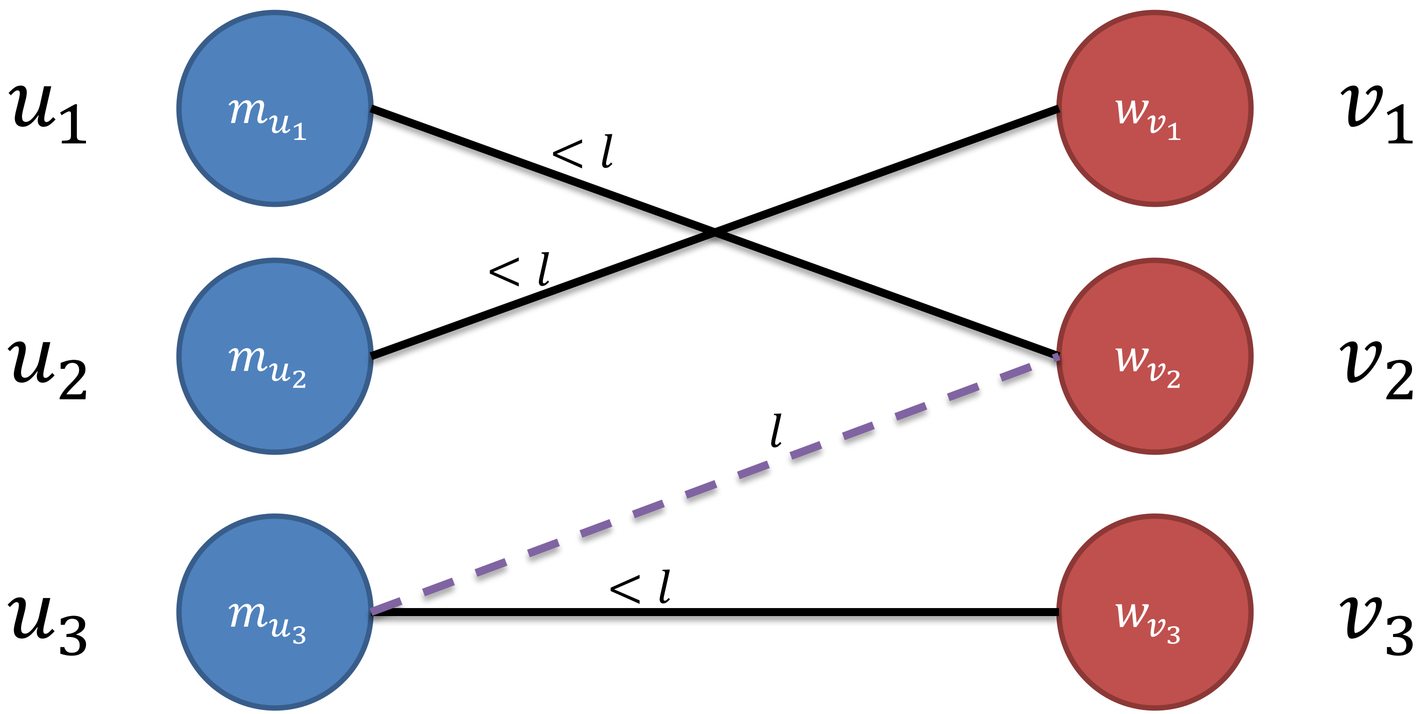
- Given:
  - sets of vectors $U, V \subseteq \{0,1\}^d$ with $|U| = |V| = n$
  - threshold l
- Decide if there is a $u \in U$ and $v \in V$ such that $\langle u, v \rangle \geq l$.

Reduction to Stable Matching

- For $u \in U$ create a man m_u with attribute values u and weight values u.
- For $v \in V$ create a woman w_v with attribute values v and weight values v.
  - Each man prefers a woman who possesses the attributes he possesses.

$$m_u \succ_{w_v} m_{u'} \Leftrightarrow \langle u, v \rangle > \langle u', v \rangle$$

