EPR ALIGNMENT OF REFERENCE FRAMES

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Application of non-locality to precision measurements?

Non-locality has applications
to a number of pure information-theoretic tasks.

Can we find applications to physical tasks, such as
clock synchronization and
direction alignment?

quantum reference frames:
Aharonov-Kaufherr PRD 1984,
Gisin-Popescu PRL 1998,
Peres-Scudo, 2001, ...
Bartlett-Rudolph-Spekkens RMP 2007

Gyroscopes

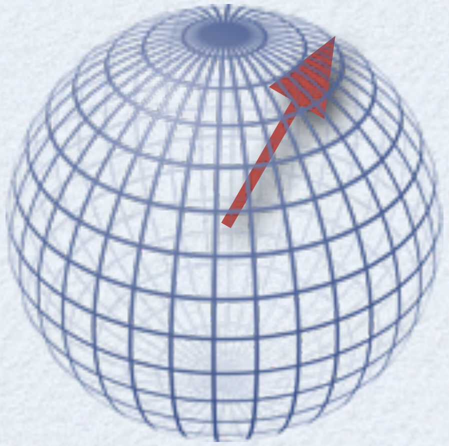
Classical gyroscope = physical system whose angular momentum indicates a direction in space

large angular momentum

$\longrightarrow$ more stable gyroscope

Quantum gyroscope = quantum system whose angular momentum indicates a direction in space

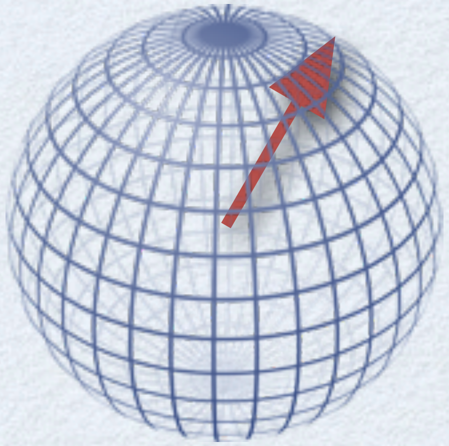
e.g. a spin-j particle



large j

$\longrightarrow$ more precise gyroscope

Spin j degrees of freedom



2j+1 Hilbert space

Rotation of an angle φ around the axis $\mathbf{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$:

$$e^{-i\varphi\,\mathbf{n}\cdot\mathbf{J}}$$

where $J_x,\ J_y,\ J_z$ are the angular momentum operators

$$[j_x, j_y] = i\,j_z \qquad [j_y, j_z] = i\,j_x \qquad [j_z, j_x] = i\,j_y$$

$$j_x^2 + j_y^2 + j_z^2 = j(j+1)\,I$$

Quantifying the error

To find out the direction, one has to perform a measurement,

mathematically described by a POVM $P\left(\mathrm{d}\widehat{\mathbf{n}}\right)$

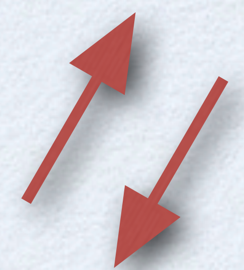
whose outcome gives an estimate $\widehat{\mathbf{n}}$ of the unknown direction.

The error is quantified by the worst-case square distance

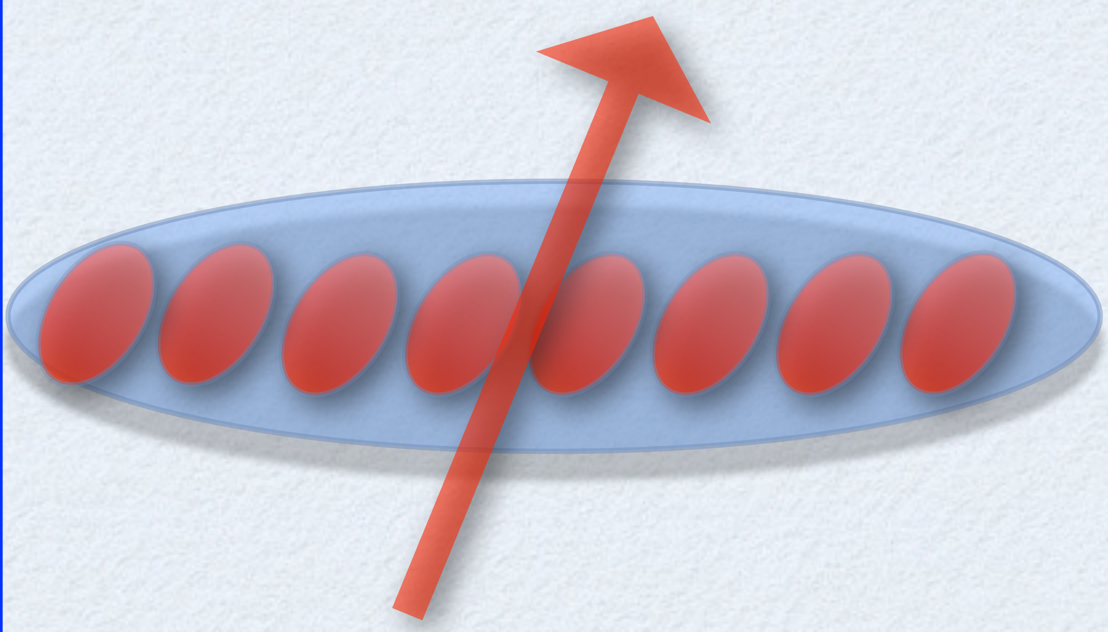
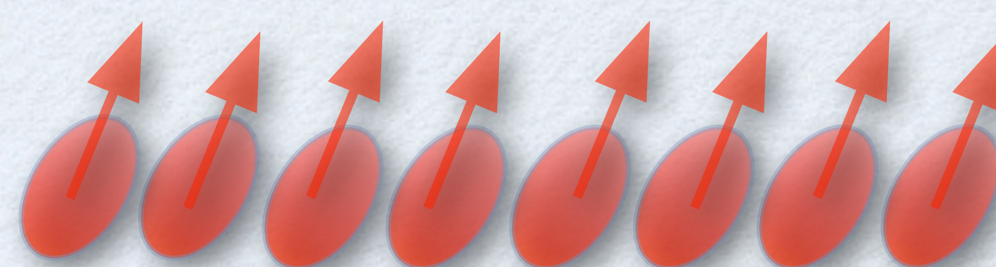
$$\langle d^2 \rangle = \sup_{\mathbf{n}} \int p\left(\mathrm{d}\widehat{\mathbf{n}} \,|\, \mathbf{n}\right) \, \|\widehat{\mathbf{n}} - \mathbf{n}\|^2$$

$$p(\mathrm{d}\widehat{\mathbf{n}}|\mathbf{n}) \; = \; \mathrm{Tr}\left[P\left(\mathrm{d}\widehat{\mathbf{n}}\right) \, \rho_{\mathbf{n}}\right]$$

