Multi-dimensional and Non-linear Mechanism Design (and Approximation)

Part I: Multi- to Single-agent Reductions

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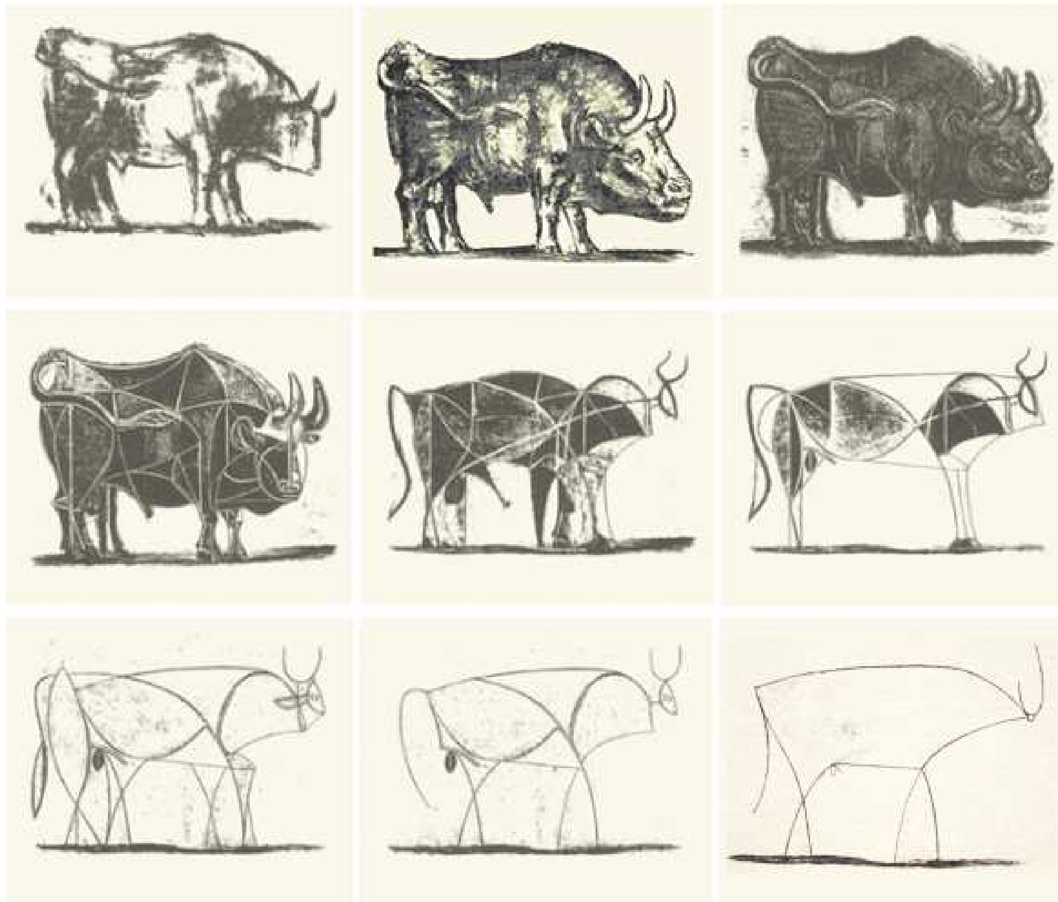
Northwestern University

August 26, 2015

Textbook: Mechanism Design and Approximation



Chapter 8: Multi-dimensional and Non-linear Preferences

(`http://jasonhartline.com/MDnA/`; coming soon)

Example Results Statements

[cf. literature on single-dimensional linear agents]

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- uniform posted pricing $\Rightarrow e/(e-1) = 1.58$ approximation.
  [cf. "correlation gap" Yan, '11]
- non-identical agents,
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  [cf. H., Roughgarden '09; Alaei, H., Niazadeh, Pountourakis, Yuan '15]

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- all-pay auction with reserve (and ironing top) $\Rightarrow$ optimal.
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- all-pay auction (no reserve) $\Rightarrow n/(n-1)$ approximation.
  [cf. Bulow, Klemperer '96]

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Multi- to Single-agent Reductions

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- preference assumption: none:
  - remaining multi-dimensional linear (utility) preferences.
  - non-linear (utility) preferences.
    (e.g., risk aversion, budgets)

Agenda:

1. Examples of optimal single-agent mechanisms.
   (derivations tomorrow)

2. Ex ante reduction (with revenue linearity)
   (e.g., unit-demand $U[0,1]^2$)

3. Interim reduction (without revenue linearity)
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Goals:

- unified framework.
- highlight differences between revenue linearity and non-linearity.

1. Examples of optimal single-agent mechanisms

[cf. Laffont, Robert '96] [cf. Armstrong '96]

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Public Budget Preferences: (single-dimensional non-linear)

- allocation: $x \in [0, 1]$; payment: p
- private value: t
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Running example: $t \sim U[0,1]$; $B = 1/4$

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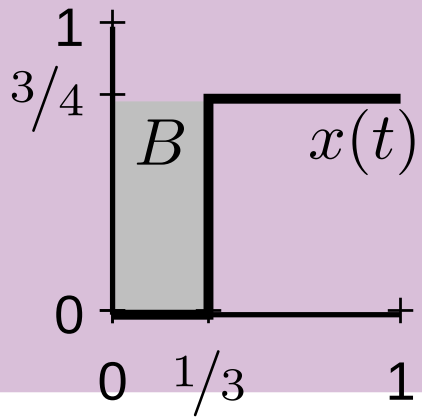

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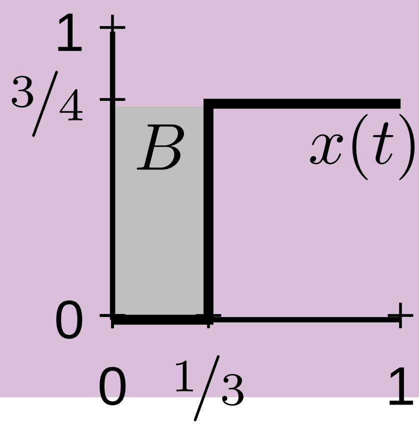

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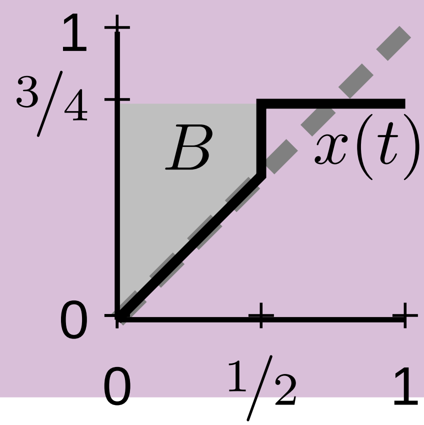

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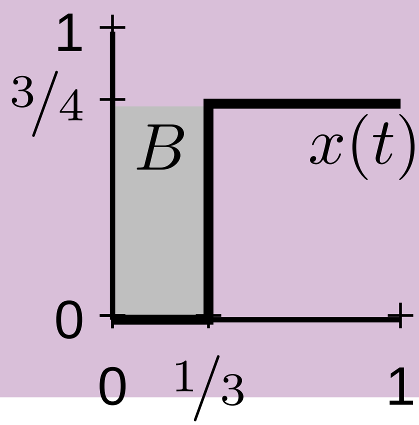

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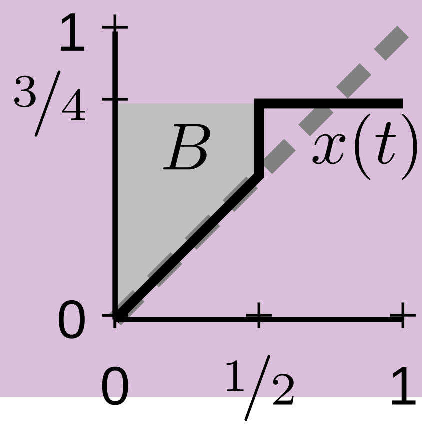

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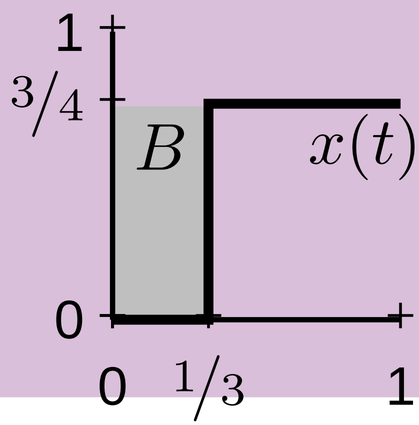

