Scarf's Lemma (1967)

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$\mathcal{P} = \{x \in \mathbb{R}^n_+ : Ax \leq b\}$.

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the set of columns j for which $a_{ij} > 0$.

A vector $x \in \mathcal{P}$ **dominates** column r if there exists a row i such that $\sum_j a_{ij}x_j = b_i$, $a_{ir} > 0$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

$\mathcal{P}$ has an extreme point that dominates every column of A.

Stable Matching-Gale & Shapley

Complete Bipartite Graph

$D \cup H =$ set of vertices (single doctors and hospitals with capacity 1)

$E =$ complete set of edges

$\delta(v) \subseteq E$ set of edges incident to $v \in D \cup H$

Each $v \in D \cup H$ has a strict ordering $\succ_v$ over edges in $\delta(v)$

Let $x_e = 1$ if we select edge $e = (d, h)$ and 0 otherwise.

Selecting edge $e = (d, h)$ corresponds to matching doctor d to hospital h.

The convex hull of all feasible matchings is given by

$$\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in D \cup H$$

A matching x is blocked by a pair $e = (d, h)$ if

1. $x_e = 0$
2. $h \succ_d h'$ where $x_{dh'} = 1$, *and*,
3. $d \succ_h d'$ where $x_{d'h} = 1$.

A matching x is called stable if it cannot be blocked by any pair.

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$$\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in D \cup H$$

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the set of columns j for which $a_{ij} > 0$.

Each $v \in D \cup H$ has a strict ordering $\succ_v$ over edges in $\delta(v)$.

$x \in \mathcal{P}$ **dominates** column r if $\exists i$ such that $\sum_j a_{ij}x_j = b_i$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

For all $e \in E$ there is a $v \in D \cup H$ such that $e \in \delta(v)$ and

$$\sum_{f \succ_v e} x_f + x_e = 1$$

Stable Roomates

Non-Bipartite Graph

$V =$ set of vertices

$E =$ complete set of edges

$\delta(v) \subseteq E$ set of edges incident to $v \in V$

Each $v \in V$ has a strict ordering $\succ_v$ over edges in $\delta(v)$

Let $x_e = 1$ if we select edge $e = (u, v)$ and 0 otherwise.

The integer points of

$$\sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V$$

form a matching.
A matching x is blocked by a pair $e = (u, v)$ if

1. $x_e = 0$
2. $u \succ_v u'$ where $x_{vu'} = 1$, *and*,
3. $v \succ_u v'$ where $x_{v'u} = 1$.

A matching x is stable if it cannot be blocked by any pair.

The extreme points of

$$\sum_{e \in \delta(v)} x_e \leq 1 \; \forall v \in V, \;\; x_e \geq 0 \; \forall e \in E$$

are $1/2$ fractional ($x_e = 0, 1, 1/2$).

If $x_e = 1/2$ in one extreme point it takes that value in all extreme points.

Edges that correspond to $x_e = 1/2$ lie on odd cycles.

Stable Partition

Stable Partition of (V, E) is a subset P of edges such that

1. Each component of P is either an edge or an odd cycle.
2. Each cycle $\{e_1, e_2, \ldots, e_k\}$ of P satisfies the following:

   $$e_1 \succ_{v_k} e_k \succ_{v_{k-1}} e_{k-1} \ldots \succ_{v_3} e_3 \succ_{v_2} e_2 \succ_{v_1} e_1$$

   'odd preference cycle'
3. $\forall e \in E \setminus P$, there exists vertex v covered by P and incident to e such that $f \succ_v e$ for all $f \in P \cap \delta(v)$.

Scarf's lemma: existence of stable partition.

Stable Partition

$A = m \times n$ nonnegative matrix and $b \in \mathbb{R}^m_+$ with $b >> 0$.

$$\sum_{e \in \delta(v)} x_e \leq 1 \; \forall v \in V$$

Each row $i \in [m]$ of A has a strict order $\succ_i$ over the set of columns j for which $a_{ij} > 0$.