What was known: $j = 1/2$

Gisin-Popescu PRL 1998:  is better than 

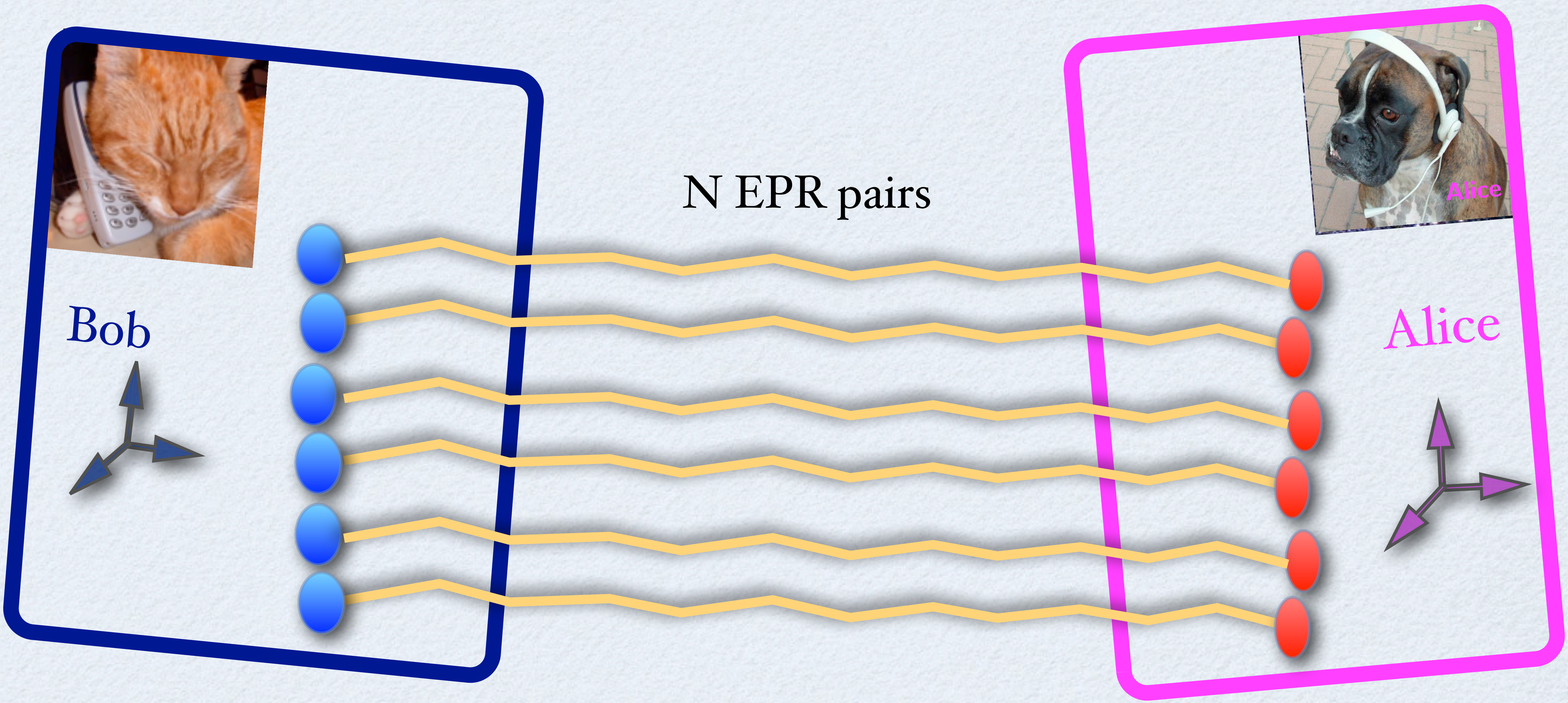
GC et al PRL 2004, Bagan et al PRA 2004, Hayashi PLA 2006

 is better than 

entangling N particles reduces the error by N^2 (instead of N)

The two-party scenario

Rudolph 1999 arXiv
Alice measures her spins along the z-axis, Bob tries to find out the direction.



Scaling up Gisin-Popescu result?

