Generalizations of the Gate Elimination Method

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Connections Between Algorithm Design and Complexity Theory
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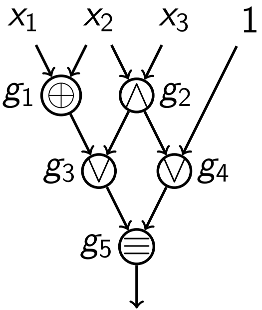
Boolean Circuits

Inputs:
$x_1, \dots, x_n, 0, 1$
Gates:
binary functions
Fan-out:
unbounded
Depth:
unbounded

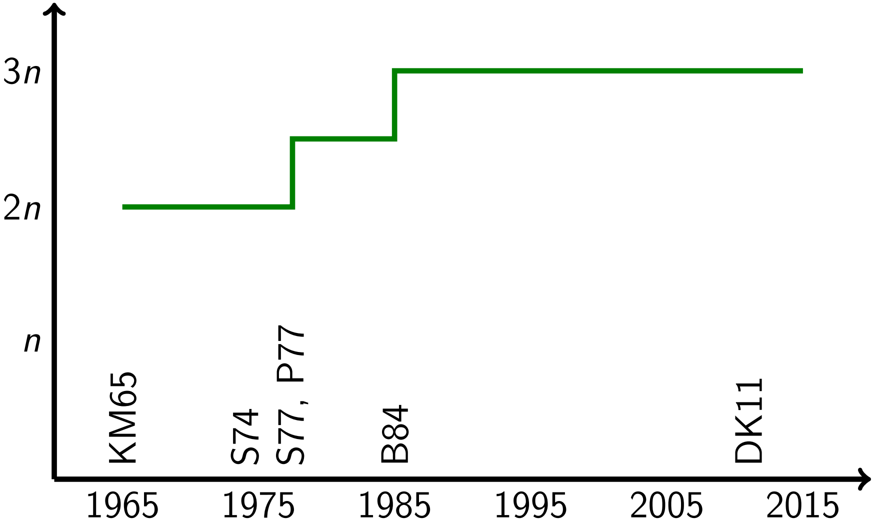
$$
\begin{aligned}
g_1 &= x_1 \oplus x_2 \\
g_2 &= x_2 \wedge x_3 \\
g_3 &= g_1 \vee g_2 \\
g_4 &= g_2 \vee 1 \\
g_5 &= g_3 \equiv g4
\end{aligned}
$$



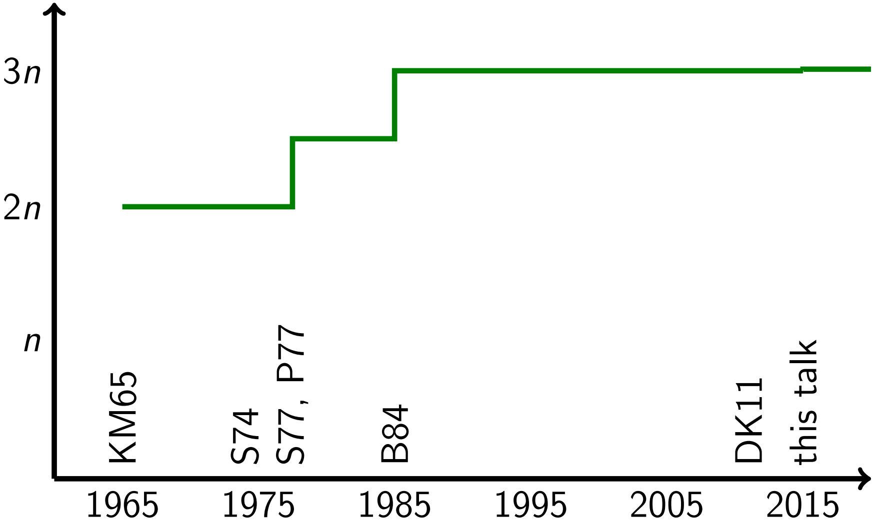
Known Lower Bounds

$2n$	$f(x) = \sum_{i<j} x_i x_j$	[Kloss, Malyshev 1965]
$2n$	$f(x) = [\sum x_i \equiv_3 0]$	[Schnorr 1974]
$2.5n$	$f(x, a, b) = x_a \oplus x_b$	[Paul 1977]
$2.5n$	symmetric	[Stockmeyer, 1977]
$3n$	$f(x, a, b, c) = x_a x_b \oplus x_c$	[Blum 1984]
$3n$	affine dispersers	[Demenkov, K 2011]
$3.011n$	affine dispersers	[this talk]
$3.11n$	quadratic dispersers (non-explicit)	[this talk]