Each $v \in V$ has a strict ordering $\succ_v$ over edges in $\delta(v)$.

$x \in \mathcal{P}$ **dominates** column r if $\exists i$ such that $\sum_j a_{ij} x_j = b_i$ and $k \succeq_i r$ for all $k \in [n]$ such that $a_{ik} > 0$ and $x_k > 0$.

For all $e \in E$ there is a $v \in V$ such that $e \in \delta(v)$ and

$$\sum_{f \succ_v e} x_f + x_e = 1$$

Quinzii Housing Model

$N = \{1, 2, \ldots, n\}$ is a set of agents.

Each $i \in \{1, \ldots, q\}$ owns house h_i.

Agents $i \in \{q + 1, \ldots, n\}$ do not own a house.

Each $i \in N$ has money in the amount w_i.

Utility of agent i for a monetary amount y and house h_j is denoted $u_i(y, h_j)$.

Each u_i is continuous and non-decreasing in money for each house.

Stable Matching with Couples

D = set of single doctors

C = set of couples, each couple $c \in C$ is denoted $c = (f_c, m_c)$

$D^* = D \cup \{m_c | c \in C\} \cup \{f_c | c \in C\}$.

H = set of hospitals

Each $s \in D$ has a strict preference relation $\succ_s$ over $H \cup \{\emptyset\}$

Each $c \in C$ has a strict preference relation $\succ_c$ over $H \cup \{\emptyset\} \times H \cup \{\emptyset\}$

Stable Matching with Couples

k_h = capacity of hospital $h \in H$

Preference of hospital h over subsets of D^* is modeled by choice function $ch_h(.) : 2^{D^*} \rightarrow 2^{D^*}$.

$ch_h(.)$ is responsive

h has a strict priority ordering $\succ_h$ over elements of $D \cup \{\emptyset\}$.

$ch_h(R)$ consists of the (upto) $\min\{|R|, k_h\}$ highest priority doctors among the feasible doctors in R.

Blocking

$\mu =$ matching

$\mu_h =$ the subset of doctors matched to h.

$\mu_s =$ position that single doctor s receives.

$\mu_{f_c}, \mu_{m_c} =$ positions that female member and male member of the couple c obtain, respectively.

Individual Rationality

μ is individually rational if

- $ch_h(\mu_h) = \mu_h$ for any hospital h
- $\mu_s \succeq_s \emptyset$ for any single doctor s
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \mu_{m_c})$
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\mu_{f_c}, \emptyset)$
- $(\mu_{f_c}, \mu_{m_c}) \succeq_c (\emptyset, \emptyset)$

Blocking #1

A matching μ can be blocked in one of three ways.

By a single doctor, $d \in D$ and a lone hospital, $h \in H$

- $h \succ_d \mu(d)$
- $d \in ch_h(\mu(h) \cup d)$

Blocking #2

By a couple $c \in C$ and a pair of distinct hospitals $h, h' \in H$

- $(h, h') \succ_c \mu(c)$
- $f_c \in ch_h(\mu(h) \cup f_c)$ when $h \neq \emptyset$
- $m_c \in ch_{h'}(\mu(h') \cup m_c)$ when $h' \neq \emptyset$

Blocking #3

By a couple $c \in C$ and a single hospital $h \in H$

- $(h, h) \succ_c \mu(c)$
- $(f_c, m_c) \subseteq ch_h(\mu(h) \cup c)$

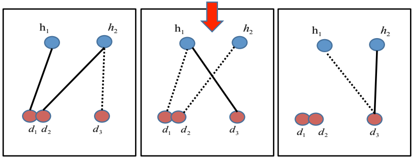
Non-Existence: Roth (84), Klaus-Klijn (05)

Hospital 1: $d_1 \succ_{h_1} \succ d_3 \succ_{h_1} \emptyset \succ_{h_1} d_2$, $k_{h_1} = 1$

Hospital 2: $d_3 \succ_{h_2} \succ d_2 \succ_{h_2} \emptyset \succ_{h_2} d_1$, $k_{h_2} = 1$

Couple $\{1, 2\}$: $(h_1, h_2) \succ_{(d_1, d_2)} \emptyset$

Single doctor d_3: $h_1 \succ h_2$



Matching with Couples

Given an instance of a matching problem with couples, determining if it has a stable matching is NP-hard.