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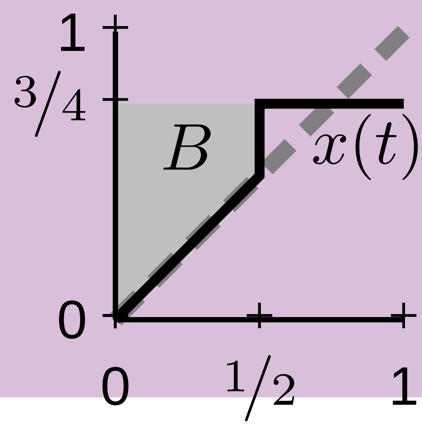

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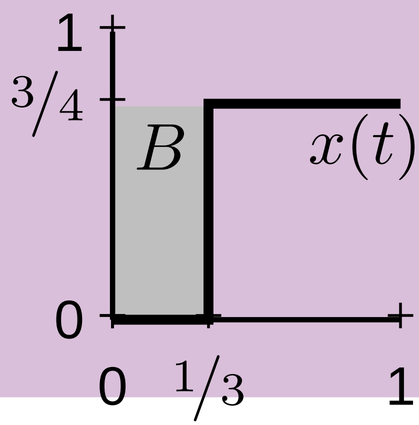

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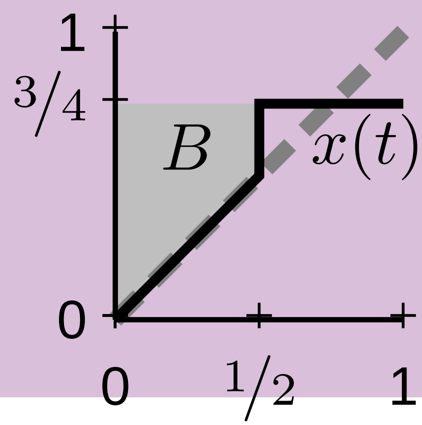

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Answer: (a)

Thm: For $t \sim U[0,1]$, revenue optimal mechanism for ex ante constraint $\hat{q} \leq 1 - B$ is "$(\hat{q} + B)$ lottery at price B."

Unit-demand Preferences

Unit-demand Preferences: (multi-dimensional linear)

- allocation: $x = (\{x\}_1, \{x\}_2)$ with $\sum_j \{x\}_j \leq 1$; payment: p
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Running Example: $t \sim U[0, 1]^2$.

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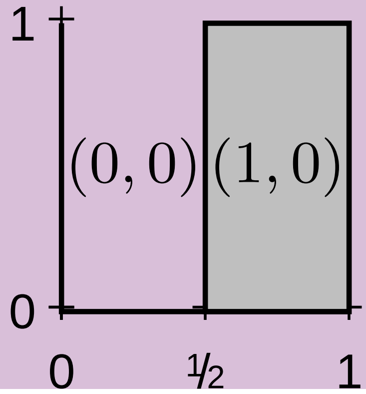

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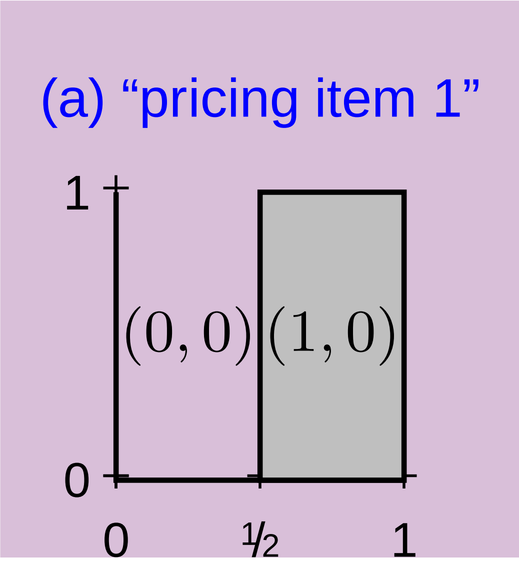

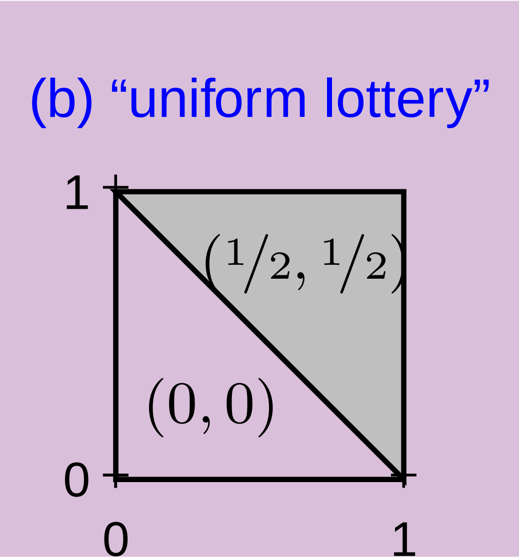

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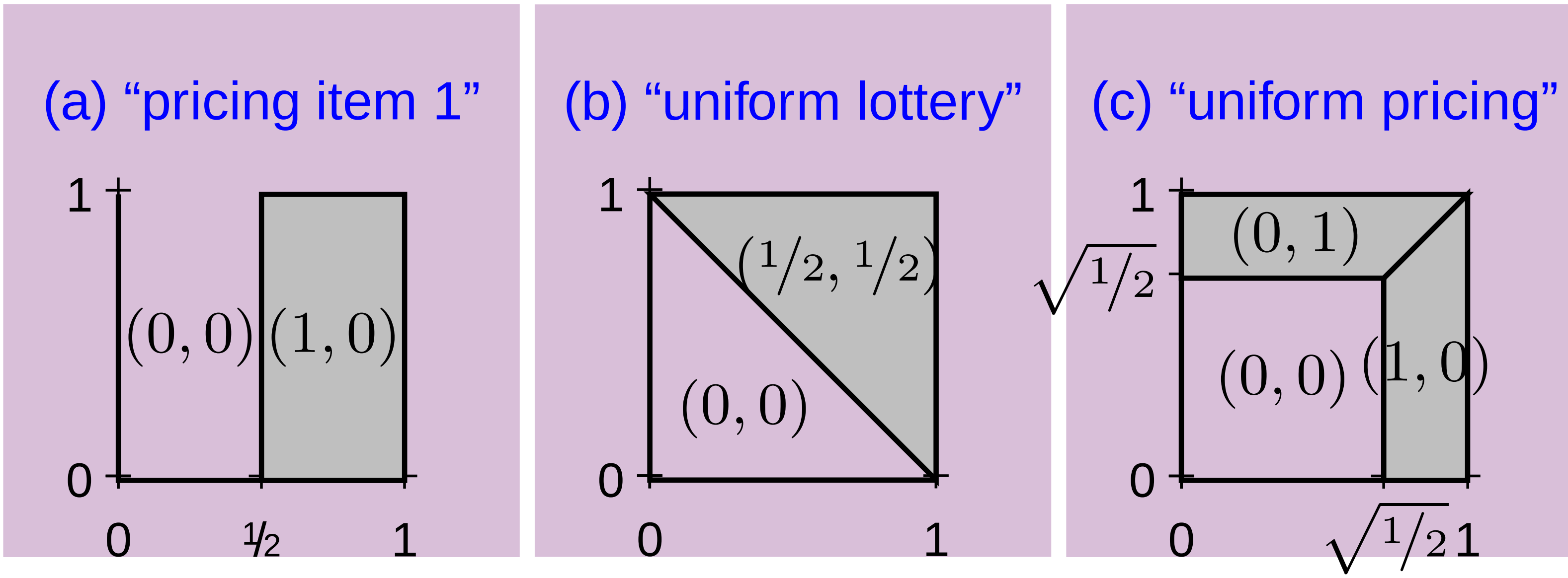