Reduction to Stable Matching



$$m_u \succ_{w_v} m_{u'} \iff \langle u, v \rangle > \langle u', v \rangle$$

d-list Stability Verification Sketch

d-list Stability Verification

- Preference Lists

σ_1	σ_2	π_1	π_2
m_1	m_3	w_1	w_4
m_2	m_1	w_2	w_3
m_3	m_4	w_3	w_2
m_4	m_2	w_4	w_1

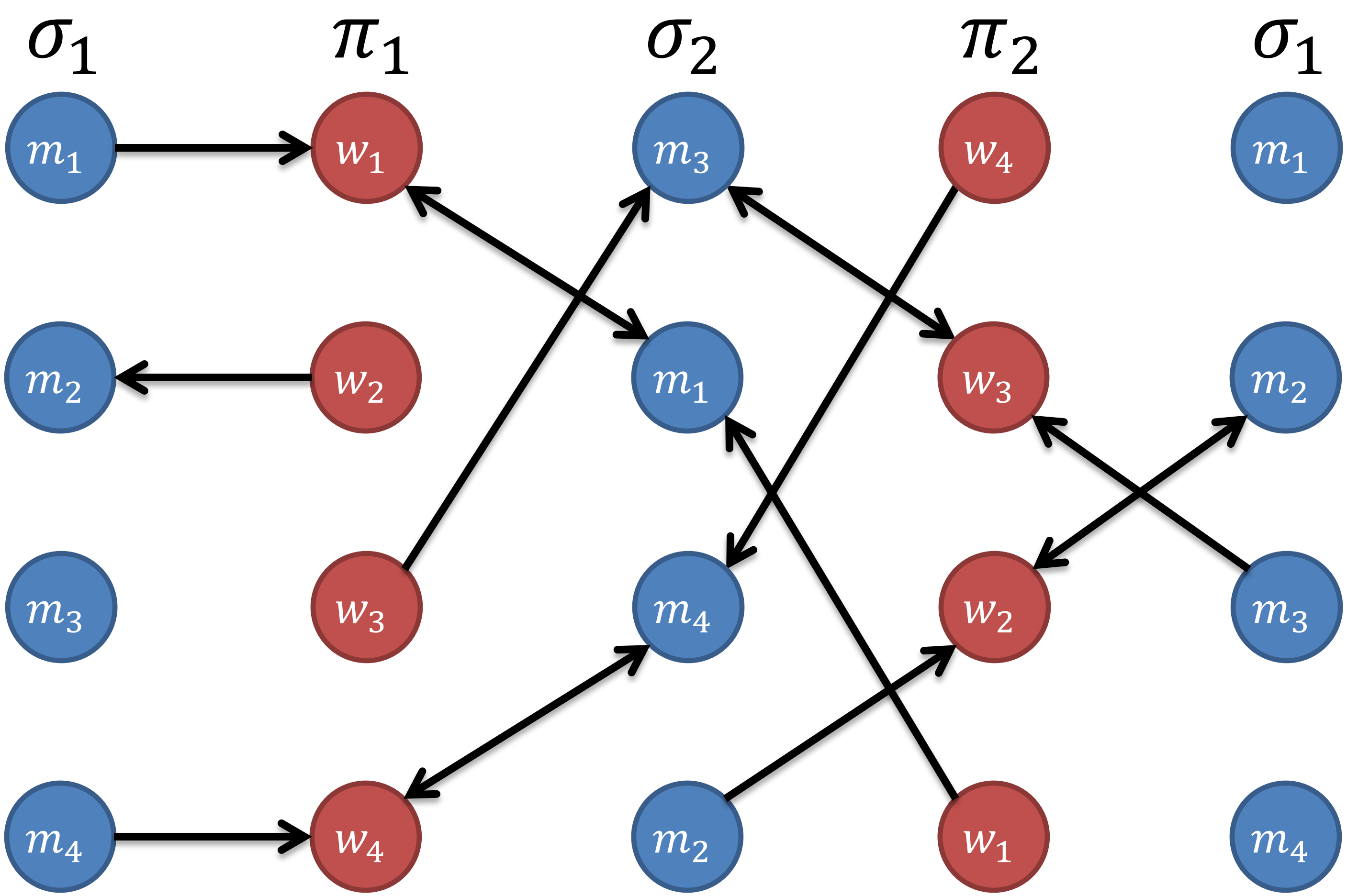
Man	List
m_1	π_2
m_2	π_1
m_3	π_2
m_4	π_1

