Lower bounds on the running time for scheduling and packing problems

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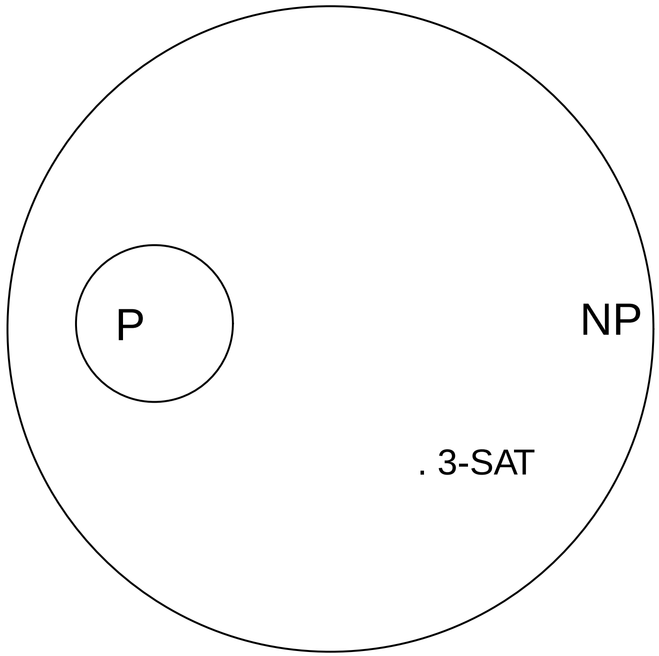
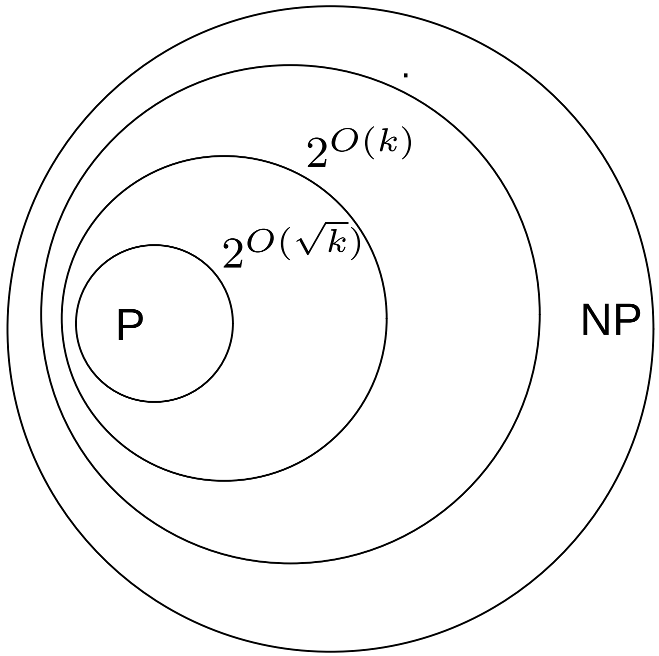
Parameterized Complexity I



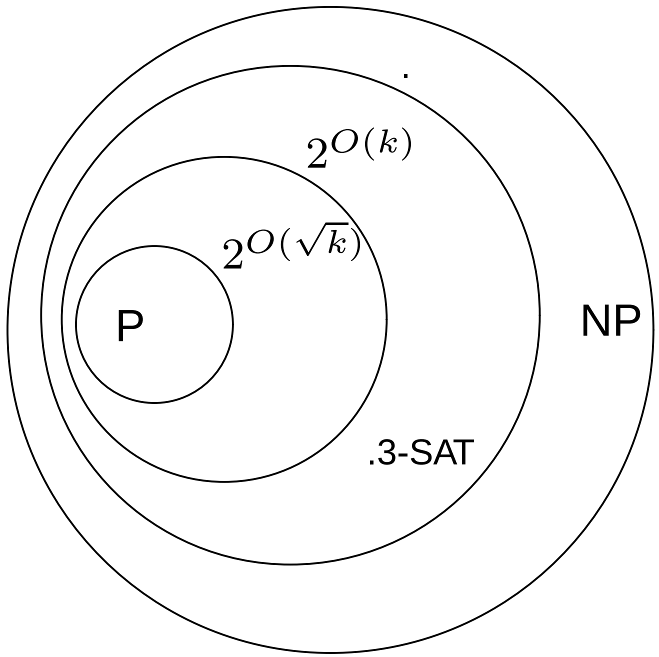
Assumption: $P \neq NP$.