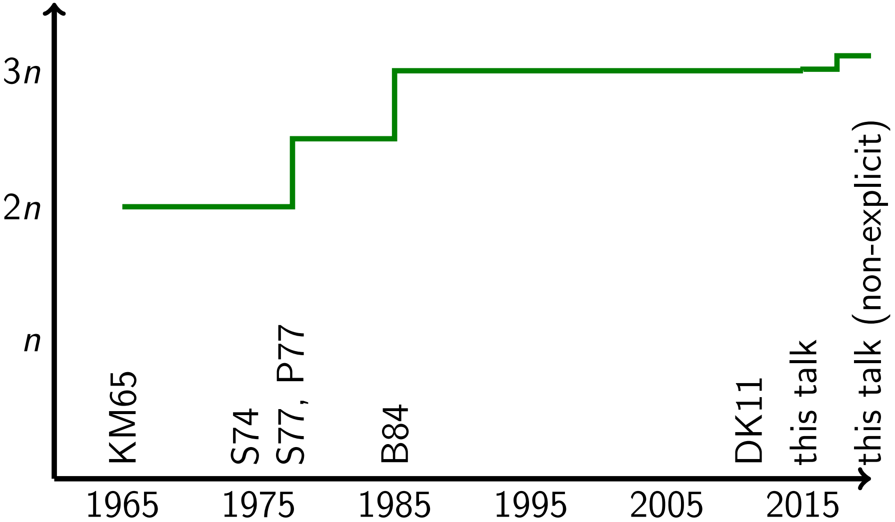
History



History


3n
2n
n
KM65
S74
S77, P77
B84
DK11
this talk
1965
1975
1985
1995
2005
2015

History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Method: Gate Elimination

To prove, say, a $3n$ lower bound for all functions f from a certain class $\mathcal{F}$:

- show that for any circuit computing f one can find a substitution eliminating at least 3 gates
- show that the resulting subfunction belongs to $\mathcal{F}$
- proceed by induction

Outline

1. $3n - o(n)$ Lower Bound for Affine Dispersers
2. $3.01n$ Lower Bound for Affine Dispersers
3. (Conditional) $3.1n$ Lower Bound for "Quadratic" Dispersers

Function: Affine Dispersers

- A function $f \in \{0,1\}^n \to \{0,1\}$ is called an affine disperser for dimension d if it is non-constant on any affine subspace of dimension at least d.
- An affine dispereser for dimension d cannot become constant after any $n - d$ linear substitutions (i.e., substitutions of the form $x_2 \oplus x_3 \oplus x_9 = 0$).
- There exist explicit constructions of affine dispersers for sublinear dimension $d = o(n)$ (e.g., [Ben-Sasson, Kopparty 2010]).

$3n - o(n)$ Lower Bound

Theorem 1 [DK11]

For a circuit C computing an A.D. for dimension d:

$$s(C) + i(C) \geq 4(n - d)\,,$$

where $i(C) = \#\text{inputs}$ and $s(C) = \#\text{gates}$.

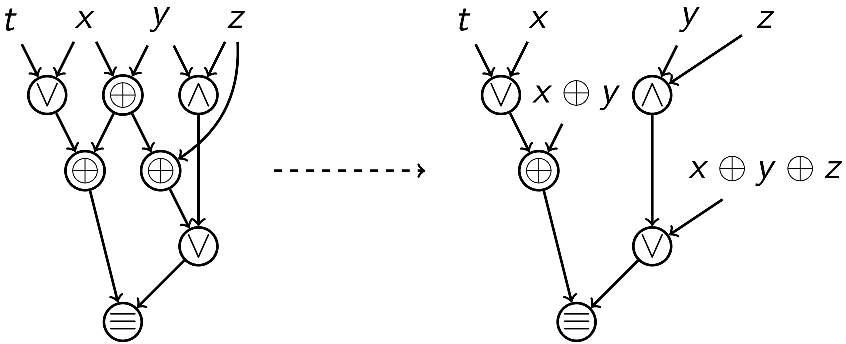
Corollary

$C(f) \geq 3n - o(n)$ for an A.D. for $d = o(n)$.

Observation

The bound is tight: $C(IP) = n - 1$ and IP is an A.D. for $d = n/2 + 1$.