...not much

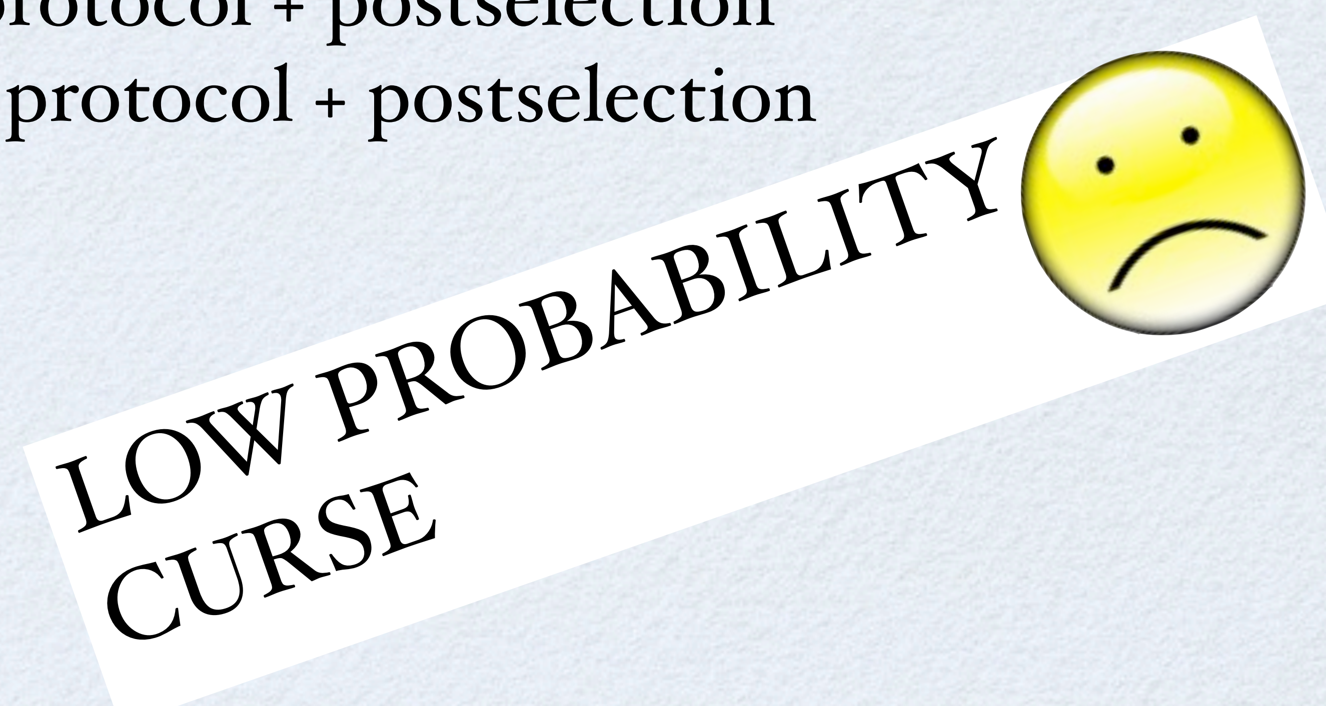
Deterministic strategies:

- O(1/N) error with Rudolph's protocol
- O(1/N) error with the optimal protocol

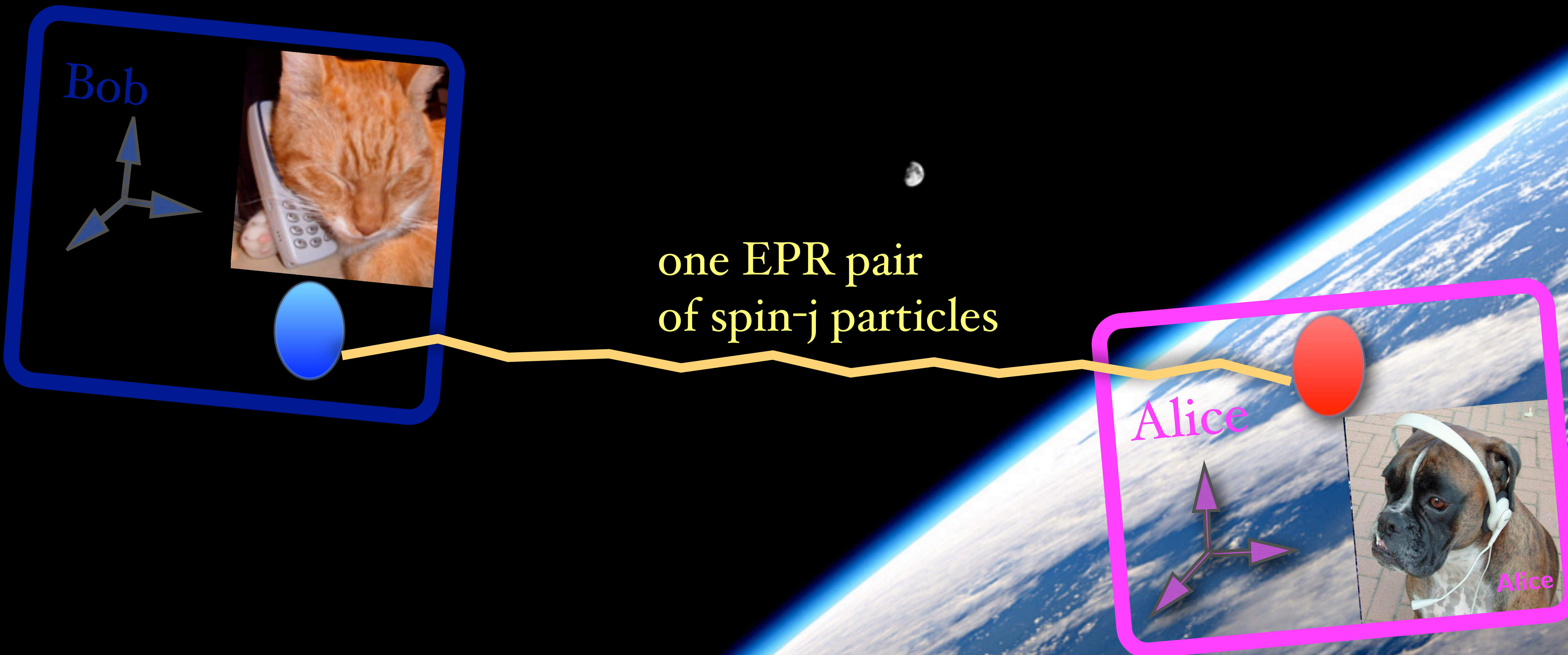
Probabilistic strategies:

- $O(1/N^2)$ error with Rudolph's protocol + postselection
- $O(1/N^2)$ error with the optimal protocol + postselection

probability of success: $O\left(2^{-N}\right)$



USING ENTANGLED GYROSCOPES OF LARGER ANGULAR MOMENTUM



$$|S_j\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^m \, |j,m\rangle \otimes |j,-m\rangle$$

$$J_z \, |j,m\rangle = m \, |j,m\rangle$$

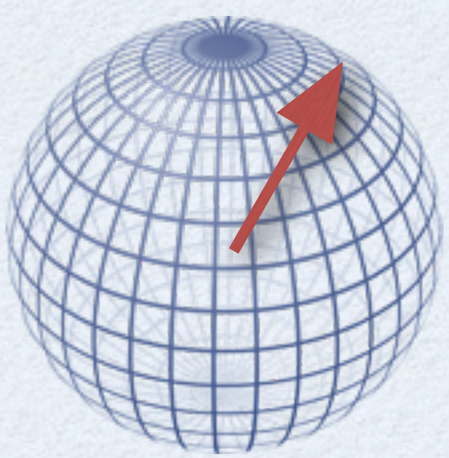
optimal state for alignment

The error

Error $\langle d^2 \rangle$

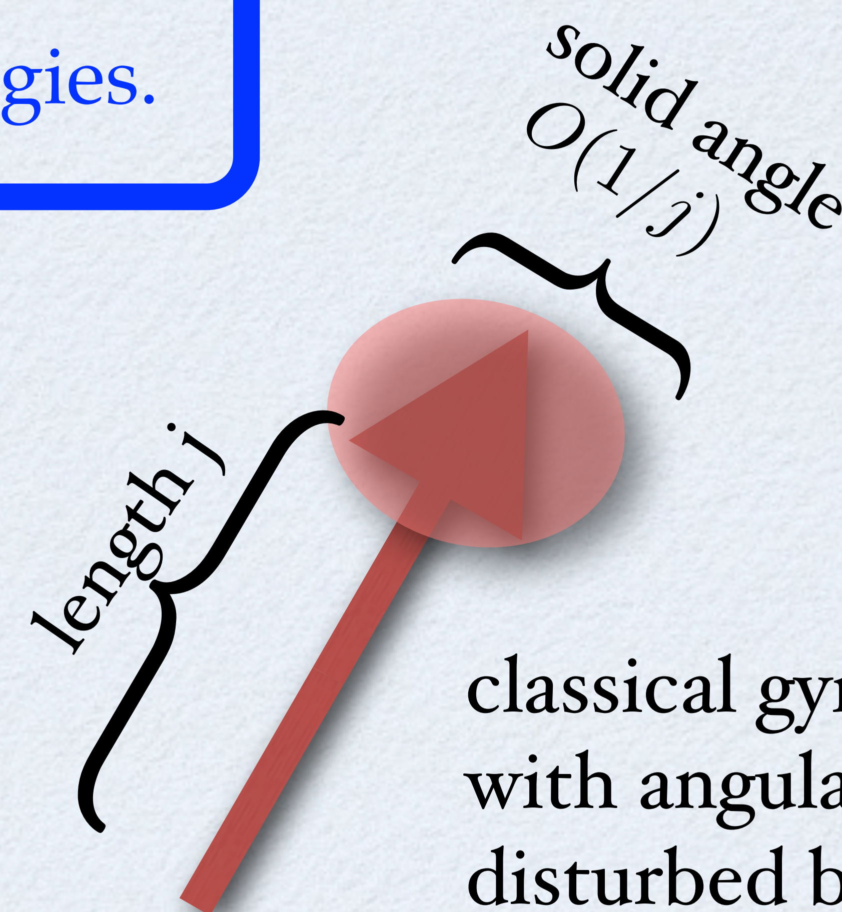
No h ... bilistic strategies.

CLASSICAL SCALING



spin j
gyroscope

$\approx$

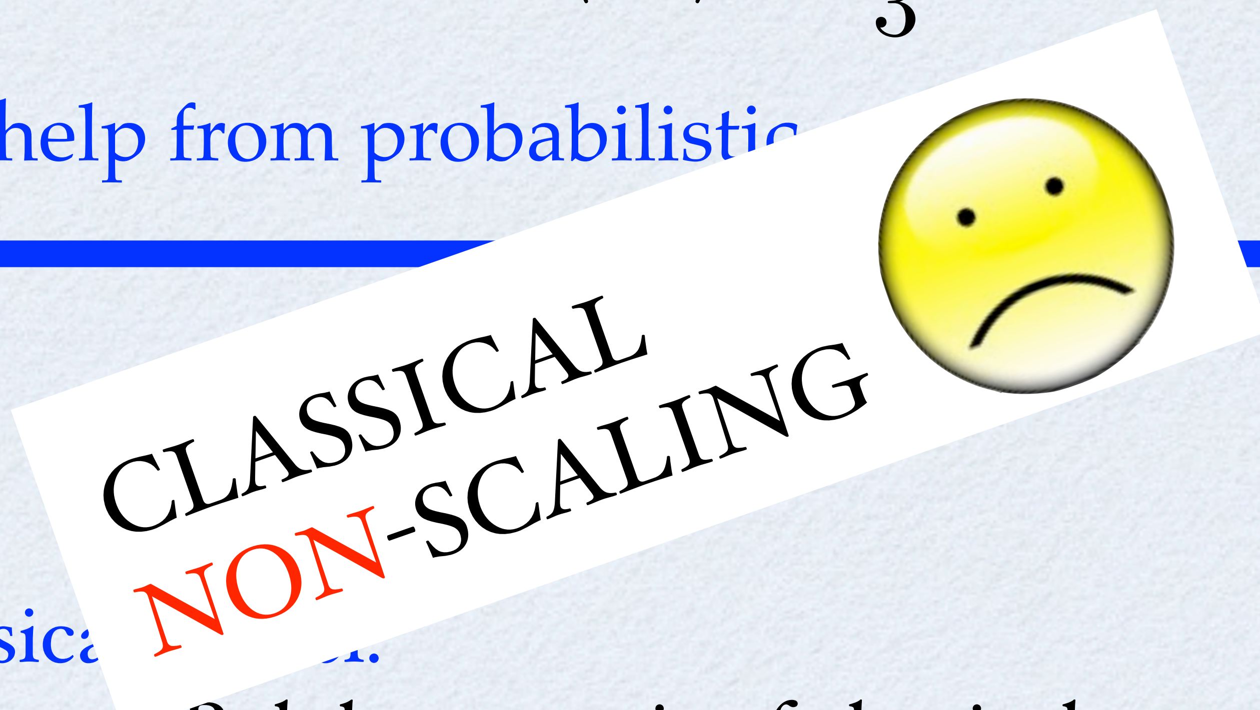


classical gyroscope
with angular momentum j,
disturbed by a random force
of fixed intensity,
error due to precession.

