# A compression algorithm for AC$^0[p]$ circuits using Certifying Polynomials

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  - Input: $f : \{0,1\}^n \to \{0,1\}$, $k$.
  - Qn: Does $f$ have a circuit of size at most $k$?

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- (Chen et al.) Compression algorithms imply circuit lower bounds.

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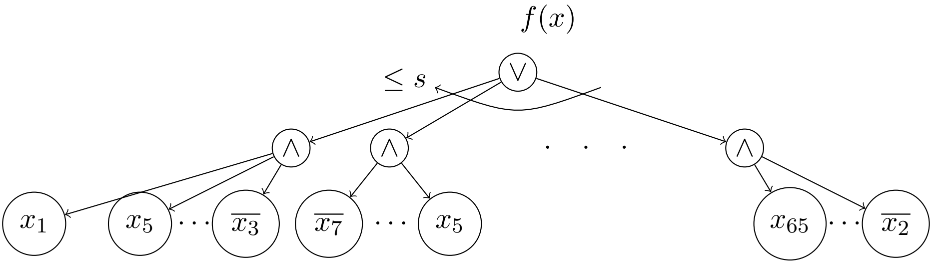

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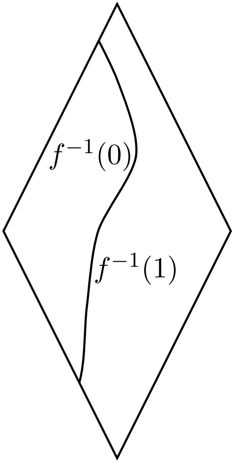

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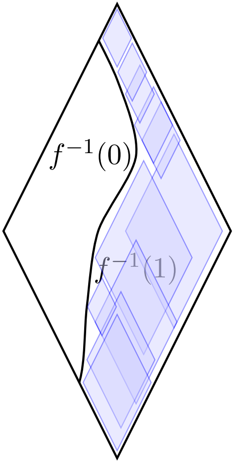

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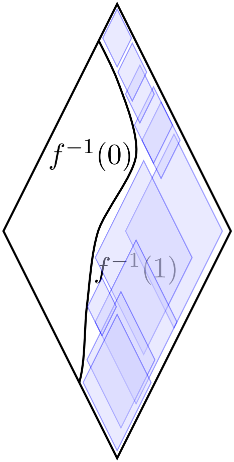

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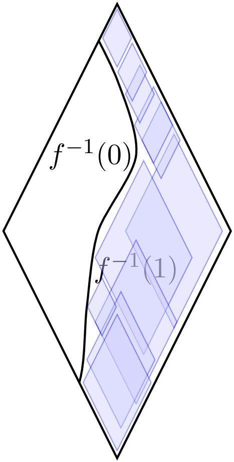

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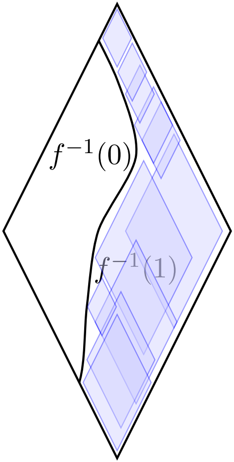
- (Lovász 1975) $O(n)$-approximation in time $2^{O(n)}$.



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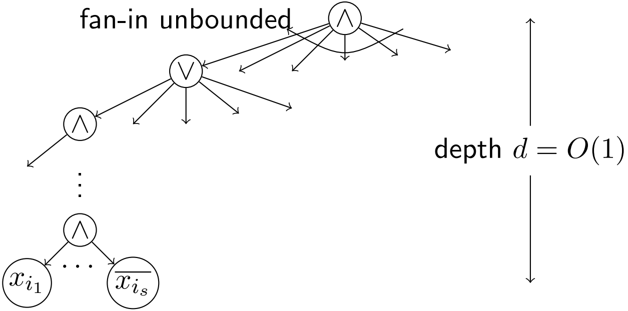

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- Further algorithms using memoization.
- General formulas, branching programs.