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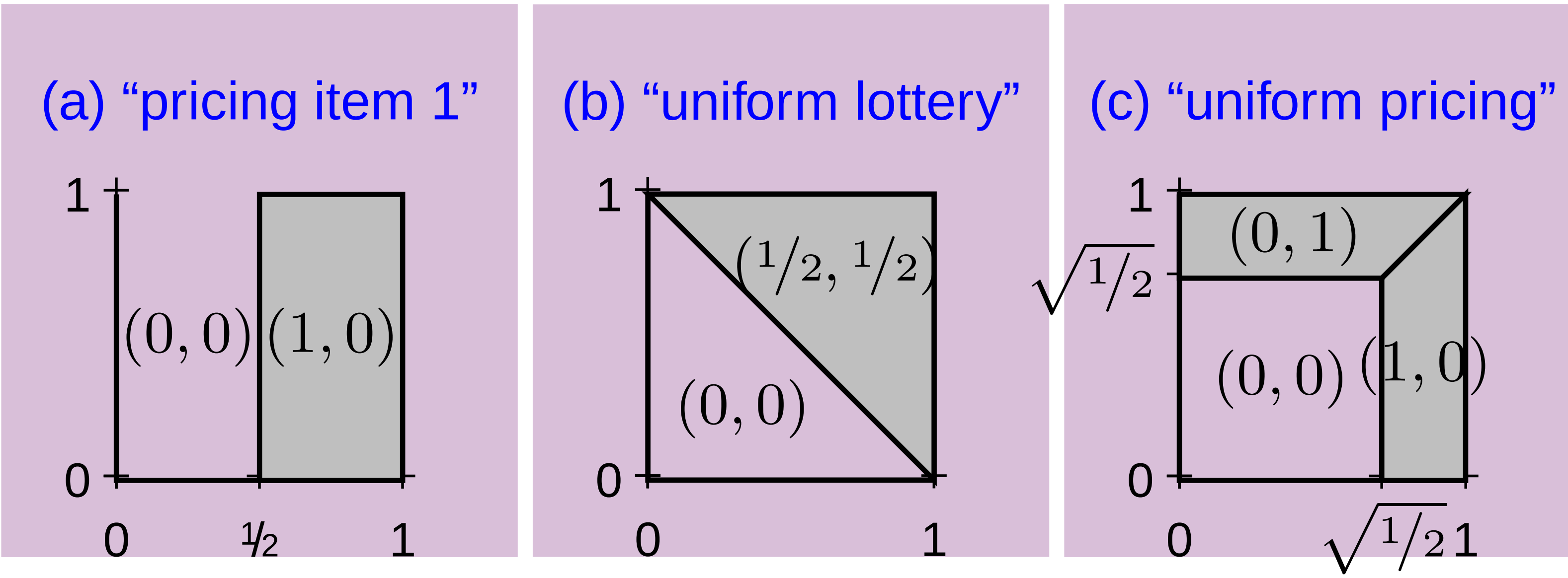

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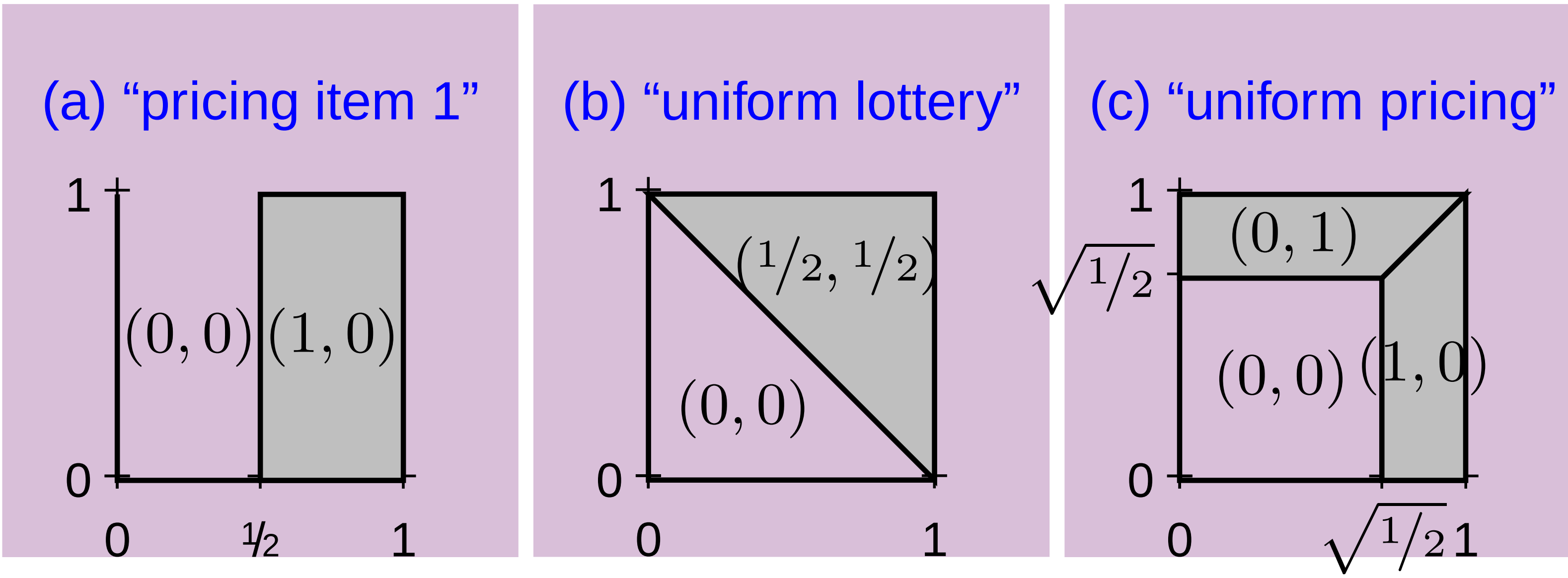

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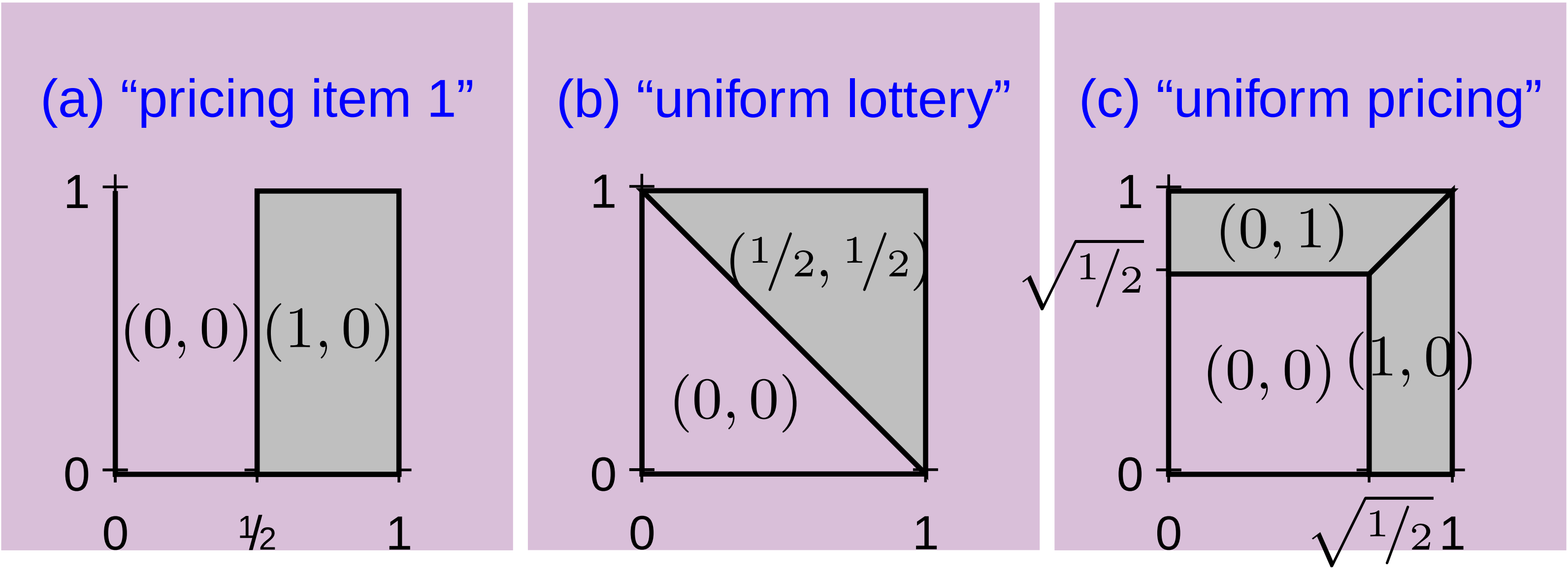

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Answer: (c)

Thm: For $t \sim U[0,1]^2$, revenue optimal mechanism for ex ante constraint $\hat{q}$ is "uniform pricing at price $\sqrt{1 - \max(\hat{q}, 2/3)}$".

Questions?