Full Cartesian frames?

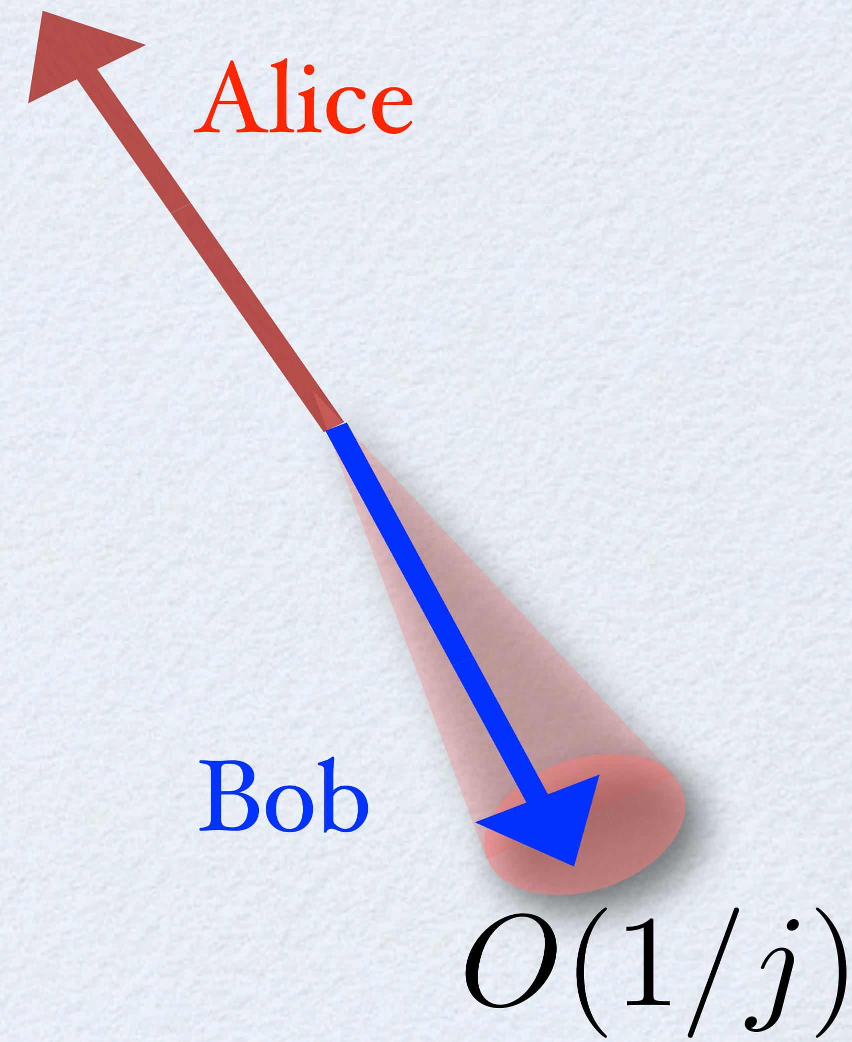
Irreducible error $\langle d^2 \rangle \geq \frac{4}{3} \qquad \forall j$

No help from probabilistic



Classica l:

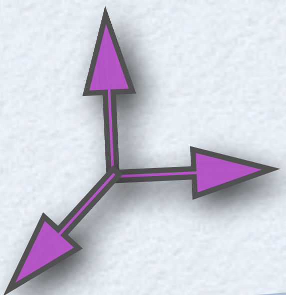
Alice and Bob have a pair of classical gyroscopes pointing in random, anti-correlated directions, up to an error O(1/j)