Parameterized Complexity II



Express running time in term of a **parameter** k.

Parameterized Complexity III



Natural parameter k for 3-SAT: the number n of variables or m of clauses.

Conjecture

Exponential Time Hypothesis (ETH) **(Impagliazzo, Paturi, Zane 2001)** There is a positive real δ such that 3-SAT with n variables and m clauses cannot be solved in time $2^{\delta n}(n+m)^{O(1)}$.

Sparsification

The ETH assumption implies that there is no algorithm for 3-SAT **(Impagliazzo, Paturi, Zane 2001)** with n variables and m clauses that runs in time $2^{\delta m}(n+m)^{O(1)}$ for a real $\delta > 0$.

Known lower bounds

- $2^{o(n)} n^{O(1)}$ for independent set, vertex cover, dominating set and hamiltonian path,
- $2^{o(k)} n^{O(1)}$ for vertex cover (where $k = OPT(I)$),
- $f(m)||I||^{o(m)}$ for $P|prec|C_{max}$ **(Chen et al. 2006)**
- $f(\epsilon)||I||^{o(\sqrt{1/\epsilon})}$ for $2D$ vector knapsack **(Kulik, Shachnai 2010)**
- $f(m)||I||^{o(m/\log m)}$ for unary bin packing **(Jansen et al. 2013)**

Goal

Find bounds for scheduling and packing problems

- prove lower bounds based on the ETH
- find algorithms to obtain upper bounds

Best results: matching lower and upper bounds

Exact algorithms

Lower bounds

Theorem: Subset Sum, Partition, Knapsack, Bin Packing and $Pm||C_{max}$ for $m \geq 2$ cannot be solved in time $2^{o(n)}||I||^{O(1)}$, unless the ETH fails.

Matching upper bounds

- naive enumeration for Subset Sum, Partition, Knapsack.
- algorithms based on subsets of job solve many scheduling problems.

Strong reduction for Subset Sum (Wegener 2003)

Variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$.

For x_i create items t_i and f_i with

$$\begin{aligned} s(t_i) &= \textstyle\sum_{j: x_i \in C_j} 10^{n+j-1} + 10^{i-1} \\ s(f_i) &= \textstyle\sum_{j: \bar{x}_i \in C_j} 10^{n+j-1} + 10^{i-1} \end{aligned}$$

Strong reduction for Subset Sum (Wegener 2003)

For C_j create items d_j and d'_j with

$$s(d_j) = s(d'_j) = 10^{n+j-1}$$

and use a capacity B with

$$B = \sum_{j=1}^{m} 3 \cdot 10^{n+j-1} + \sum_{i=1}^{n} 10^{i-1}.$$

Reduction for $(\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2)$

$s(t_1)$	1	0	0	0	1
$s(t_2)$	0	1	0	1	0
$s(t_3)$	0	1	1	0	0
$s(f_1)$	0	1	0	0	1
$s(f_2)$	1	0	0	1	0
$s(f_3)$	0	0	1	0	0
$s(d_1)$	0	1	0	0	0
$s(d_2)$	1	0	0	0	0
$s(d'_1)$	0	1	0	0	0
$s(d'_2)$	1	0	0	0	0
B	3	3	1	1	1

Notice: there is no carry over.

Truth assignment

$s(t_1)$	**1**	0	0	0	**1**
$s(t_2)$	0	1	0	1	0
$s(t_3)$	0	**1**	**1**	0	0
$s(f_1)$	0	1	0	0	1
$s(f_2)$	**1**	0	0	**1**	0
$s(f_3)$	0	0	1	0	0
$s(d_1)$	0	1	0	0	0
$s(d_2)$	1	0	0	0	0
$s(d_1')$	0	1	0	0	0
$s(d_2')$	1	0	0	0	0
B	3	3	1	1	1

Assignment: $\phi(x_1) = \phi(x_3) = true$ and $\phi(x_2) = false$.