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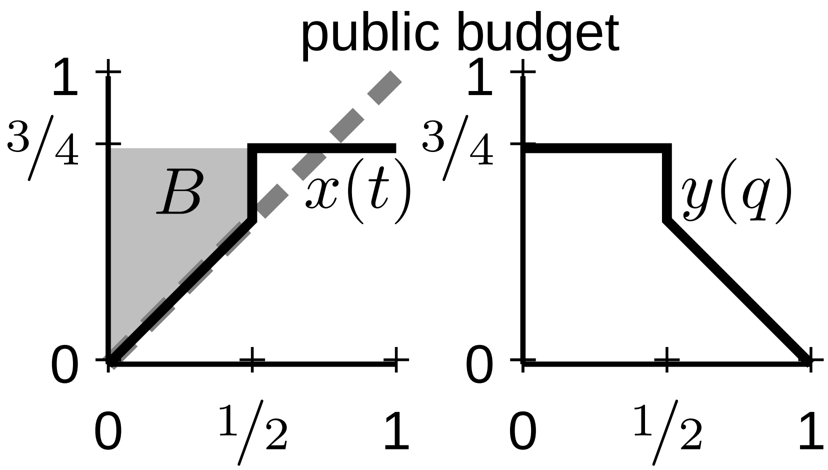

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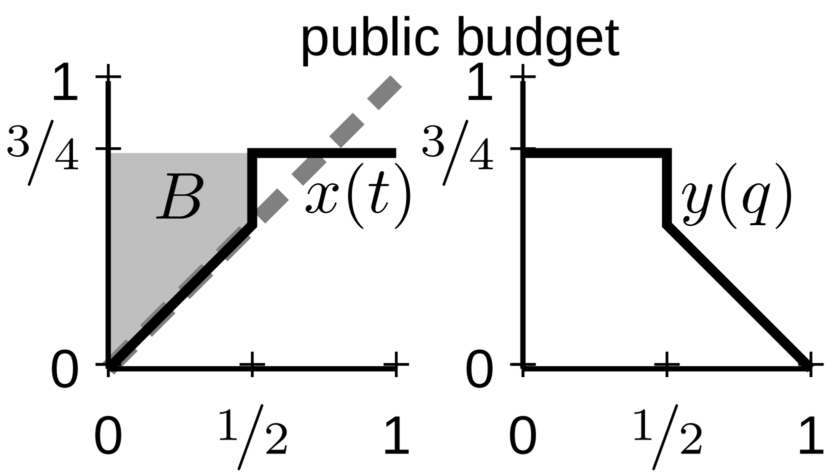

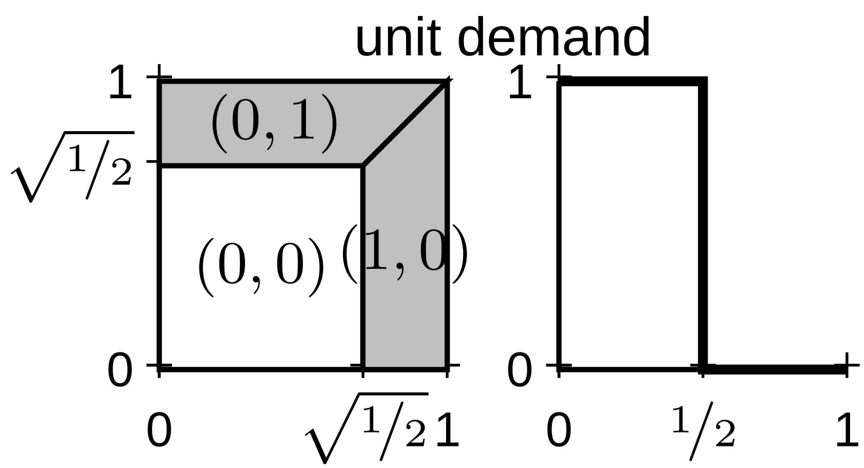

Interim Pricing

Interim Pricing Problem: for allocation constraint $\hat{y}$, find

- stationary transformation $\sigma : [0,1] \to \Delta([0,1])$, and (with $\sigma(q) \sim U[0,1]$ for $q \sim U[0,1]$)
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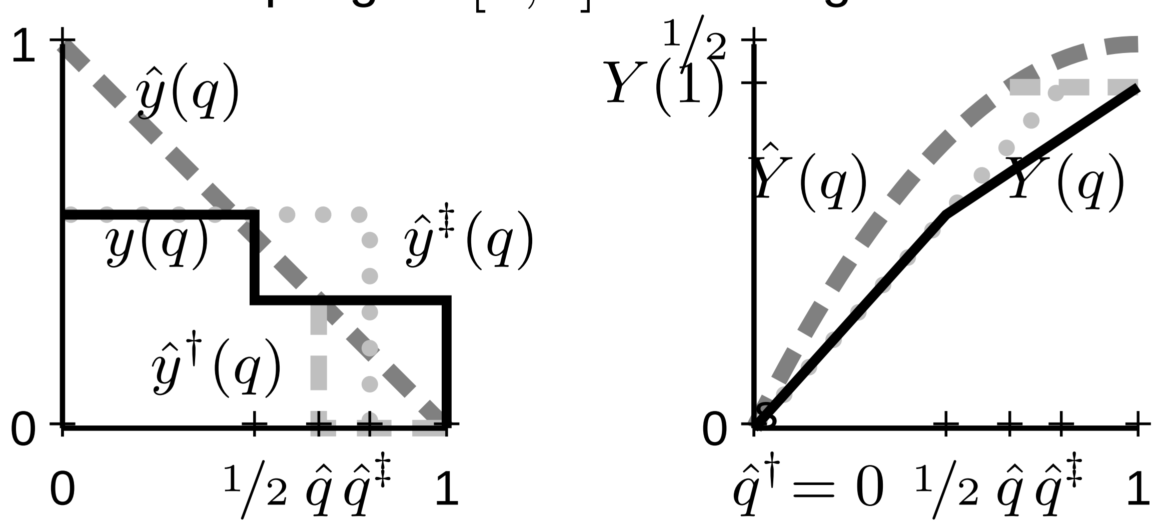

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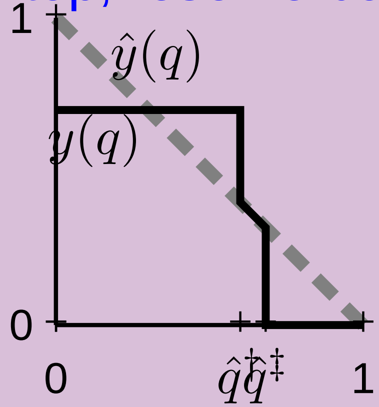

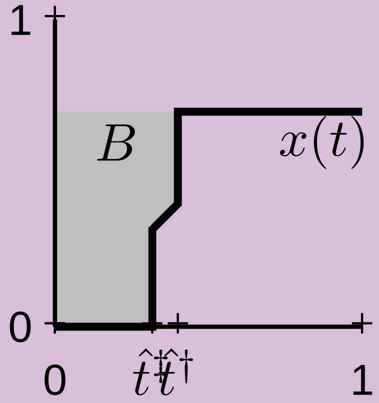

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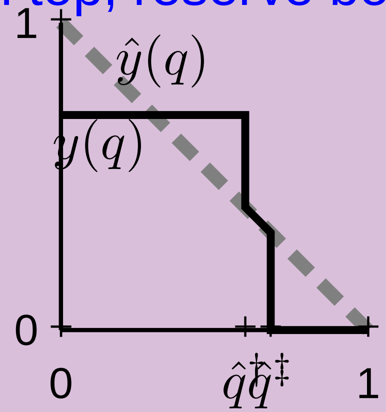

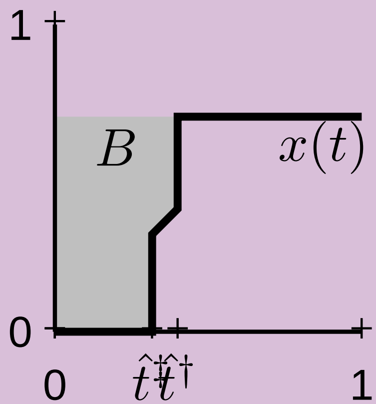

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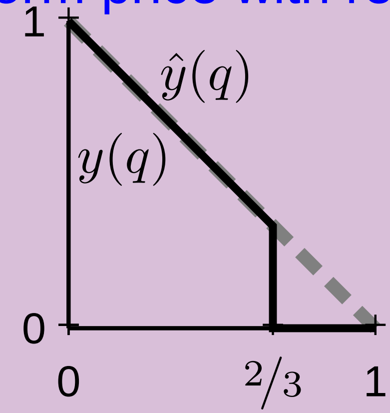
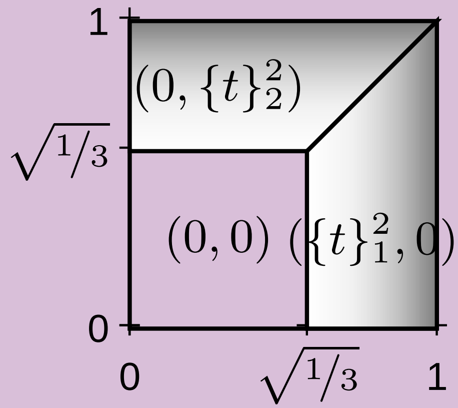
unit-demand

"uniform price with reserve"





Note: $\sqrt{1/3}$ reserve for all $\hat{y}$.