Roth & Peranson algorithm (1999): Heuristic modification of Gale-Shapley

On all recorded instances in NRMP, it returns a matching that is stable wrt reported preferences.

Kojima-Pathak-Roth (QJE 2013); Ashlagi-Braverman-Hassidim (OR 2014)

- Randomized preferences,
- Market size increases to infinity
- Fraction of couples goes to 0

Probability that Roth & Peranson gives a stable matching approaches 1.

Matching with Couples

US: resident matching: 40,000 doctors participate every year, fraction of couples couples can be upto 10%.

Ashlagi et al: If the fraction of couple does *not* goes to 0, the probability of no stable matching is positive.

Simulations by Biro et. al.: when the number of couples is large, Roth & Perason algorithm fails to find a stable match.

(Thanh Nguyen & Vohra)

Given any instance of a matching problem with couples, there is a 'nearby' instance that is guaranteed to have stable matching.

For any capacity vector k, there exists a k' and a stable matching with respect to k', such that

1. $|k_h - k'_h| \leq 4 \ \forall h \in H$
2. $\sum_{h\in H} k_h \leq \sum_{h\in H} k'_h \leq \sum_{h\in H} k_h + 9.$

$x(d,h) = 1$ if single doctor d is assigned to hospital $h \in H$ and zero otherwise.

$x(c,h,h') = 1$ if f_c is assigned to h and m_c is assigned to h' and zero otherwise.

$x(c,h,h) = 1$ if f_c and m_c are assigned to hospital $h \in H$ and zero otherwise.

Every 0-1 solution to the following system is a feasible matching and vice-versa.

$$\sum_{h \in H} x(d,h) \leq 1 \ \forall d \in D \tag{1}$$

$$\sum_{h,h' \in H} x(c,h,h') \leq 1 \ \forall c \in D \tag{2}$$

$$\sum_{d \in D} x(d,h) + \sum_{c \in C} \sum_{h' \neq h} x(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} x(c,h',h) + \sum_{c \in C} 2x(c,h,h) \leq k_h \ \forall h \in H \tag{3}$$

Scarf's Lemma (1967)

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$\mathcal{P}$ has an extreme point that dominates every column of A.

Matching with Couples

Constraint matrix and RHS satisfy conditions of Scarf's lemma.

Each row associated with single doctor or couple (1-2) has an ordering over the variables that 'include' them from their preference ordering.

Row associated with each hospital (3) does not have a natural ordering over the variables that 'include' them.

(1-3) *not* guaranteed to have integral extreme points.

Matching with Couples

Use choice function to induce an ordering over variables for each hospital.

Construct ordering so that a dominating solution wrt this ordering will correspond to a stable matching.

Apply Scarf's Lemma to get a 'fractionally' stable solution.

Round the fractionally stable solution into an integer solution that preserves stability (sharpening of Shapley-Folkman-Starr Lemma).

Shapley-Folkman-Starr (Starr (1969))

$\{S^j\}_{j=1}^n$ be a collection of sets in $\Re^m$ with $n > m$.

$S = \sum_{i=1}^m conv(S^i)$

Every $b \in S$ can be expressed as $\sum_{j=1}^n x^j$ where $x^j \in conv(S^j)$ for all $j = 1, \dots, n$ and $|\{j : x^j \in S^j\}| \geq n - m$.

Shapley-Folkman-Starr

Let A be an $m \times n$ 0-1 matrix and $b \in \Re^m$ with $n > m$.