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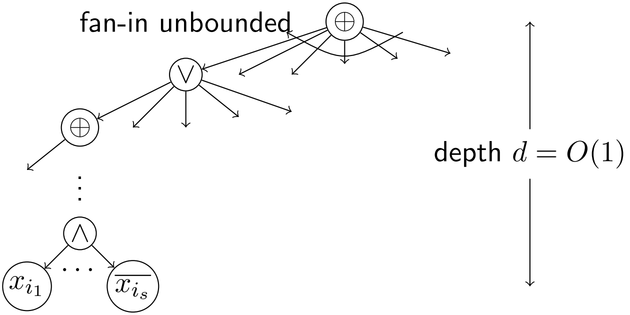
- Natural next question: AC$^0[2]$: AC$^0$ augmented with $\oplus$ gates.



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Also works for $\mathsf{AC}^0[p]$ ($p$ prime).

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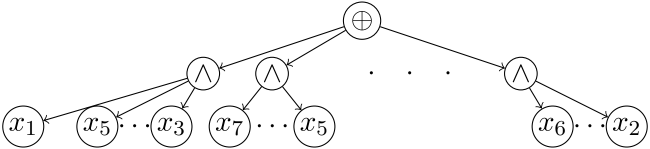

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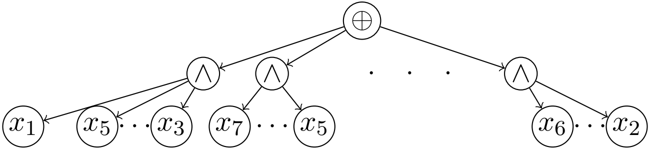

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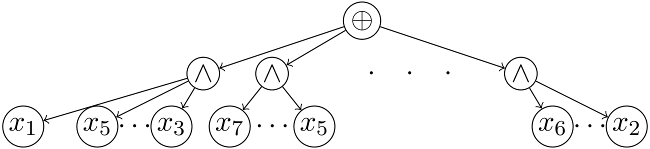
- Say that $P$ $\varepsilon$-approximates $f$ if $\Pr_{x \in \{0,1\}^n}[P(x) \neq f(x)] \le \varepsilon$.



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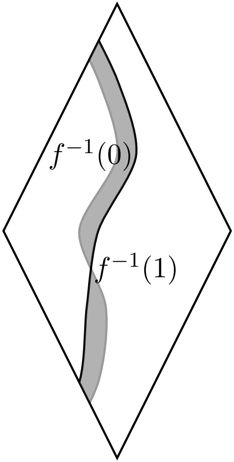

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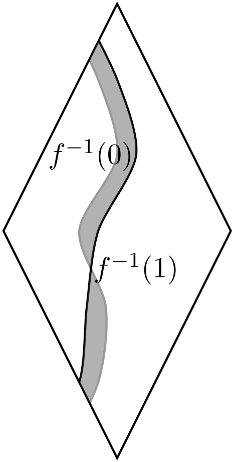

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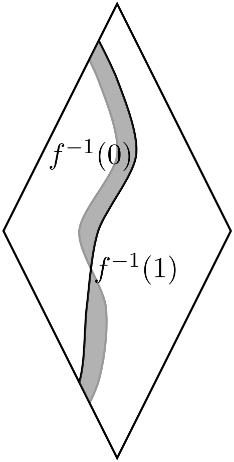

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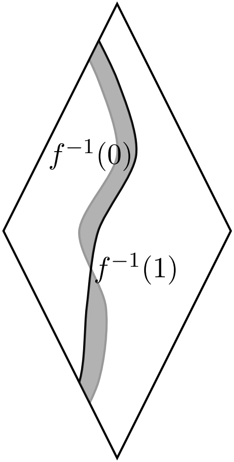

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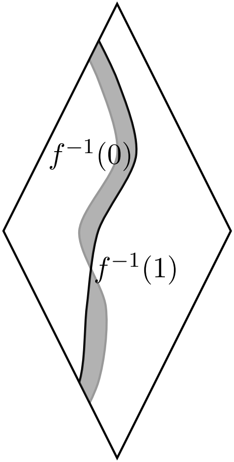

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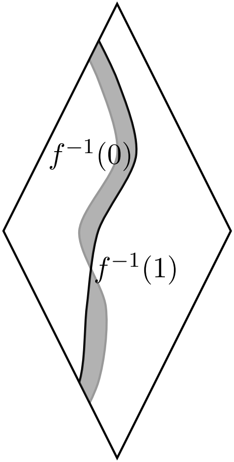
- Bottleneck: How to find $P$?