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Thm: revenue linearity implies orderability.

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[Alaei, Fu, Haghpanah, H ’13] [cf. Myerson ’81; Bulow, Roberts ’89]

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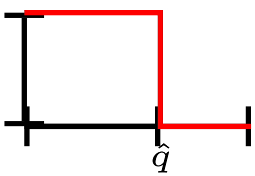

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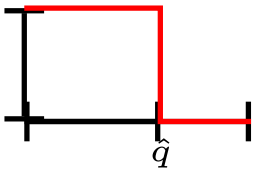

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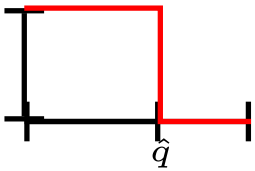

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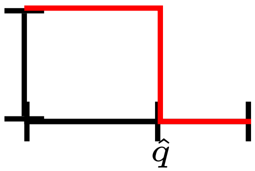

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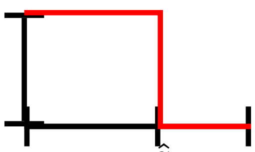

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Optimal Multi-agent Mechanisms

Marginal Revenue Mechanism: (for orderable agents)

1. map agent types to quantiles via ordering: $\boldsymbol{t} \to \boldsymbol{q} = (q_1, \ldots, q_n)$
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- revenue curves are concave; marginal revenue curves are monotone; critical quantiles exist; mechanism is incentive compatible.

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Example: red-blue car, types i.i.d. $U[0,1]^2$.

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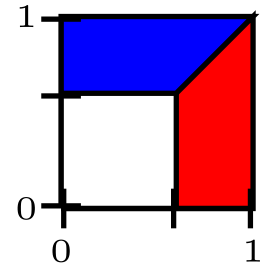

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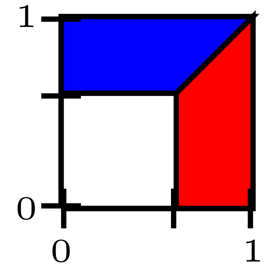

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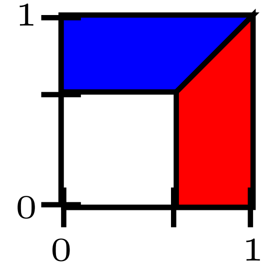

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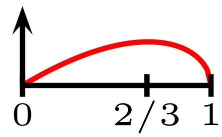

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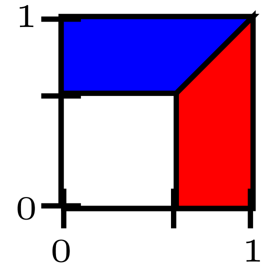

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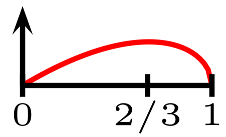

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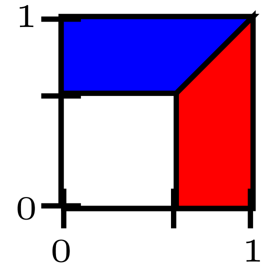

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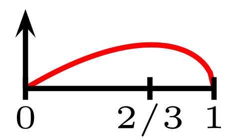

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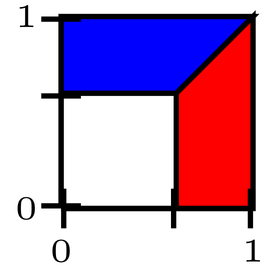

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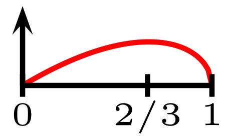

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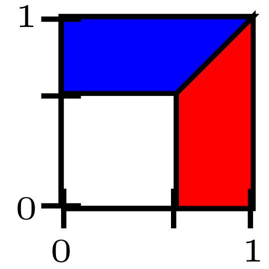

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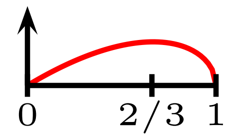

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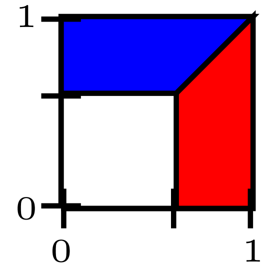

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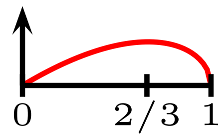

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- **Cor:** the marginal revenue mechanism is revenue optimal.

Multi-dimensional and Non-Linear Mechanism Design (and Approximation)

Part II: Solving Single-agent Problems

Jason Hartline

Northwestern University

August 27, 2015

Multi- to Single-agent Reductions

Ex ante Reduction: [cf. Myerson '81; Bulow and Roberts '89]

- single-agent problem: constraint on ex ante allocation probability.
- multi-agent composition: marginal revenue mechanism.
- preference assumption: *revenue linearity*
  - single-dimensional linear (utility) preferences.
  - some multi-dimensional linear (utility) preferences.

Interim Reduction: [cf. Border; Alaei et al; Cai et al]

- single-agent problem: constraint on entire *allocation rule*.
- multi-agent composition: stochastic weighted optimization.
- preference assumption: none:
  - remaining multi-dimensional linear (utility) preferences.
  - non-linear (utility) preferences.
    (e.g., risk aversion, budgets)

Loose Ends

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   - yes, but this must be proved. [later today]

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Examples: posted pricing; anonymous pricing.

3. Interim Reduction (without revenue linearity)

[Alaei, Fu, Haghpanah, H, Malekian ’12]

[cf. Cai, Daskalakis, Weinberg ’12,’13]
[cf. Maskin, Riley ’84; Matthews ’84; Border ’91,’07; Mierendorff ’11]

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Def: Interim allocation constraints $\hat{\boldsymbol{y}}$ (with $\hat{y}_i : [0,1] \to [0,1]$) is *interim feasible* if exists ex post feasible mechanism $\hat{\boldsymbol{y}}^{EP} : [0,1]^n \to \mathcal{X} \subset [0,1]^n$ that induces them.

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Agenda:

- theorem proof sketch.
- understanding interim feasibility.
- characterizing ex post mechanisms.
- optimization subject to interim feasibility.

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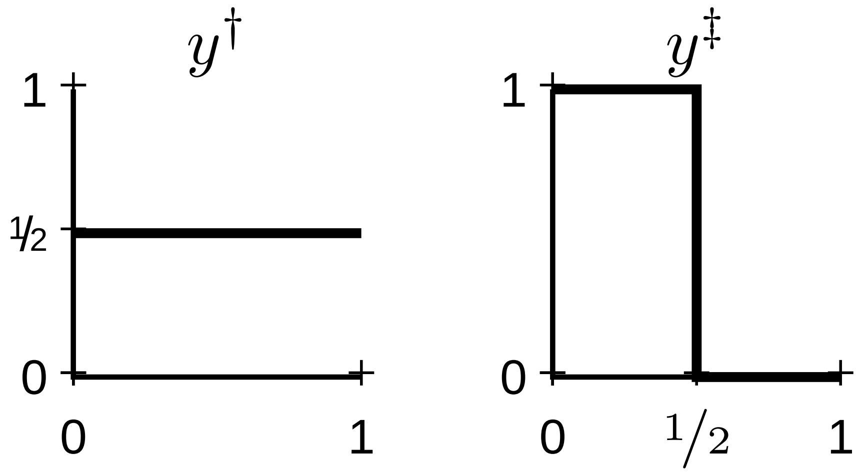
Proof: from definition of interim pricing problem.

