Lower bounds for the parameterized complexity of Minimum Fill-in and other completion problems

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Joint work with Ivan Bliznets, Marek Cygan, Paweł Komosa and Lukáš Mach

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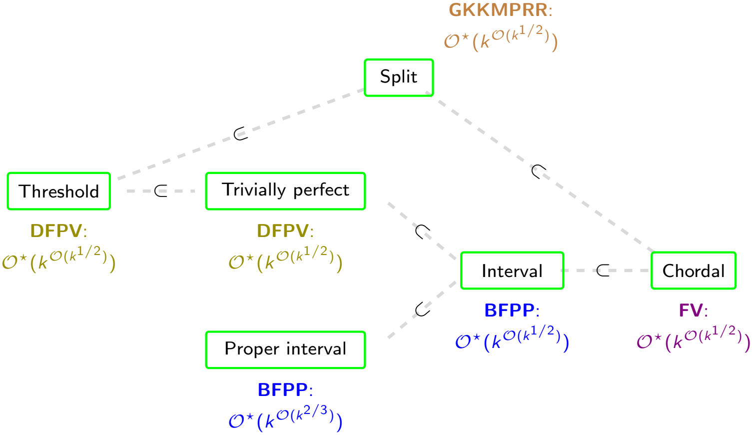
Motivation


GKKMPRR: $\mathcal{O}^\star(k^{\mathcal{O}(k^{1/2})})$
Split
Threshold
DFPV: $\mathcal{O}^\star(k^{\mathcal{O}(k^{1/2})})$
Trivially perfect
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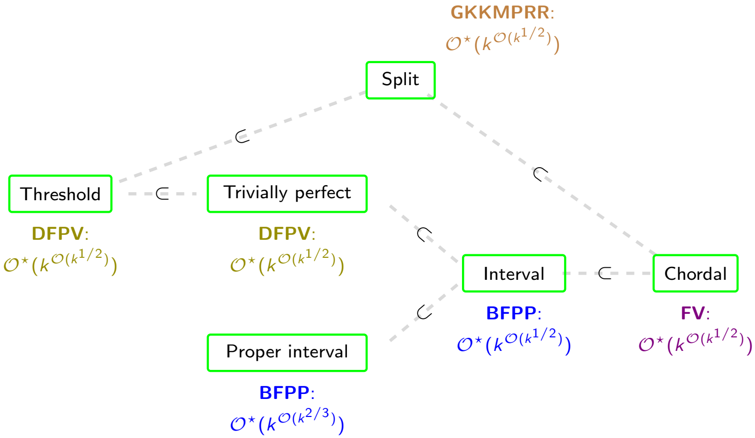

Is $\mathcal{O}^\star(2^{\tilde{\mathcal{O}}(k^{1/2})})$ the correct answer?

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- **Goal**: Prove a $2^{o(n)}$ lower bound for MINIMUM FILL-IN.

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- Same lower bounds for all the other completion problems.

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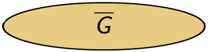
- **Note**: It can be as large as cubic.

Reduction MaxCut⇝OLA

- Complement the graph.



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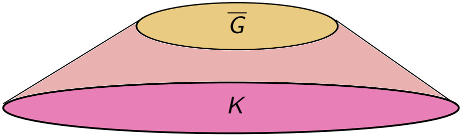

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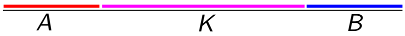

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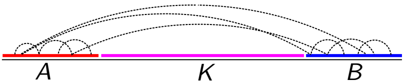

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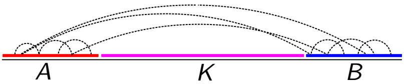

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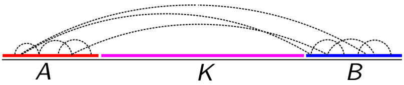

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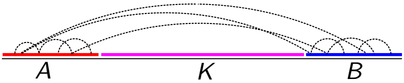

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- Maximum noise is smaller than nm, so the gap amortizes the noise.

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- **Cor**: Under ETH, there is no $2^{\mathcal{O}(n/\log^c n)}$ algorithm for OLA, for some c.

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- Let's look at the reduction OLA⇝Minimum Fill-in first.

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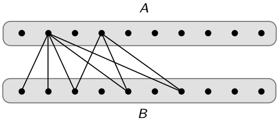

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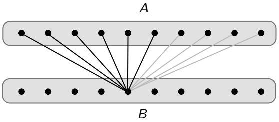

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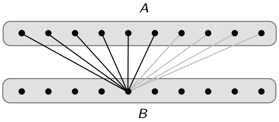

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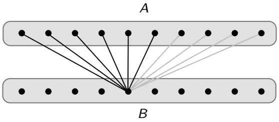

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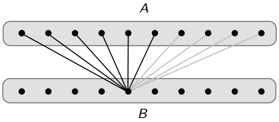

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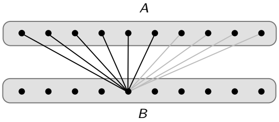

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- **Cor**: Suffices to get reduction OLA⇝Chain Completion

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- **Ergo**: Minimization of the number of fill edges is equivalent to minimization of the OLA cost.

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- **Intuition**: MinBisection on bounded degree graphs does not admit a subexponential-time approximation scheme.

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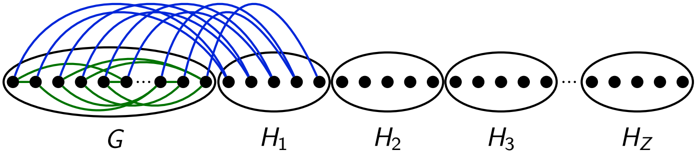

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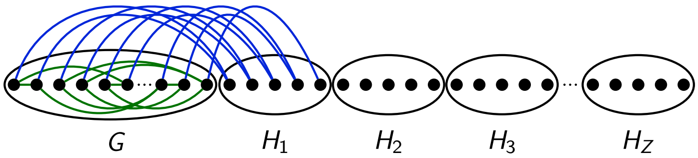

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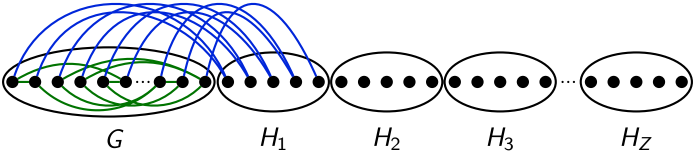

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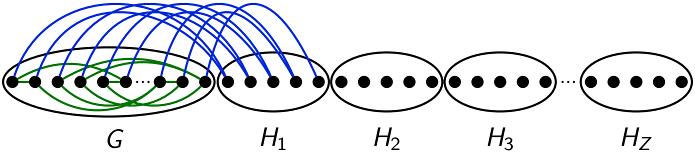

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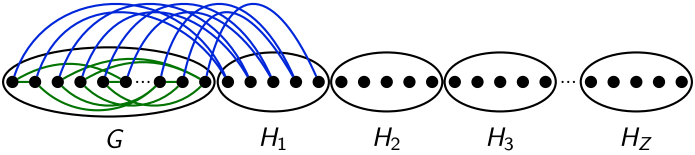

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