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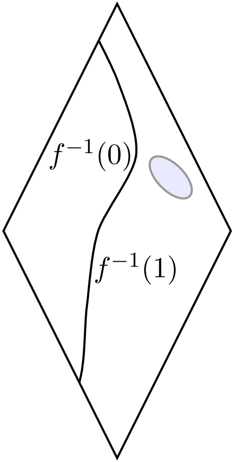

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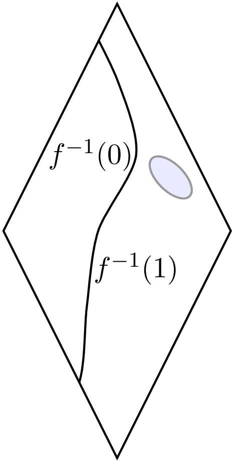

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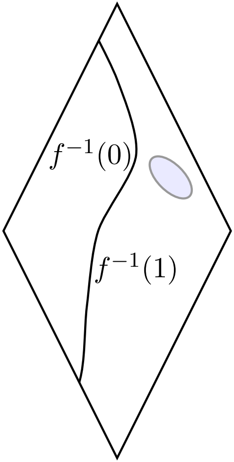

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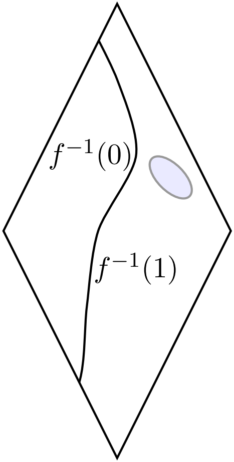

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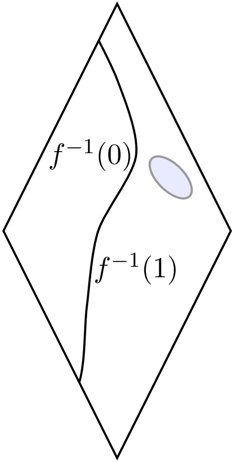

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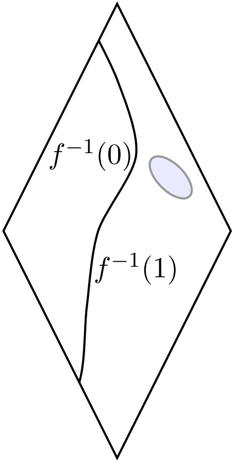

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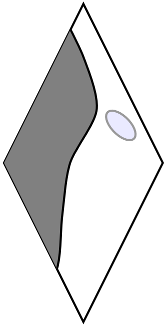

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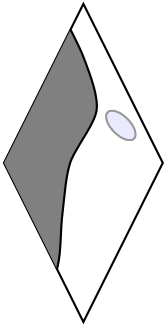

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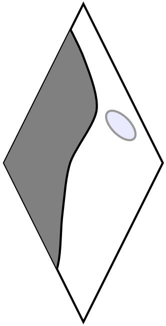

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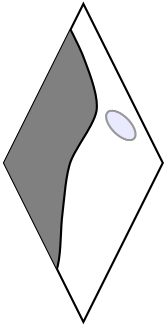
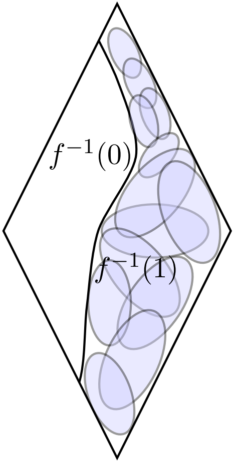


# Using $V_D$ to compress

- Each $R \in V_D$ covers a subset of $f^{-1}(1)$.
- Select a few $R_1, \ldots, R_m \in V_D$ such that $\bigvee_i R_i = f$.