Interim Feasibility: Examples

Question: Consider single-item and allocation rules:



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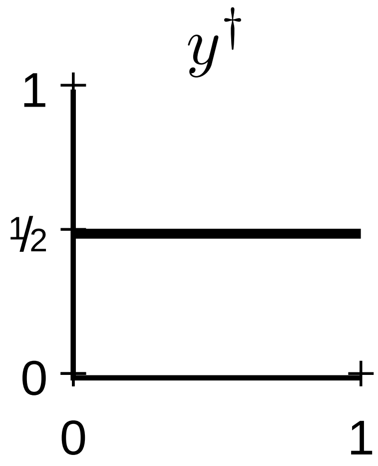

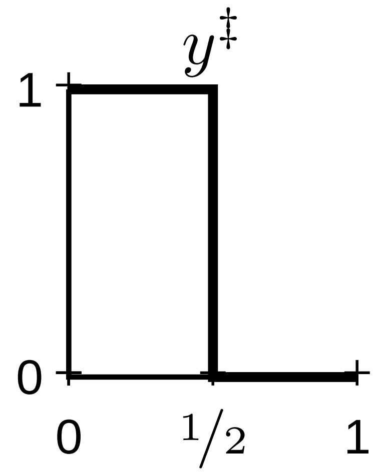

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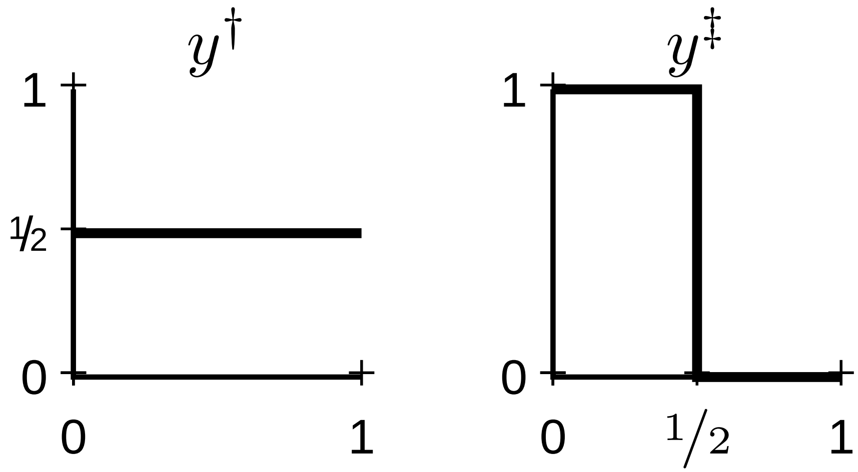

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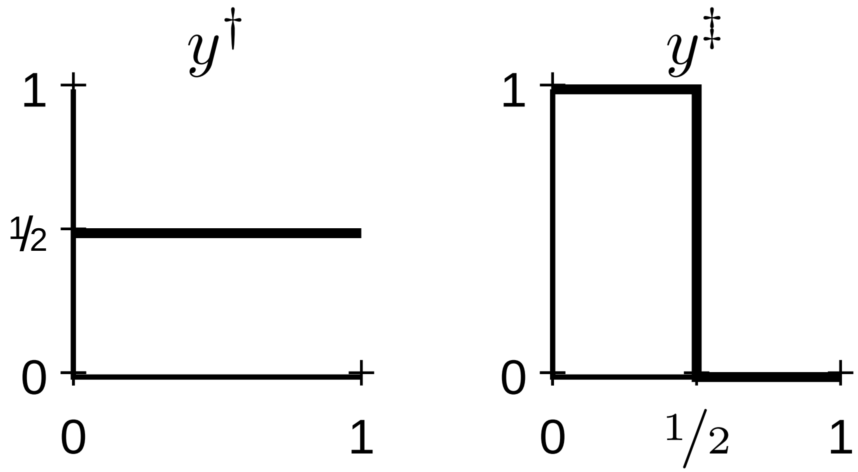

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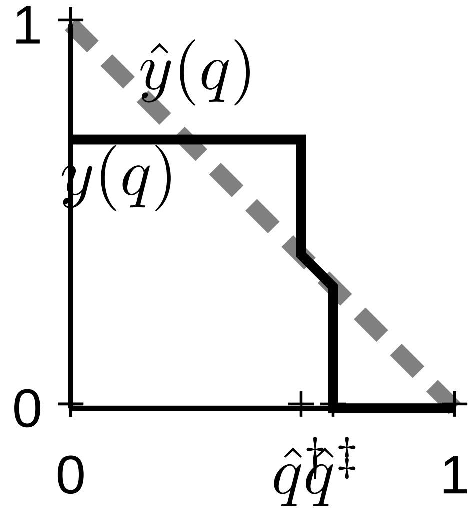

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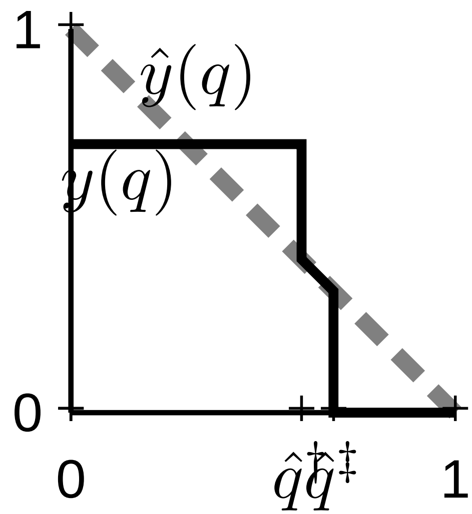

Note: almost all positive results in literature for non-linear mechanism design are based on this fact. (e.g., budget, risk aversion.)

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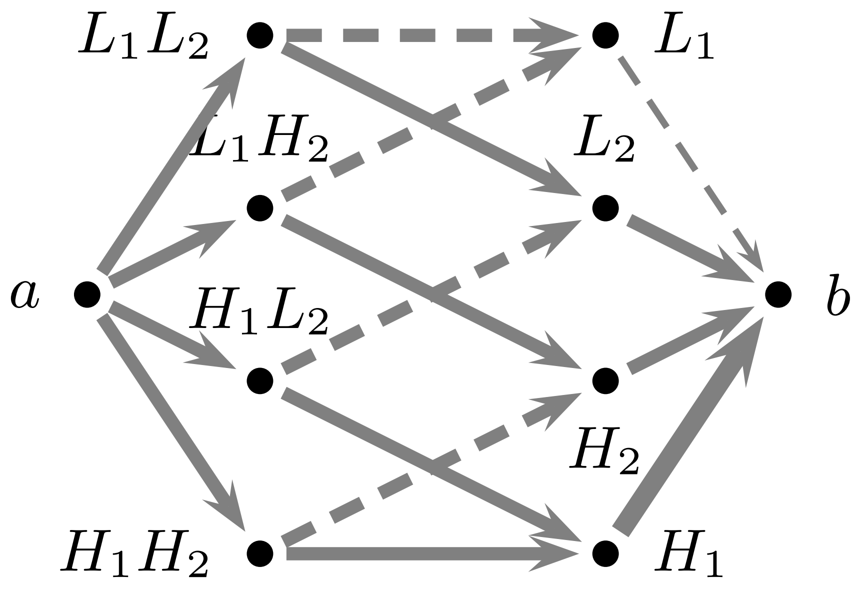

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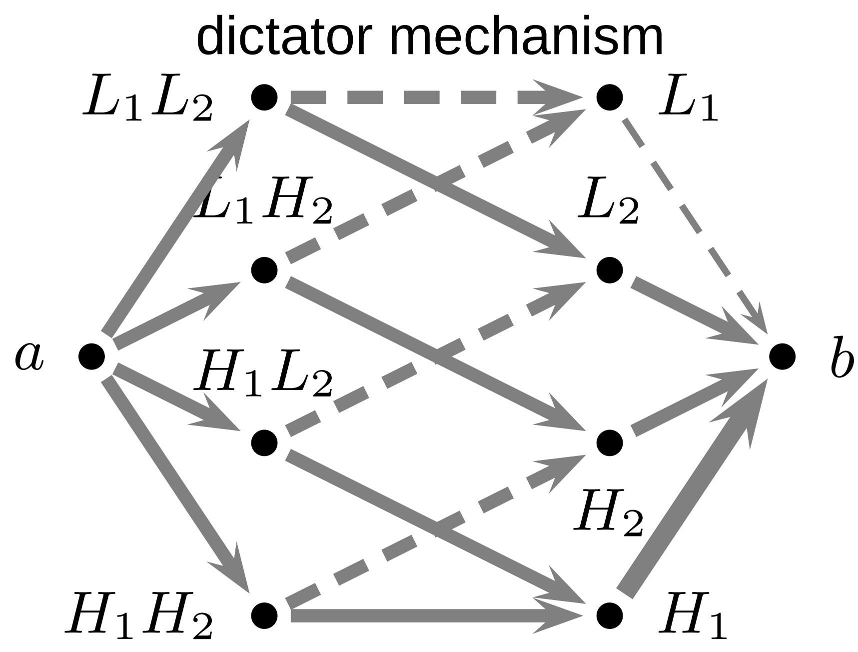