Truth assignment

$s(t_1)$	**1**	0	0	0	1
$s(t_2)$	0	0	0	1	0
$s(t_3)$	0	**1**	**1**	0	0
$s(f_1)$	0	1	0	0	1
$s(f_2)$	**1**	0	0	**1**	0
$s(f_3)$	0	0	1	0	0
$s(d_1)$	0	**1**	0	0	0
$s(d_2)$	**1**	0	0	0	0
$s(d'_1)$	0	**1**	0	0	0
$s(d'_2)$	1	0	0	0	0
B	3	3	1	1	1

Subset Sum solution: $A = \{t_1, t_3, f_2, d_1, d_2, d'_1\}$.

Properties of reduction

(a) 3-SAT instance is satisfiable, iff the constructed subset sum instance has a solution.

(b) constructed instance has $2n + 2m \leq 8m$ items, using $n \leq 3m$ (i.e. a strong **linear reduction**),

(c) the existence of an algorithm for Subset Sum in time $2^{o(n)}||I||^{O(1)}$ implies that 3-SAT can be decided in time $2^{o(m)}(n+m)^{O(1)}$.

Size of constructed instance

$s(t_1)$	1	0	0	0	1
$s(t_2)$	0	1	0	1	0
$s(t_3)$	0	1	1	0	0
$s(f_1)$	0	1	0	0	1
$s(f_2)$	1	0	0	1	0
$s(f_3)$	0	0	1	0	0
$s(d_1)$	0	1	0	0	0
$s(d_2)$	1	0	0	0	0
$s(d'_1)$	0	1	0	0	0
$s(d'_2)$	1	0	0	0	0
B	3	3	1	1	1

Notice: $||I|| \leq (2n + 2m + 1)(n + m) = O(m^2)$.

Further results I

Theorem: Subset Sum, Partition, Knapsack, Bin Packing and $Pm||C_{max}$ for $m \geq 2$ cannot be solved in time $2^{o(\sqrt{||I||})}$, unless the ETH fails.

Matching upper bounds

- Subset Sum, Partition **(O'Neil, Kerlin 2010)**,
- Knapsack, Bin Packing **(O'Neil 2011)**.

Further results II

Theorem: For any $\delta > 0$, there is no $2^{O(m^{1/2-\delta}\sqrt{||I||})}$ time algorithm for $Pm||C_{max}$, unless the ETH fails.

Upper bound: $2^{O(\sqrt{m\log^2(m)||I||})}$ for $Pm||C_{max}$.

Approximation schemes

Lower bounds

Theorem: There is no EPTAS for multiple knapsack (MK) with running time $2^{o(1/\epsilon)}||I||^{O(1)}$, unless the ETH fails, even for 2 knapsacks of equal capacity and when either

(i) all items have the same profit or

(ii) the profit of each item equals its size.

Upper bound for MK: $2^{O(1/\epsilon \log^4(1/\epsilon))} + ||I||^{O(1)}$ **(Jansen 2012)**.

Proof sketch I

Consider a restricted version $MK_{res}(\alpha, C)$, where

(i) I has $m = 2$ knapsacks of capacity $\frac{1}{2}s(A)$ (where $s(A)$ must be even).

(ii) $||C|| \leq ||A||^{O(1)}$,

(iii) $profit(A) \leq \alpha C n$ where $\alpha = O(1)$,

(iv) $profit(a) \geq C$ for all $a \in A$

Proof Sketch II

Idea: reduce an instance of Partition to this restricted version of MK where the sizes remain the same.

Notice: If there is a solution for Partition, then there is a packing into 2 knapsacks.

Suppose that there is an approximation scheme A_ϵ for MK that finds an $(1+\epsilon)$ solution in time $2^{o(1/\epsilon)}||I||^{O(1)}$. Set $\epsilon = 1/(\alpha n)$.