Woman	List
w_1	σ_2
w_2	σ_1
w_3	σ_2
w_4	σ_2

- Candidate Matching

$$(m_1 w_1, m_2 w_2, m_3 w_3, m_4 w_4)$$

d-list Stability Verification

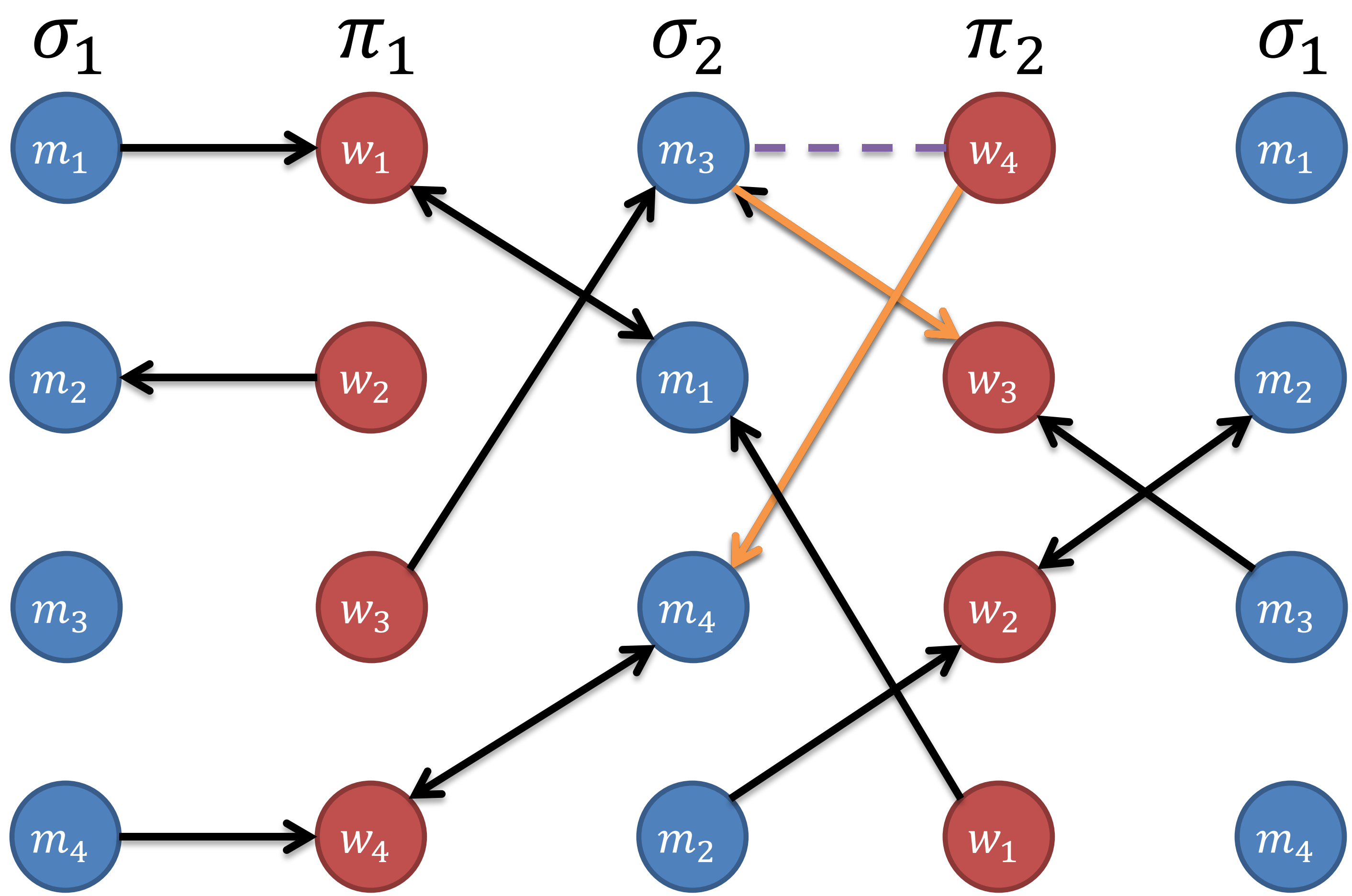


Man	List
m_1	π_1
m_2	π_2
m_3	π_2
m_4	π_1

Woman	List
w_1	σ_2
w_2	σ_1
w_3	σ_2
w_4	σ_2

$$(m_1w_1, m_2w_2, m_3w_3, m_4w_4)$$

d-list Stability Verification

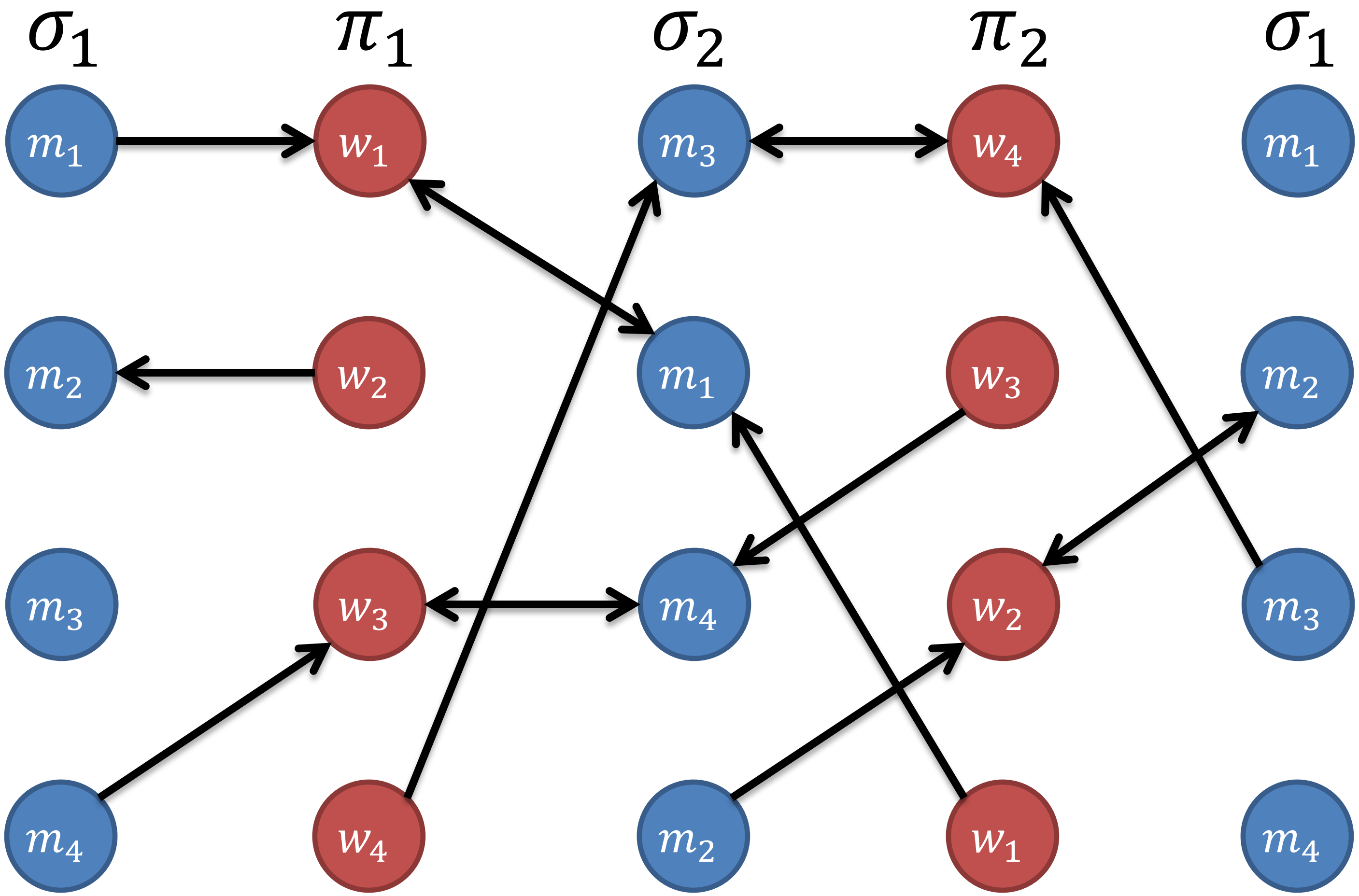


Man	List
m_1	π_1
m_2	π_2
m_3	π_2
m_4	π_1

Woman	List
w_1	σ_2
w_2	σ_1
w_3	σ_2
w_4	σ_2