Denote each column j of the A matrix by a^j.

$S^j = \{a^j, 0\}$.

Suppose $b = Ax^* = \sum_{j=1}^n a^j x_j^*$ where $0 \leq x_j^* \leq 1 \ \forall j$.

SFS $\Rightarrow b = \sum_{j=1}^n a^j y_j$ where each $y_j \in [0, 1]$ with at at least $n - m$ of them being 0-1.

y has at most m fractional components.

Let y^* be obtained by rounding up each fractional component.

$||Ay^* - b||_\infty \leq m$

Shapley-Folkman-Starr

Suppose $b = \sum_{j=1}^{n} a^j y_j$ with at least $m+1$ components of y being fractional.

Let C be submatrix of A that corresponds to integer components of y.

Let $\bar{C}$ be submatrix of A that corresponds to fractional components of y.

$$b = \bar{C} y_{\bar{C}} + C y_C$$

$$b - Cy_C = \bar{C}y_{\bar{C}}$$

Suppose (for a contradiction) that $\bar{C}$ has more columns ($\geq m+1$) than rows (m).

$\Rightarrow \; ker(\bar{C}) \neq 0 \; \Rightarrow \; \exists z \in ker(\bar{C})$

Consider $y_{\bar{C}} + \lambda z$.

$$b - Cy_C = C[y_{\bar{C}} + \lambda z]$$

Choose λ to make at least one component of $y_{\bar{C}}$ take value 0 or 1.

Shapley-Folkman-Starr

A gulf profound as that Serbonian Bog
Betwixt Damiata and Mount Casius old,
Where Armies whole have sunk.

Exchange economy with non-convex preferences i.e., upper contour sets of utility functions are non-convex.

n agents and m goods with $n \geq m$.

Starr (1969) identifies a price vector p^* and a feasible allocation with the property that at most m agents do not receive a utility maximizing bundle at the price vector p^*.

u_i is agent i's utility function.

e^i is agent i's endowment

Replace the upper contour set associated with u_i for each i by its convex hull.

Let u_i^* be the quasi-concave utility function associated with the convex hull.

p^* is the Walrasian equilibrium prices wrt $\{u_i^*\}_{i=1}^n$.

x_i^* be the allocation to agent i in the associated Walrasian equilibrium.

Shapley-Folkman-Starr

For each agent i let

$$S^i = \arg\max\{u_i(x) : p^* \cdot x \leq p^* \cdot e^i\}$$

$w =$ vector of total endowments and $S^{n+1} = \{-w\}$.

Let $z^* = \sum_{i=1}^n x_i^* - w = 0$ be the excess demand with respect to p^* and $\{u_i^*\}_{i=1}^n$.

z^* is in convex hull of the Minkowski sum of $\{S^1, \ldots, S^n, S^{n+1}\}$.

By the SFS lemma $\exists x_i \in conv(S^i)$ for $i = 1, \ldots, n$, such that $|\{i : x_i \in S^i\}| \geq n - m$ and $0 = z^* = \sum_{i=1}^n x_i - w$.

Shapley-Folkman-Starr

$$\max \sum_{j=1}^{n} f_j(y_j)$$

$$\text{s.t. } Ay = b$$

$$y \geq 0$$

A is an $m \times n$ matrix with $n > m$.

$f_j^*(\cdot)$ is the smallest concave function such that $f_j^*(t) \geq f_j(t)$ for all $t \geq 0$

Shapley-Folkman-Starr

Solve the following to get y^*:

$$\max \sum_{j=1}^{n} f_j^*(y_j)$$

$$\text{s.t. } Ay = b$$

$$y \geq 0$$

$e_j = \sup_t [f_j^*(t) - f_j(t)]$

Sort e_j's in decreasing order.

$$\sum_{j=1}^{n} f_j(y_j^*) \geq \sum_{j=1}^{n} f_j^*(y_j^*) - \sum_{j=1}^{m} e_j$$

Shapley-Folkman-Starr

Let A be an $m \times n$ 0-1 matrix and $b \in \Re^m$ with $n > m$.