Proof Sketch III

Claim: the approximation scheme packs all items (if Partition has a solution).

$$profit(A) \leq \alpha Cn \Longleftrightarrow \frac{1}{\alpha n} profit(A) \leq C$$

If all items can be packed, then A_ϵ has profit at least

$$\tfrac{1}{1+\epsilon} OPT(I) = (1 - \tfrac{\epsilon}{1+\epsilon}) profit(A) = profit(A) - \tfrac{1}{1+\alpha n} profit(A)$$
$$> profit(A) - \tfrac{1}{\alpha n} profit(A) \geq profit(A) - C.$$

Since $profit(a) \geq C$ for all $a \in A$, there is no unpacked item.

Proof Sketch IV

Consequence: We can decide whether a partition instance admits a solution by running a $(1+\epsilon)$ approximation algorithm.

Since $profit(A) \leq \alpha C n$ and $||C|| \leq ||A||^{O(1)}$, we have $||I|| = ||A||^{O(1)}$. Using $\epsilon = 1/(\alpha n)$, the approximation scheme A_ϵ runs in time $2^{o(1/\epsilon)}||I||^{O(1)} = 2^{o(n)}||A||^{O(1)}$. This gives a contradiction.

MK with $profit(a) = 1$ for all $a \in A$

By a reduction from Partition with even $s(A)$ to MK with $profit(a) = 1$ for all $a \in A$.

Then, $profit(A) = n$ and $profit(a) \geq 1$. This means $\alpha = 1$ and $C = 1$ works. Therefore, we obtain a **instance of $MK_{res}(1, 1)$**.

Notice: If $s(A)$ is odd then we have a no-instance.

MK with $profit(a) = s(a)$ for all $a \in A$

By a reduction from Partition-ψ, where there exists a $C \in \mathbb{N}$ such that $C \leq s(a) \leq 3C$ for all $a \in A$.

The property above implies $s(A) \leq 3Cn$. Using $profit(a) = s(a)$, we get $profit(a) \geq C$ and $profit(A) \leq 3Cn$. This means $\alpha = 3$ and the value C works. We obtain a **instance of $MK_{res}(3, C)$**.

Notice: There is also no algorithm that decides Partition-ψ in time $2^{o(n)}||A||^{O(1)}$.

Other results

Theorem: There is no PTAS for 2D vector knapsack with running time $n^{o(1/\epsilon)}||I||^{O(1)}$, unless the ETH fails.

Matching upper bound: $n^{O(1/\epsilon)}||I||^{O(1)}$ **(Caprara et al. 2010)**.

Other results

Theorem: For any $\delta > 0$, there is no $2^{O((1/\epsilon)^{1-\delta})} + n^{O(1)}$ EPTAS for $P||C_{max}$, unless the ETH fails.

Upper bound: $2^{O(1/\epsilon^2 \log^3(1/\epsilon))} + ||I||^{O(1)}$ for $P||C_{max}$ and $Q||C_{max}$ **(Jansen 2010)**.

Other results

Theorem: For any $\delta > 0$, there is no $2^{O((1/\epsilon)^{1-\delta})} + n^{O(1)}$ EPTAS for $P||C_{max}$, unless the ETH fails.

Improved upper bound: $2^{O(1/\epsilon \log^4(1/\epsilon))} + ||I||^{O(1)}$ for $P||C_{max}$ and $Q||C_{max}$ **(Jansen, Klein, Verschae 2015)**.

Other results

Theorem: For any $\delta > 0$, there is no $(1/\epsilon)^{O(m^{1-\delta})} + n^{O(1)}$ FPTAS for $Pm||C_{max}$, unless the ETH fails.

Upper bound: FPTAS for $Rm||C_{max}$ with running time $(m/\epsilon)^{O(m)} + O(n)$ and $(1/\epsilon)^{O(m)} + O(n)$ for $\epsilon < 1/m$ **(Jansen, Mastrolilli 2010)**.

Summary and Open problems

For further results we refer to:

- K. Jansen, F. Land, K. Land: Bounding the running time for scheduling and packing problems, WADS 2013.
- L. Chen, K. Jansen, G. Zhang: On the optimality of approximation schemes for scheduling, SODA 2014.

Open problems:

- Show a lower bound for d dimensional vector knapsack.
- Close the gaps for MK and $P||C_{max}$.