Not Stable $(m_1w_1, m_2w_2, m_3w_3, m_4w_4)$

d-list Stability Verification



Man	List
m_1	π_1
m_2	π_2
m_3	π_2
m_4	π_1

Woman	List
w_1	σ_2
w_2	σ_1
w_3	σ_2
w_4	σ_2

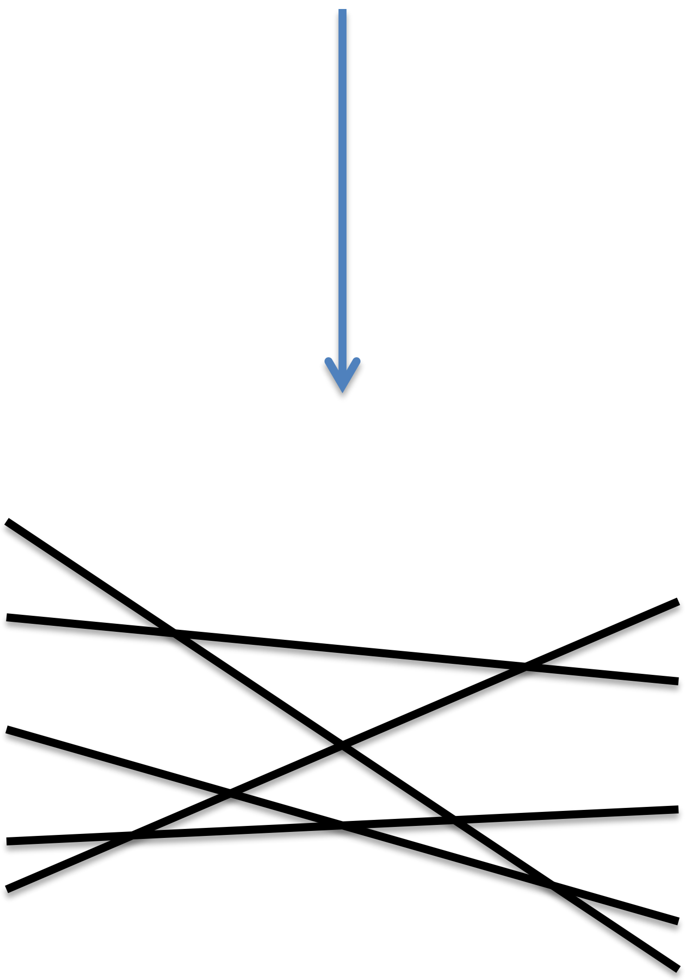
Stable

$(m_1w_1, m_2w_2, m_3w_4, m_4w_3)$

$\boldsymbol{O(dn)}$

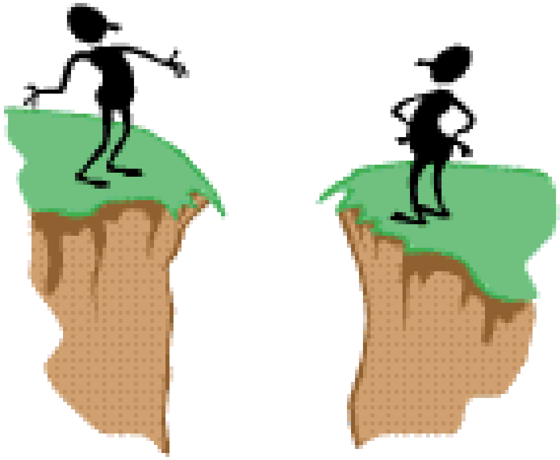
d-attribute Algorithms

- Convert to Ray-shooting
  - Dynamic data structures
    - [Matousek, Schwarzkopf '92]
- Finding a stable matching
  - $\tilde{O}(n^{2-1/\lfloor d/2 \rfloor})$
  - One-sided
- Verifying a stable matching
  - $\tilde{O}(n^{2-1/2d})$



Finding vs. Verifying

- d-attribute
  - One-sided vs. two-sided
- d-list
  - $O(n^2)$ vs. $O(dn)$



Future Directions

- Subquadratic algorithm for finding a stable matching in the full d-attribute case
  - 2-list case is still open
- Other succinct preference models
- Applying attributes to other preference markets
  - Stable Roommates
  - Housing Allocation

Thank you!