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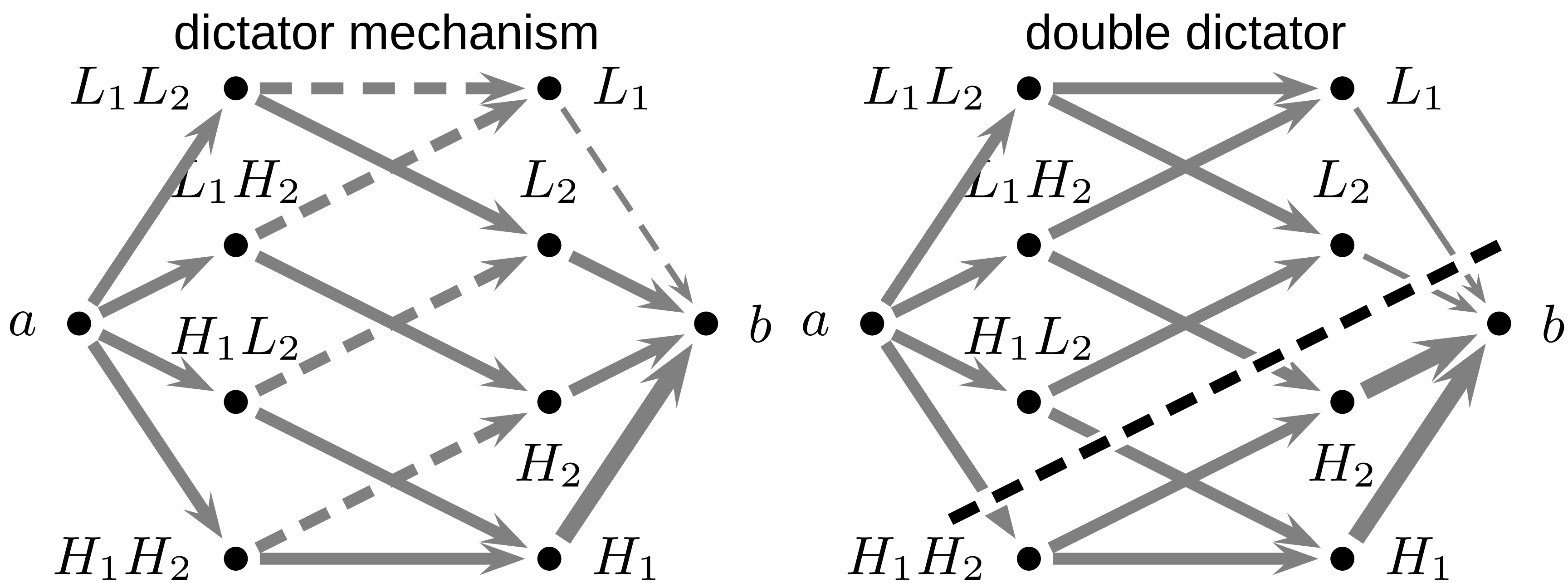

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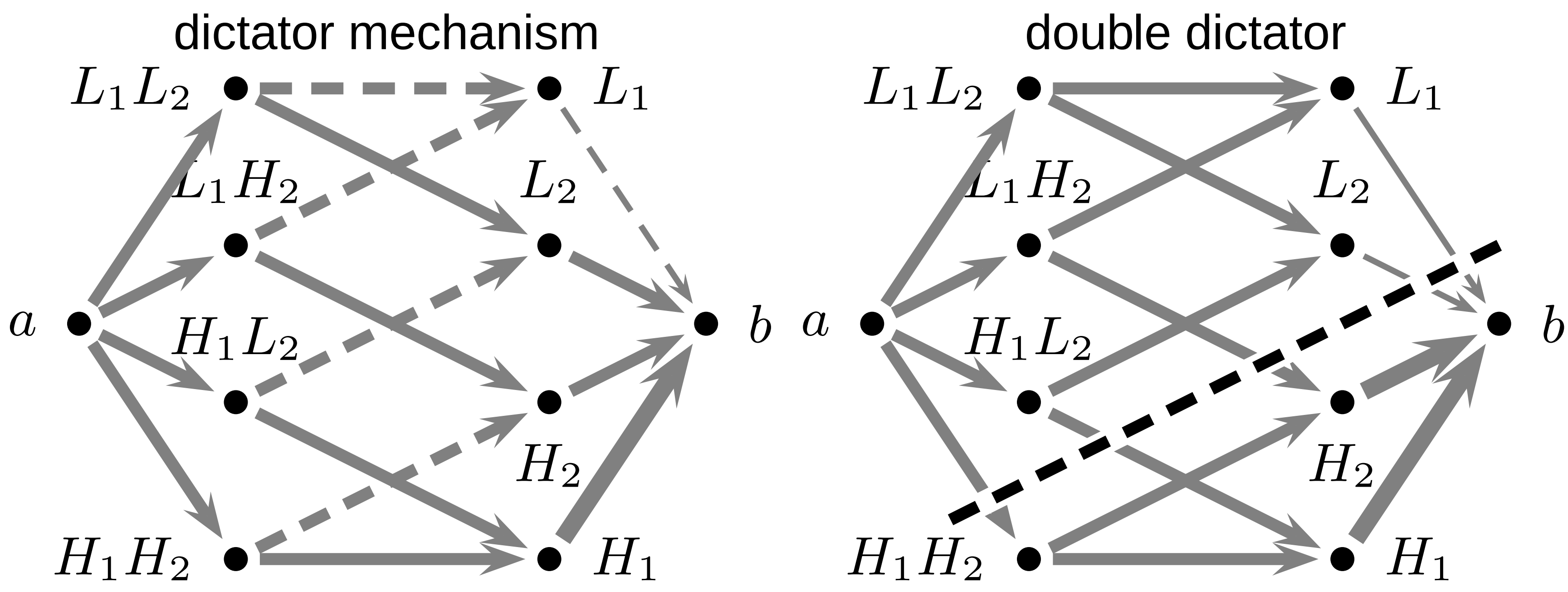

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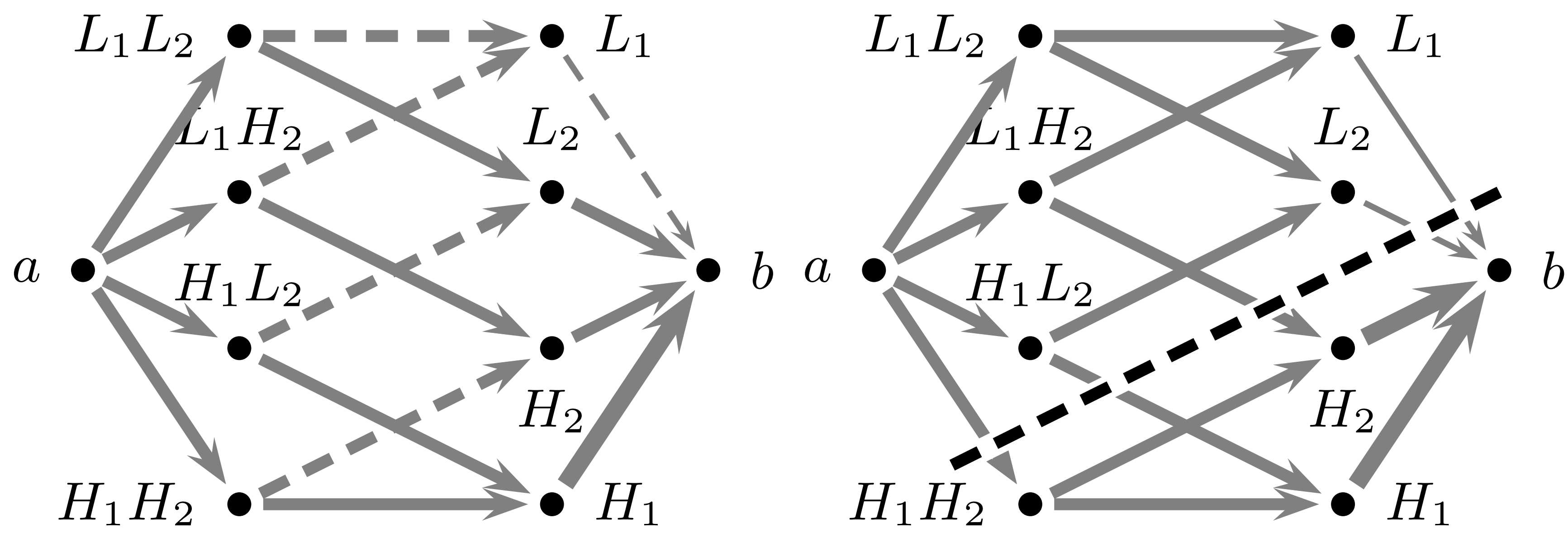