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- There are points in $f^{-1}(1)$ that are never covered by $R \in V_D$.
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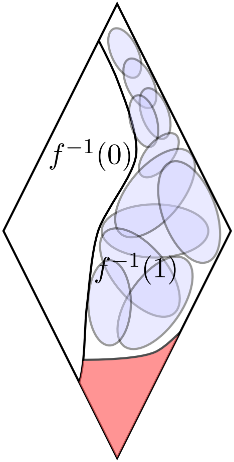

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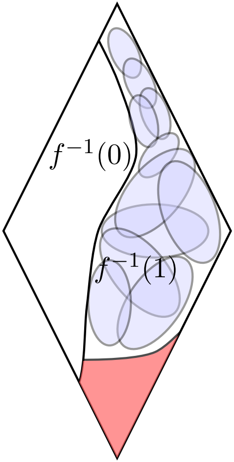

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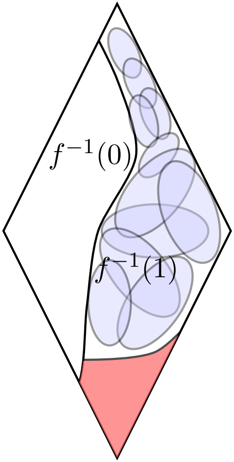

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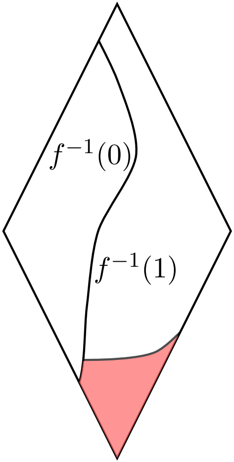

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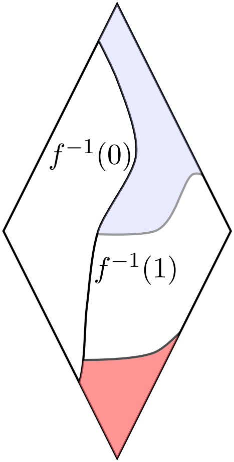

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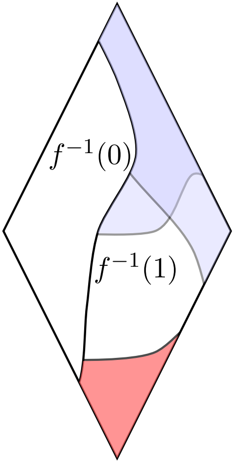

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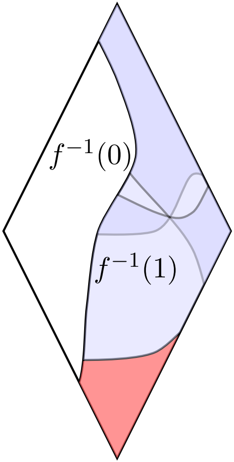

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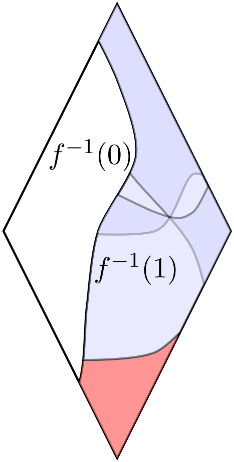

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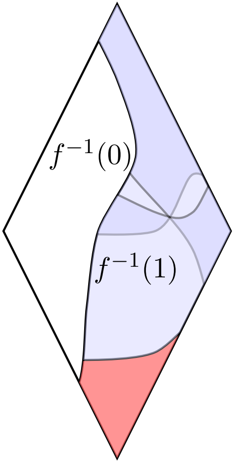

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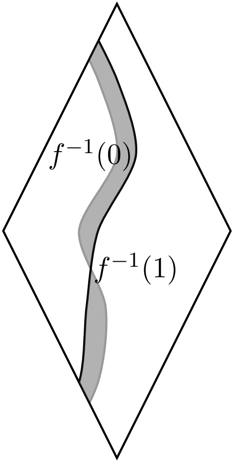