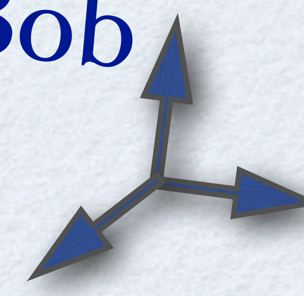
TWO EPR PAIRS
WITH
THE ASSISTANCE
OF
LOGICAL ENTANGLEMENT

The teleportation trick

Alice



Bob



Bell measurement

EPR pair of spin-j gyroscopes

EPR pair of spin-j gyroscopes

Log(2j+1) logical EPRpairs

Log(2j+1) logical EPR pairs

Correction operation

$|S_{j,g}\rangle$

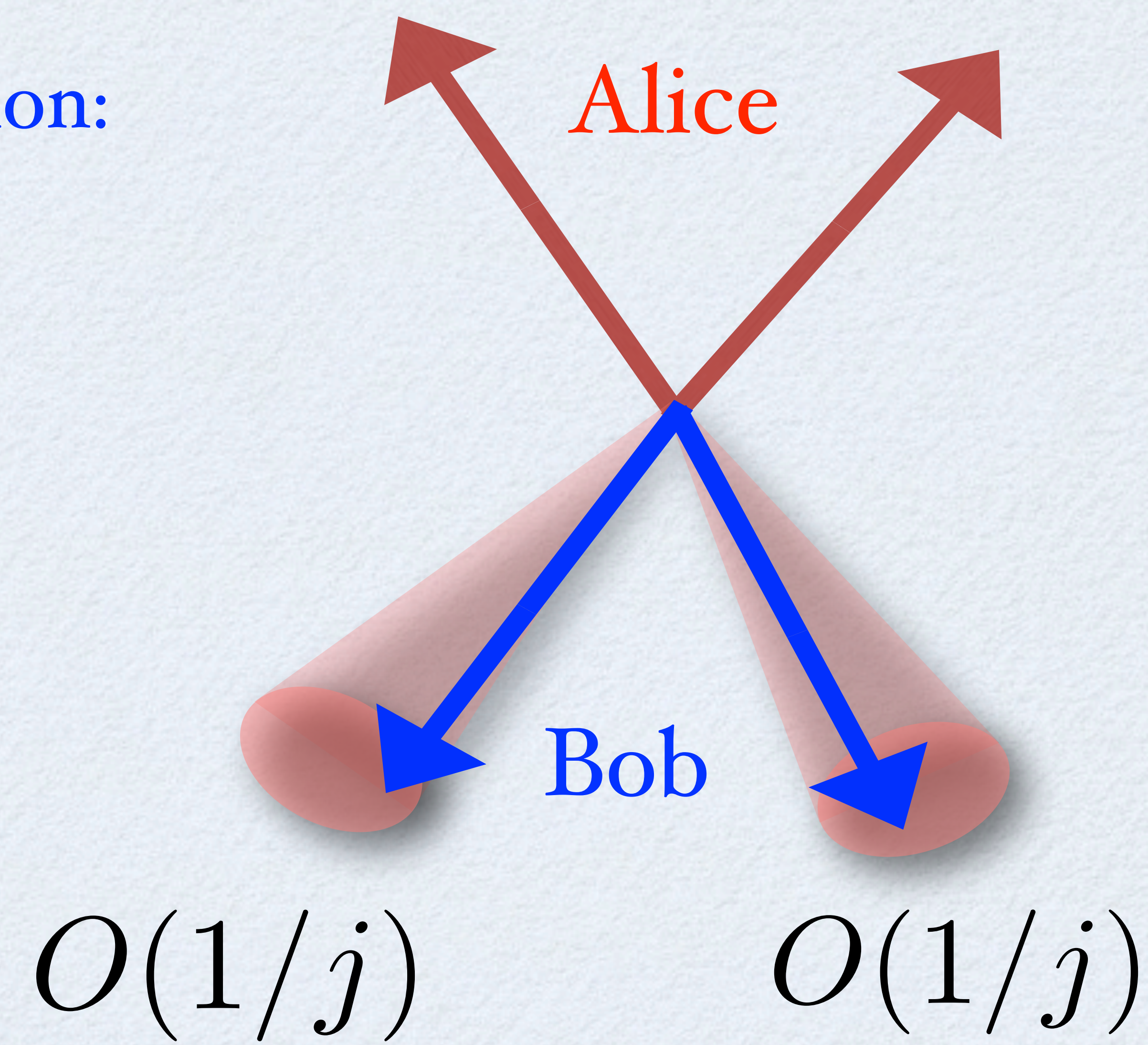
$$|S_{j,g}\rangle = \left(e^{-i\varphi\, \mathbf{n}\cdot\mathbf{j}} \otimes I\right) |S_j\rangle$$

MORE EPR PAIRS

Error for the full Cartesian frame

$$\langle d^2 \rangle = \frac{2}{3j} + O\left(\frac{\log j}{j^2}\right)$$

Again, classical explanation:



Probabilistic strategies

Suppose that Bob uses a probabilistic filter, with two outcomes “yes” and “no”

The yes case

The filter implements the transformation

$$|S_{j,g}\rangle^{\otimes 2} = \bigoplus_{k=0}^{2j} \sqrt{\frac{2k+1}{2j+1}}\, |S_{k,g}\rangle$$

$$\downarrow$$

$$|\Phi^{\mathrm{yes}}_{j,g}\rangle = \bigoplus_{k=0}^{2j} \frac{\sin\frac{\pi(k+1)}{2(j+1)}}{\sqrt{j+1}}\, |S_{k,g}\rangle$$

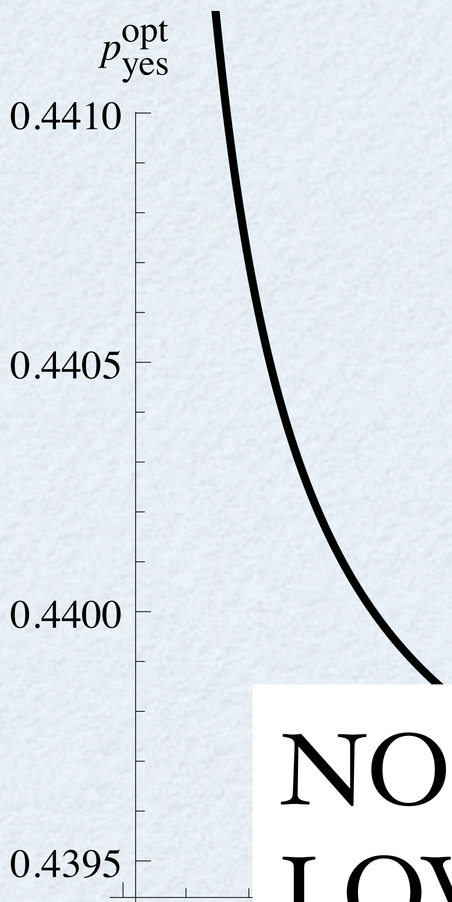
= optimal state for transmitting a Cartesian frame (GC, D'Ariano, Perinotti, Sacchi PRL 2004, Bagan et al PRA 2004, Hayashi PLA 2006)

$$\Longrightarrow \langle d^2\rangle = \frac{\pi^2}{6j^2} + O\left(\frac{1}{j^3}\right)$$

Probabilistic super-activation

In the classical model, Alice and Bob cannot align their axes with an error smaller than $O(1/j)$

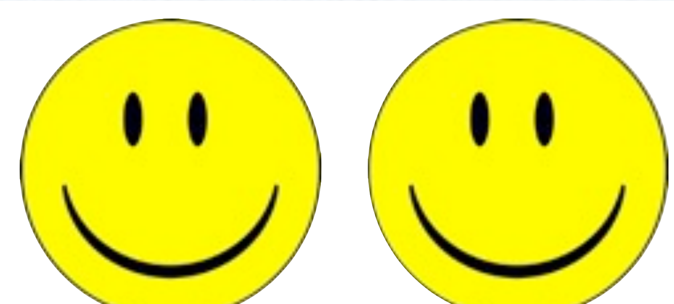
"Probabilistic super-activation" ...what is the probability of seeing it?



$$p_{\mathrm{yes}}^{\mathrm{opt}} = \min_k \frac{(2k+1)(j+1)}{(2j+1)^2 \sin\left[\frac{\pi(k+1)}{2(j+1)}\right]^2}$$

$\rightarrow 43.9\%$ for $j \rightarrow \infty$!