Denote each column j of the A matrix by a^j.

$S^j = \{a^j, 0\}$.

Suppose $b = Ax^* = \sum_{j=1}^n a^j x_j^*$ where $0 \le x_j^* \le 1 \; \forall j$.

SFS $\Rightarrow b = \sum_{j=1}^n a^j y_j$ where each $y_j \in [0,1]$ with at at least $n-m$ of them being 0-1.

y has at most m fractional components.

Let y^* be obtained by rounding up each fractional component.

$||Ay^* - b||_\infty \le m$

Shapley-Folkman-Starr: Weakness

Does not allow you to control which constraints to violate.

Want to satisfy (1-2) but are willing to violate (3).

Degree of violation is large because it makes no use of information about A matrix. In our case each variable intersects exactly two constraints.

We use this sparsity to show that no constraint can contain many occurences of a fractional variable.

Kiralyi, Lau & Singh (2008)

Gandhi, Khuller, Parthasarathy & Srinivasan (2006)

(Thanh Nguyen & Vohra)

Given any instance of a matching problem with couples, there is a 'nearby' instance that is guaranteed to have stable matching.

For any capacity vector k, there exists a k' and a stable matching with respect to k', such that

$$|k_h - k'_h| \leq 3 \ \forall h \in H.$$

Iterative Rounding

Step 0: Choose extreme point $x^* \in \arg\max\{w \cdot x : Ax \leq b, x \geq 0\}$.

Step 1: If x^* is integral, output x^*, otherwise continue to either Step 2a or 2b.

Step 2a: If any coordinate of x^* is integral, fix the value of those coordinates, and update the linear program and move to step 3.

Iterative Rounding: Updating LP

$C =$ columns of A that correspond to the non-integer valued coordinates of x^*.

$\overline{C} =$ columns of A that correspond to the integer valued coordinates of x^*.

A_C and $A_{\overline{C}}$ be the sub-matrices of A that consists of columns in C and the complement $\overline{C}$, respectively.

Let x_C and $x_{\overline{C}}$ be the sub-vector of x that consists of all coordinates in C and $\overline{C}$. The updated LP is:

$$\max\{w_C \cdot x_C : \text{ s.t. } D_C \cdot x_C \leq d - D_{\overline{C}} \cdot x_{\overline{C}}^{opt}\}.$$

Step 2b: If all coordinates of x^* fractional, delete *certain* rows of A (to be specified later) from the linear program. Update the linear program, move to Step 3.

Step 3: Solve the updated linear program $\max\{w \cdot x \text{ s.t. } Ax \leq b\}$ to get an extreme point solution. Let this be the new x^* and return to Step 1.

Rounding & Matching

Start with an extreme point solution x^* to the following:

$$\sum_{h \in H} x(d,h) \leq 1 \ \forall d \in D$$

$$\sum_{h,h' \in H} x(c,h,h') \leq 1 \ \forall c \in D$$

$$\sum_{d \in D} x(d,h) + \sum_{c \in C} \sum_{h' \neq h} x(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} x(c,h',h) + \sum_{c \in C} 2x(c,h,h) \leq k_h \ \forall h \in H$$

Round x^* into a 0-1 solution $\bar{x}$ such that

$$\sum_{h \in H} \bar{x}(d,h) \leq 1 \ \forall d \in D$$

$$\sum_{h,h' \in H} \bar{x}(c,h,h') \leq 1 \ \forall c \in D$$

$$\sum_{d \in D} \bar{x}(d,h) + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h',h) + \sum_{c \in C} 2\bar{x}(c,h,h) \leq k_h + 3 \ \forall h \in H$$

$$\sum_{d \in D} \bar{x}(d,h) + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h,h') + \sum_{c \in C} \sum_{h' \neq h} \bar{x}(c,h',h) + \sum_{c \in C} 2\bar{x}(c,h,h) \geq k_h - 3 \ \forall h \in H$$