XOR-layered Circuits



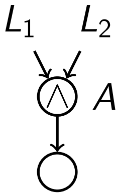
$n = 4$ inputs
$s = 7$ gates

$n = 6$ inputs
$s = 5$ gates

Proof

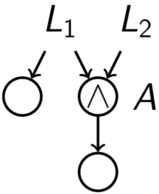
- Need to show that $s(C) + i(C) \geq 4(n - d)$.
- For this, make $n - d$ affine restrictions each time reducing $s + i$ by at least 4.
- Convert C to XOR-layered and take a top-gate A:

Case 1



$L_1 \leftarrow 0$:
$\Delta s = 2$
$\Delta i = 2$

Case 2



$L_1 \leftarrow 0$:
$\Delta s = 3$
$\Delta i = 1$

Outline

$3.01n$ Lower Bound

Theorem 2 [FGHK15]

The circuit complexity of an affine disperser for sublinear dimension is at least

$$\left(3 + \frac{1}{86}\right) n - o(n)\,.$$

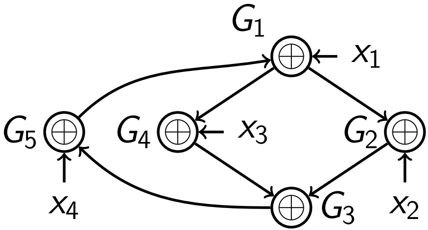
Main Ingredients of the Proof

Delayed linear substitutions: we make substitutions like $x_3 \leftarrow 0$, $x_5 \leftarrow x_7 \oplus x_{10} \oplus 1$, and $x_3 \leftarrow x_4 x_7$. For each quadratic substitution of the form $x_3 \leftarrow x_4 x_7$ we will later assign either x_4 or x_7 a constant making this quadratic substitution linear.

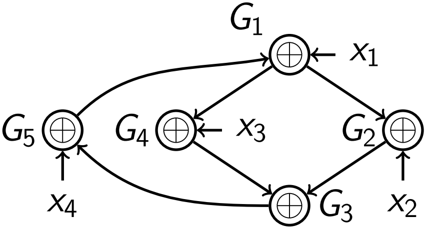
Cyclic circuits: for the induction to go through, we consider a more general model — circuits with cycles.

Circuit complexity measure: we use a carefully chosen circuit complexity measure to estimate the progress of gate elimination.

Cyclic Circuits

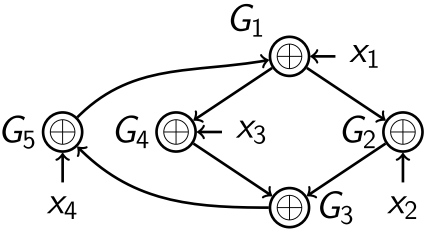


Cyclic Circuits



$$
\begin{aligned}
G_1 &= x_1 \oplus G_5 \\
G_2 &= x_2 \oplus G_1 \\
G_3 &= G_2 \oplus G_4 \\
G_4 &= x_3 \oplus G_1 \\
G_5 &= x_4 \oplus G_3
\end{aligned}
$$

Cyclic Circuits



$$
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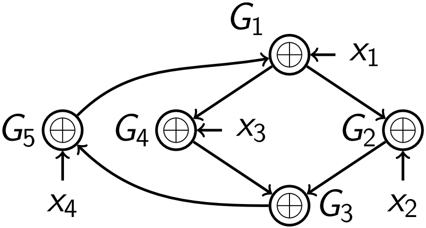
G_2 &= x_2 \oplus G_1 \\
G_3 &= G_2 \oplus G_4 \\
G_4 &= x_3 \oplus G_1 \\
G_5 &= x_4 \oplus G_3
\end{aligned}
\qquad
\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \\ G_5 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ x_3 \\ x_4 \end{bmatrix}
$$

Cyclic Circuits



$$
\begin{aligned}
G_1 &= x_1 \oplus x_2 \oplus x_3 \oplus x_4 \\
G_2 &= x_1 \oplus x_3 \oplus x_4 \\
G_3 &= x_2 \oplus x_3 \\
G_4 &= x_1 \oplus x_2 \oplus x_4 \\
G_5 &= x_2 \oplus x_3 \oplus x_4
\end{aligned}
$$

Circuit Complexity Measure

$$\mu = s + \frac{65}{43} \cdot q + \frac{1}{43} \cdot b + \frac{260}{43} \cdot i$$

where

- s is the number of gates
- q is the number of quadratic substitutions
- b is the number of “bottleneck” gates in the circuit
- i is the number of inputs

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From Affine to Quadratic Dispersers

Theorem 3 [GK15]

Let $f\colon \{0,1\}^n \to \{0,1\}$ be a function that is not constant on any set $S \subseteq \{0,1\}^n$ of size at least $2^{n/100}$ that can be defined as

$$S = \{x\colon p_1(x) = \dots = p_{2n}(x) = 0\},\ \deg(p_i) \le 2.$$

Then

$$C(f) \geq 3.1n\,.$$

Open problem

Explicit construction of such f (even in NP, even with $o(n)$ outputs).