NO
LOW PROBABILITY CURSE



The no case

In the unfavorable instance, the error is $\langle d^2 \rangle \approx \frac{1.189}{j}$

Of course, no gain on average

$$\langle d^2 \rangle_{\text{average}} \approx p_{\text{yes}}^{\text{opt}} \frac{\pi^2}{6j^2} + \left(1 - p_{\text{yes}}^{\text{opt}}\right) \frac{1.189}{j}$$

$$\approx \left(1 - p_{\text{yes}}^{\text{opt}}\right) \frac{1.189}{j}$$

$$\approx \frac{0.668}{j} \geq \frac{0.666}{j} = \frac{2}{3j}$$

BUT 1) almost same average performance

2) the quadratic improvement is heralded

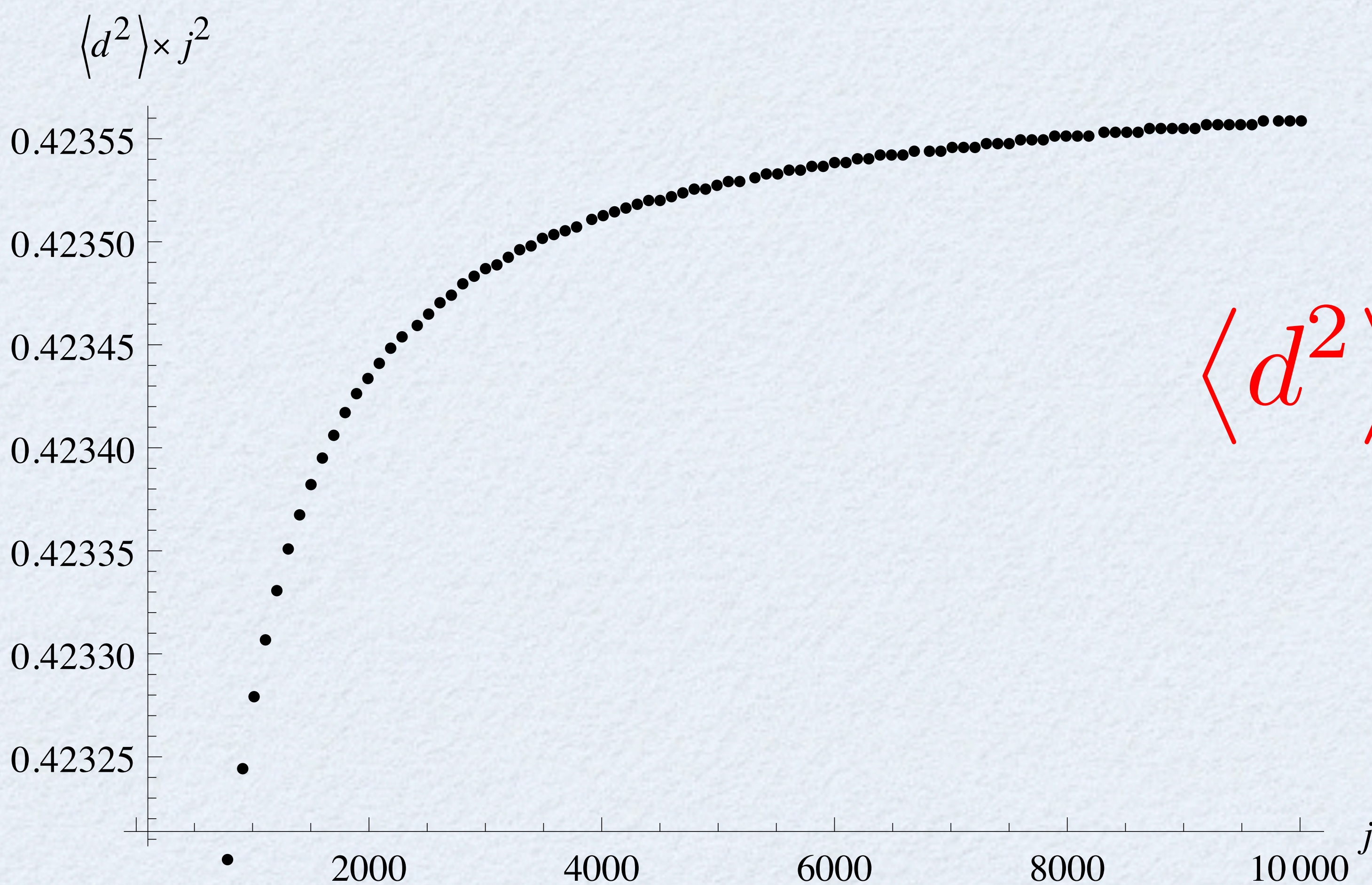
MORE
EPR PAIRS

Deterministic super-activation

2N EPR pairs $\longrightarrow$ Heisenberg scaling with probability

$$p_{\text{yes}} \geq 1 - (0,561)^N \quad \text{(brute force repetition)}$$

In fact, the optimal strategy can do much better:
for 4 pairs the Heisenberg scaling is achieved deterministically