If every component of x^* is < 1 (all fractional), there must be a hospital h where

$$\sum_{d\in D}\lceil x^*(d,h)\rceil+\sum_{c\in C}\sum_{h'\neq h}\lceil x^*(c,h,h')\rceil+\sum_{c\in C}\sum_{h'\neq h}\lceil x^*(c,h',h)\rceil+\sum_{c\in C}2\lceil x^*(c,h,h)\rceil \leq k_h+3$$

OR

$$\sum_{d\in D}\lfloor x^*(d,h)\rfloor+\sum_{c\in C}\sum_{h'\neq h}\lfloor x^*(c,h,h')\rfloor+\sum_{c\in C}\sum_{h'\neq h}\lfloor x^*(c,h',h)\rfloor+\sum_{c\in C}2\lfloor x^*(c,h,h)\rfloor \geq k_h-3$$

In setp 2(b) of the iterative rounding method, delete this row/constraint.

If false, then for *every* hospital h

$$\sum_{d \in D} \lceil x^*(d,h) \rceil + \sum_{c \in C} \sum_{h' \neq h} \lceil x^*(c,h,h') \rceil + \sum_{c \in C} \sum_{h' \neq h} \lceil x^*(c,h',h) \rceil + \sum_{c \in C} 2 \lceil x^*(c,h,h) \rceil \geq k_h + 4$$

AND

$$0 = \sum_{d \in D} \lfloor x^*(d,h) \rfloor + \sum_{c \in C} \sum_{h' \neq h} \lfloor x^*(c,h,h') \rfloor + \sum_{c \in C} \sum_{h' \neq h} \lfloor x^*(c,h',h) \rfloor + \sum_{c \in C} 2 \lfloor x^*(c,h,h) \rfloor \leq k_h - 4$$

1. Every column of A contains at most 3 non-zero entries.
2. The columns of A that correspond to non-zero entries of x^* are linearly independent and form a basis.
3. The number of non-zero entries that intersect row h is at least $k_h + 4$.

TU Co-op Game

$N =$ set of players

Value function $V : 2^N \rightarrow \Re$ is monetary value of subset S forming a coalition.

$V(N) \geq \max_{S \subset N} V(S)$.

TU Co-op Game

An allocation z specifies a division of the total surplus:

$$\sum_{i \in N} z_i = V(N).$$

An allocation z is **blocked** by a coalition $S \subset N$ if

$$\sum_{i \in S} z_i < V(S).$$

The **Core** of (v, N) is the set of unblocked allocations:

$$C(V, N) = \{z \in \Re^n : \sum_{j \in N} z_j = V(N), \sum_{j \in S} z_j \geq V(S)\ \forall S \subset N\}.$$

TU Co-op Game

Let $B(N)$ be feasible solutions to the following:

$$\sum_{S:i\in S} \delta_S = 1 \quad \forall i \in N$$

$$\delta_S \geq 0 \quad \forall S \subset N$$

Each $\delta \in B(N)$ are called balancing weights. $C(V, N) \neq \emptyset$ iff

$$V(N) \geq \sum_{S \subset N} V(S)\delta_S \quad \forall \delta \in B(N).$$

NTU Co-op Game

A game with nontransferable utility is a pair (N, V) where N is a finite set of players, and, for every coalition $S \subseteq N$, $V(S)$ is a subset of $\Re^n$ satisfying:

1. If $S \neq \emptyset$, then $V(S)$ is non-empty and closed; and $V(\emptyset) = \emptyset$.
2. For every $i \in N$ there is a V_i such that for all $x \in \Re^n$, $x \in V(i)$ if and only if $x_i \leq V_i$.
3. If $x \in V(S)$ and $y \in \Re^n$ with $y_i \leq x_i$ for all $i \in S$ then $y \in V(S)$ (lower comprehensive).
4. The set $\{x \in V(N) : x_i \leq V_i\}$ is compact.