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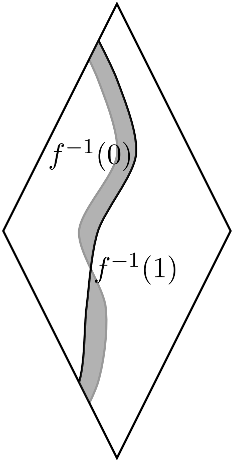

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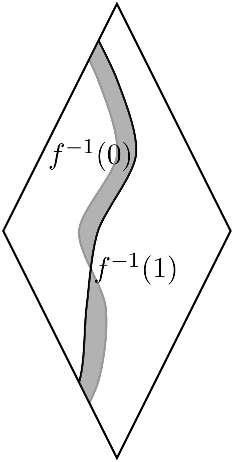

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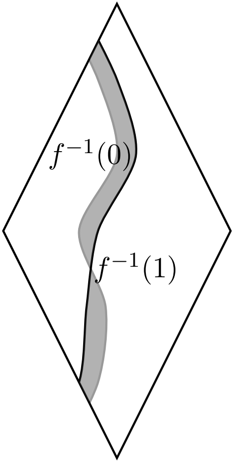

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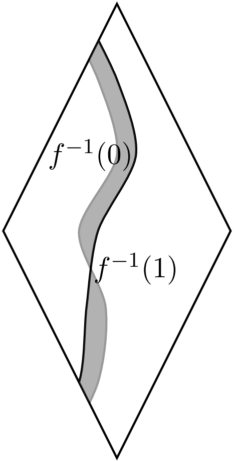

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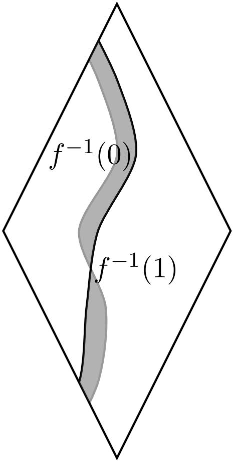

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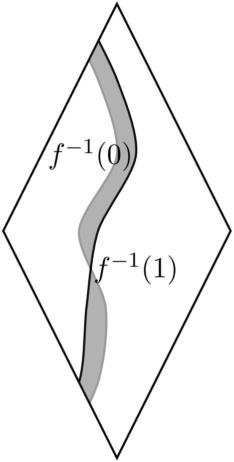

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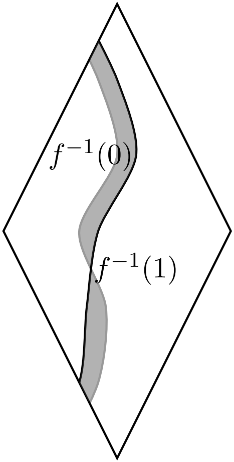

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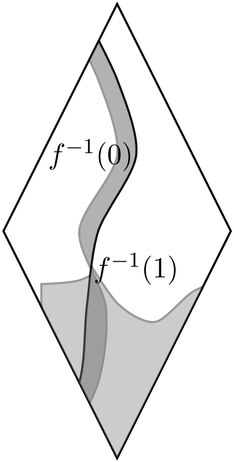

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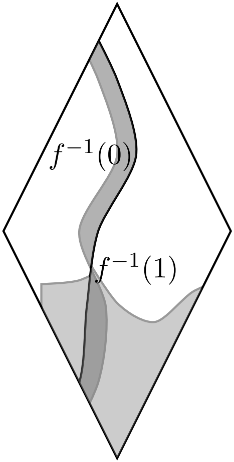

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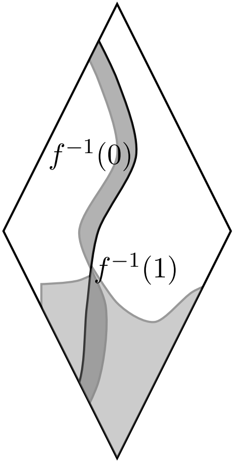

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- How large can $|F'|$ be?

**Theorem (Nie-Wang 2014)**

$$\frac{|F'|}{2^n} \leq \frac{|E|}{\binom{n}{\leq D_2}}.$$

Choose $D_2$ so that $\binom{n}{\leq D_2} = \sqrt{\varepsilon} 2^n$.

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