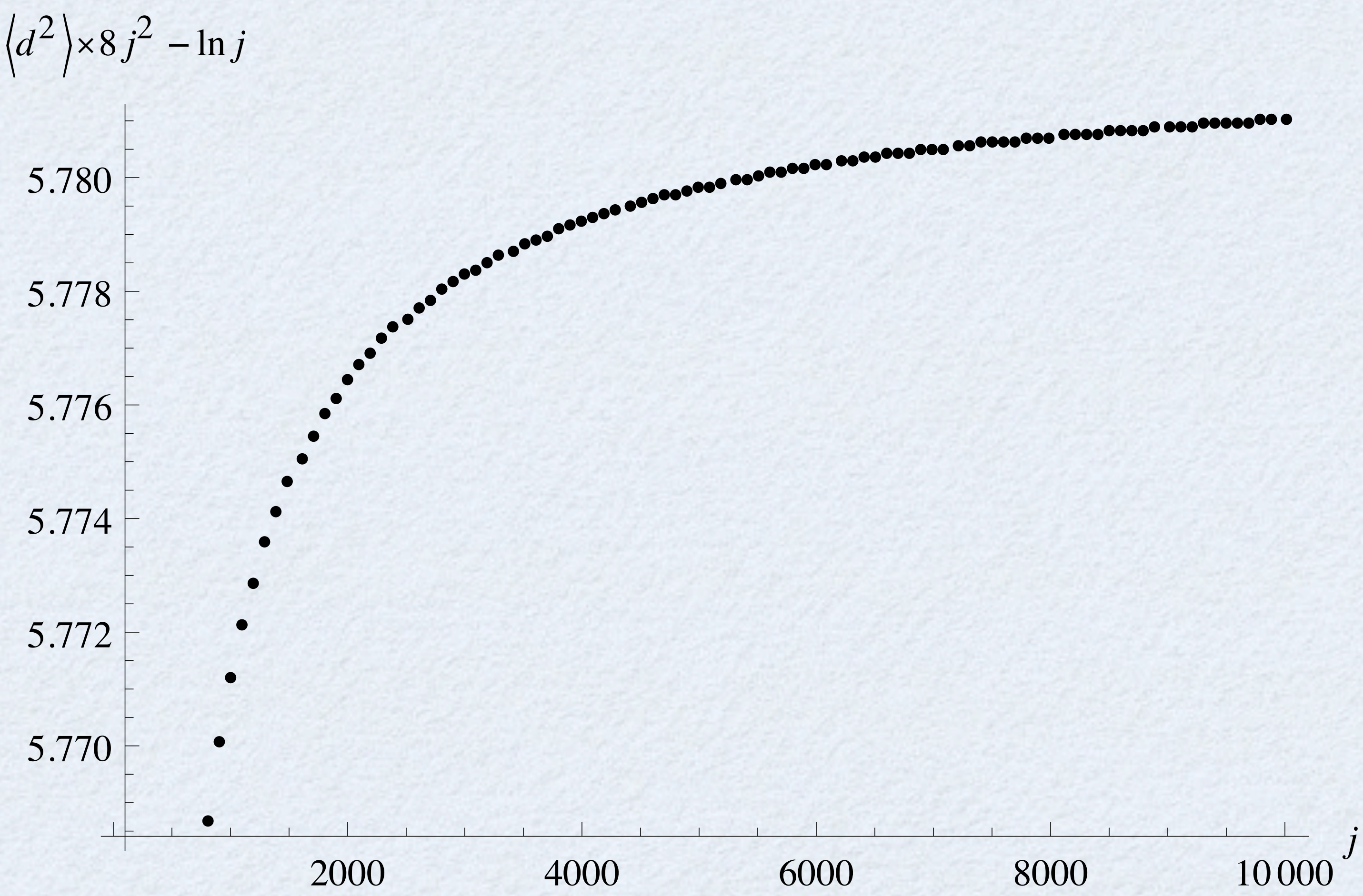


$$\langle d^2 \rangle = \frac{11 \ln 2}{18 j^2} + O\left(\frac{1}{j^3}\right)$$

Quasi-Heisenberg scaling for 3 EPR pairs

4 EPR pairs are the minimum to achieve the Heisenberg scaling deterministically.

Still, if Alice and Bob have only 3 pairs, they can still beat the classical scaling. The optimal strategy yields:



$$\langle d^2 \rangle = \frac{\ln j}{8j^2} + O\left(\frac{1}{j^2}\right)$$

(quasi-Heisenberg scaling)

A CAUTIONARY TALE ABOUT THE QUANTUM CRAMÈR-RAO BOUND IN NON-ASYMPTOTIC SCENARIOS

Cramèr-Rao bound

State parametrization $|S_{j,\boldsymbol{\theta}}\rangle = \left(e^{-i\boldsymbol{\theta}\cdot\mathbf{j}} \otimes I\right) |S_j\rangle \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \end{pmatrix}$

Quantum Fisher Information matrix $F_Q = \frac{4j(j+1)}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Quantum Cramèr-Rao bound $V_{\boldsymbol{\theta}} \geq F_Q^{-1} = \frac{3}{4j(j+1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

achievable in the asymptotic regime of large number of copies

Caveat

A naive application of the CRB
would promise Heisenberg scaling of the error:

$$V_{\boldsymbol{\theta}} \approx O\left(\frac{1}{j^2}\right)\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \Longrightarrow \quad \langle d^2 \rangle \approx O\left(\frac{1}{j^2}\right)$$

in contradiction with the analytical results for 1,2 and 3 pairs

For spin-j singlets,
the CRB is not achievable in the finite-copy regime

Asymptotic achievability of CRB

To achieve the CRB, the number of copies must be large.
But how large?

Hopefully not large compared to j^2 ...

Covariance matrix for the optimal measurement:

$$V^{\mathrm{opt}}_{\boldsymbol{\theta},n} = \frac{3}{4nj(j+1)}\, I + O\left(\max\left\{n^{-3/2} j^{-3}, n^{-2} j^{-2}\right\}\right).$$

CRB achieved whenever $n \gg 1$, uniformly in j

CONCLUSIONS

Conclusions

Super-activation of quantum gyroscopes:

1 EPR pair $\longrightarrow$ no scaling with the size
2 EPR pairs $\longrightarrow$ Heisenberg scaling with non-vanishing prob.
3 EPR pairs $\longrightarrow$ quasi-Heisenberg
≥4 EPR pairs $\longrightarrow$ Heisenberg with certainty

The moral:

- not having a pre-defined direction helps
- two spin-j particles are more useful than a single spin-(2j) particle
- logical qubits help
- ...be careful using the quantum CRB in non-asymptotic scenarios!

THANK YOU

FOR YOUR

ATTENTION!