NTU Co-op Game

Core of an NTU-game (N, V) is all payoff vectors that are feasible for the grand coalition N and that cannot be improved upon by any coalition, including N itself.

If $x \in V(N)$, then coalition S can improve upon x if there is a $y \in V(S)$ with $y_i > x_i$ for all $i \in S$.

Core of the game (N, V) is

$$V(N) \setminus \cup_{S \subseteq N} intV(S).$$

NTU Co-op Game

NTU game (N, V) is balanced if for any balanced collection $\mathcal{C}$ of subsets of N,

$$\cap_{S \in \mathcal{C}} V(S) \subseteq V(N).$$

Scarf's lemma: a balanced NTU game has a non-empty core.

Proof of Scarf's Lemma

A is an $m \times n$ *positive* matrix.

$\mathcal{P} = \{x \geq 0 : Ax \leq e\}$.

$U = \{u_{ij}\}$ is an $m \times n$ positive matrix.

$x \in F$ is **dominating** if for each column index k there is a row index i such that

1. $\sum_{j=1}^{n} a_{ij}x_j = 1$ and
2. $u_{ik} \leq \min_{j:x_j>0} u_{ij}$.

Proof of Scarf's Lemma

Associate a 2 person game with the pair (U, A).

Fix a large integer t, let $w_{ij} = -\frac{1}{u_{ij}^t}$.

Payoff matrix for row player will be A.

Payoff matrix for the column player will be W.

Proof of Scarf's Lemma

(x^*, y^*) is an equilibrium pair of mixed strategies for the game.

x^* is mixed strategy for column (payoff matrix W).

y^* is mixed strategy for row player (payoff matrix A).

For all pure strategies q for row:

$$\sum_{i=1}^{m} y_i^* [\sum_{j=1}^{n} a_{ij} x_j^*] \geq \sum_{j=1}^{n} a_{qj} x_j^*, \ \forall q. \tag{4}$$

(4) will bind when $y_q^* > 0$.

Proof of Scarf's Lemma

For all pure strategies r for column

$$\sum_{i=1}^{m} y_i^*[\sum_{j=1}^{n}(u_{ij})^{-t}x_j^*] = \sum_{j=1}^{n} x_j^*[\sum_{i=1}^{m}(u_{ij})^{-t}y_i^*] \leq \sum_{i=1}^{m}(u_{ir})^{-t}y_i^*, \ \forall r \tag{5}$$

Therefore, for each column r there is a row index i_r with $y_{i_r}^* > 0$ such that

$$\sum_{j=1}^{n}(u_{i_rj})^{-t}x_j^* \leq (u_{i_rr})^{-t} \Rightarrow u_{i_rr} \leq (\frac{1}{\sum_{j=1}^{n}(u_{i_rj})^{-t}x_j^*})^{1/t}.$$

Proof of Scarf's Lemma

$u_{i_r r} \leq \frac{u_{i_r k}}{(x_k^*)^{1/t}}$ for all $x_k^* > 0$.

$t \to \infty$, x^* and y^* will converge to some $\bar{x}$ and $\bar{y}$ respectively.

For sufficiently large t, $\bar{x}_j > 0 \;\Rightarrow x_j^* > 0$ and $\bar{y}_j > 0 \;\Rightarrow y_j^* > 0$.

$$u_{i_r r} \leq \frac{u_{i_r k}}{(x_k^*)^{1/t}} \to u_{i_r k} \;\forall x_k^* > 0$$

Recall that for index i_r we have $y_{i_r}^* > 0$. Therefore, $\sum_{j=1}^m a_{i_r j} x_j^* = \sum_{i=1}^n y_i^* [\sum_{j=1}^m a_{ij} x_j^*] = v$.

$x = \frac{\bar{x}}{v} \in F$ and $\sum_{j=1}^n a_{i_r j} x_j = 1$.