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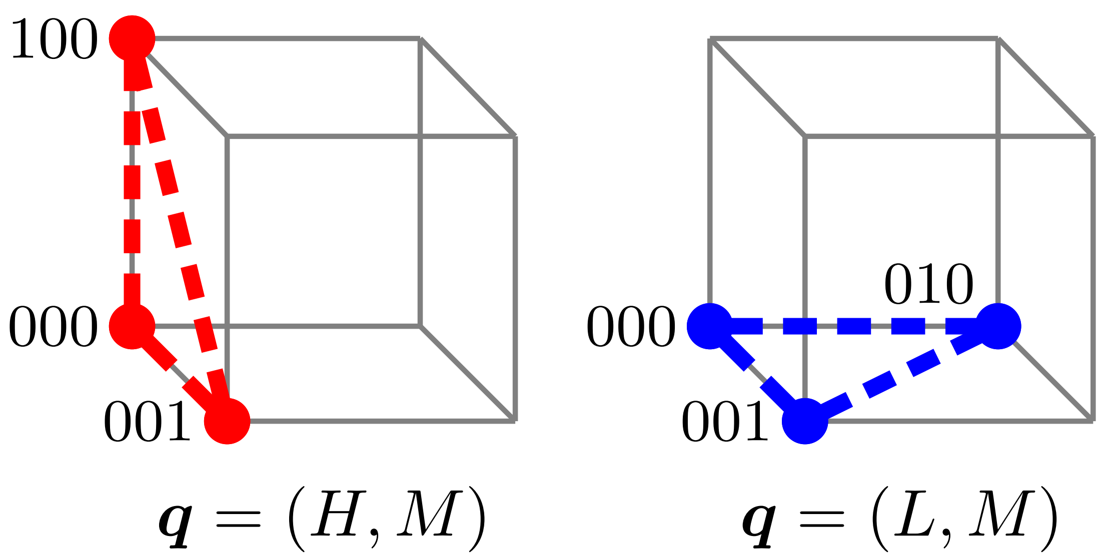

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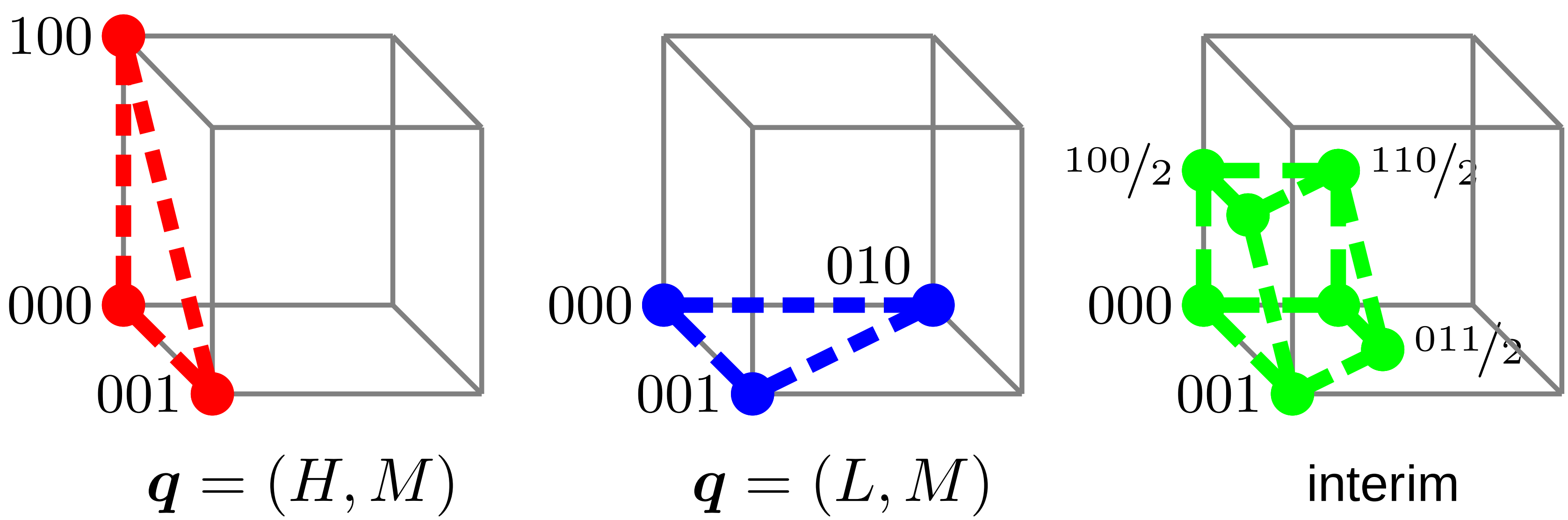

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Conclusions: Multi- to Single-agent Reductions

Ex ante Reduction: [cf. Myerson '81; Bulow and Roberts '89]

- single-agent problem: constraint on ex ante allocation probability.
- multi-agent composition: marginal revenue mechanism.
- preference assumption: *revenue linearity*
  - single-dimensional linear (utility) preferences.
  - some multi-dimensional linear (utility) preferences.

Interim Reduction: [cf. Border; Alaei et al; Cai et al]

- single-agent problem: constraint on entire *allocation rule*.
- multi-agent composition: stochastic weighted optimization.
- preference assumption: none:
  - remaining multi-dimensional linear (utility) preferences.
  - non-linear (utility) preferences.
    (e.g., risk aversion, budgets)

4. Solving Public Budget Single-agent Problem

[cf. Laffont, Robert ’96; Bulow, Roberts ’89; Devanur, Ha, H. ’13]

[cf. Bulow, Klemperer ’96]

5. Solving Unit-demand Single-agent Problem

[Haghpanah, H. ’15]

[cf. Daskalakis, Deckelbaum, Tzamos ’13,’14] [cf. Wang, Tang ’14]
[cf. Giannakopoulos, Koutsoupias ’14]

[cf. Armstrong ’96; Rochet, Chone ’98]

Unit-demand Preferences:

- m items.
- allocation: $x = (\{x\}_1, \ldots, \{x\}_m)$ with $\sum_j \{x\}_j \leq 1$;
  payment: p
- private type: $t = (\{t\}_1, \ldots, \{t\}_m)$ in type space $\mathcal{T} = [0,1]^m$
- utility: $u = \sum_j t \cdot x - p$.
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Assumption: item-symmetric distributions; wlog $\{t\}_1 \geq \{t\}_j$.

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Thm: For item-semetric distributions, favorite-item projection is optimal if **Dist**$_t[\{t\}_2/\{t\}_1 \mid \{t\}_1]$ is ordered according to $\{t\}_1$ by first-order stochastic dominance.

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Approach: solve on rays from origin; check consistency [Armstrong '96]

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- optimal auction with known θ is independent of θ; therefore, it is optimal without knowledge of θ.

Beyond Rays from Origin

Challenges for Generalization:

- must consider paths other than rays from origin
  (but there are many, and most "do not work")

- must solve mechanism design problem on general paths
  (argument for rays does not generalize)

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(b) is *incentive compatible* if pointwise *virtual surplus maximizer* $x(t) \in \operatorname{argmax}_{x^\dagger} \phi(t) \cdot x^\dagger$ is incentive compatible.
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Conclusion: virtual values reduce optimization in expectation to pointwise.

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Lemma: $\phi(t) = t - \frac{1-F(t)}{f(t)}$ is an amortization of revenue.

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E.g., $t \sim U[0,1]$; $\quad F(t) = t$; $\quad f(t) = 1$; $\quad \phi(t) = 2t - 1$.

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Note: multi-dimensional amortizations of revenue are not generally incentive compatible. (thus, are not generally virtual value functions)

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Consistency: identify sufficient conditions on distribution by checking consistency, i.e.,

(a) when positive, virtual value for favorite item $\geq$ virtual value for other item.

(b) when negative, both are negative.

Results of Analysis ($m = 2$)

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Thm: favorite item project is optimal if slope of equi-quantile curve at t is at least $\{t\}_2/\{t\}_1$.

Conclusions

multi-dimensional and non-linear mechanism design theory that mirrors single-dimensional linear theory

1. multi- to single-agent reductions
2. marginal revenue
3. multi-dimensional virtual values