Derandomization via Robust Algebraic Circuit Lower Bounds

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based on various work with Amir Shpilka

(I am on the job market this year)

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Theme

Question

*When do lower bounds for a circuit class $\mathcal{C}$ yield efficient **deterministic** algorithms for deciding properties of circuits from $\mathcal{C}$?*

Motivations

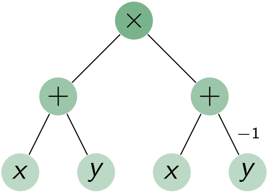
- *Derandomize algorithms*
- *$\mathcal{C}$ may be too weak for derandomization via hardness versus randomness paradigm*
- *Better understanding*

Polynomial Identity Testing (PIT)

Question (Polynomial Identity Testing (PIT))

Given a polynomial $f \in \mathbb{Q}[x_1, \ldots, x_n]$, is $f \not\equiv 0$?

Polynomials are given by a small computational device, e.g.
$x^2 - y^2 = (x + y)(x - y)$ can be given by



Polynomial Identity Testing (ii)

Lemma (Schwartz-Zippel)

$f \not\equiv 0$ iff $f(\vec{\alpha}) \neq 0$ for random $\vec{\alpha}$.

- Gives a randomized algorithm for testing if $f \not\equiv 0$, only uses f as a **black-box**. Deterministic algorithms?
- **hitting set** ($\equiv$ **black-box PIT**): set of points $\mathcal{H} \subseteq \mathbb{Q}^n$ such that

$$f \not\equiv 0 \text{ iff } f|_{\mathcal{H}} \not\equiv 0, \text{ for computationally simple } f.$$

- Non-constructively $|\mathcal{H}| =$ small, constructively?

Depth-3 Powering Formulas

Algebraic formulas typically use addition ($\sum$) and multiplication ($\prod$), but we can also use addition ($\sum$) and **powering** ($\bigwedge$)

$$xy = \frac{1}{4}\left((x+y)^2 - (x-y)^2\right) \ ,$$

Have equivalence for arbitrary formulas, but not for low-depth.
A **depth-3 powering formula** ($\sum \bigwedge \sum$) is a sum of powers of linear forms

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{s} (\alpha_{i,0} + \alpha_{i,1}x_1 + \cdots + \alpha_{i,n}x_n)^{d_i} \ , \qquad \alpha_{i,j} \in \mathbb{Q} \ .$$

$\sum \bigwedge \sum$ are a moral analogue of CNFs/DNFs from boolean complexity.

Lower Bounds for Depth-3 Powering Formulas

Theorem (NisanWigderson96,Kayal08,Sylvester1851)

The monomial $x_1 \cdots x_n$ requires $2^{\Omega(n)}$ size to be computed as a $\sum \bigwedge \sum$ formula.

Theorem (F-Shpilka13)

If $f(x_1, \ldots, x_n) = \sum_{\vec{a}} \alpha_{\vec{a}} x_1^{a_1} \cdots x_n^{a_n}$ is computed by a size-s $\sum \bigwedge \sum$ formula, then f is determined by its restriction to monomials involving $O(\log s)$ variables. This implies a $\mathrm{poly}(n, s)^{O(\lg s)}$ size hitting set.

This is proven by making the above lower bound **robust**.

Lower Bounds for Depth-3 Powering Formulas (ii)

The lower bound follows from a *complexity measure* argument. For $f \in \mathbb{Q}[x_1, \ldots, x_n]$ define

$$\mu(f) := \dim \operatorname{span} \left\{ \frac{\partial}{\partial_{x_1}^{a_1} \cdots \partial_{x_n}^{a_n}} f \right\}_{a_1, \ldots, a_n \geq 0}$$

facts:

- $\mu(f + g) \leq \mu(f) + \mu(g)$.
- $\mu\left((\alpha_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n)^d\right) \leq d + 1$.
- $\mu(x_1 \cdots x_n) = 2^n$.

$\Longrightarrow$ $x_1 \cdots x_n$ needs size $2^{\Omega(n)}$ to be computed as

$$x_1 \cdots x_n = \sum_{i=1}^{s} (\alpha_{i,0} + \alpha_{i,1} x_1 + \cdots + \alpha_{i,n} x_n)^{d_i} \ ,$$

if $d_i \leq \operatorname{poly}(n)$.

Lower Bounds for Depth-3 Powering Formulas (iii)

lower bound: small $\sum \bigwedge \sum$ formula cannot *exactly* equal large monomials. Approximately?

$$(x_1 + \cdots + x_n)^n = x_1 \cdots x_n + \cdots + x_1^n + \cdots + x_n^n + \cdots \ .$$

Express $f \neq 0$ as

$$f = \alpha x_1^{a_1} \cdots x_n^{a_n} + \text{lower order terms} \ ,$$

where monomials are ordered lexicographically with $x_1 \succ \cdots \succ x_n$

fact: $\mu(f) \geq \mu(x_1^{a_1} \cdots x_n^{a_n})$ — the measure μ is robust

$\implies$ the leading monomial of a small $\sum \bigwedge \sum$ formula involves few variables [**F**-Shpilka13]

$\implies$ quasipolynomial time deterministic blackbox PIT for $\sum \bigwedge \sum$

Conclusions

Robust Lower Bound:

$$\mu(\text{extrema}(f) + \text{lower terms}) \geq \mu(\text{extrema}(f)) \; .$$

Other PIT via Robust Lower Bounds:

- [SV09]: Read-Once Formula
- [**F**S12]: (commutative) read-once algebraic branching programs
- [MRS14,**F**15]: sums of powers of low-degree polynomials
- [GKST15,**F**15]: sparse polynomials

Open questions:

- polynomial-size hitting set for $\sum \bigwedge \sum$ formula? best known is $s^{\mathcal{O}(\lg \lg s)}$ for size s [**F**SS14]

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