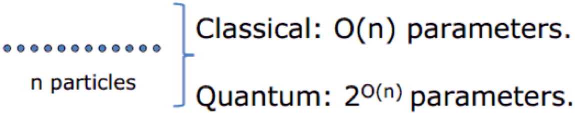
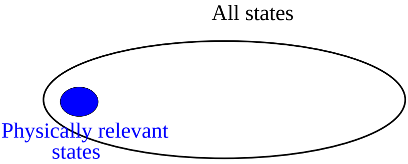
Local Algorithms for 1D Quantum Systems

I. Arad, Z. Landau, U. Vazirani, T. Vidick

Many-body physics



Are physical systems special?



Ground States of Local Hamiltonians

- Hilbert space of particles $\mathcal{H} = (\mathbb{C}^d)^{\otimes n}$,
- Local terms $0 \leq H_i \leq 1$,
- Interested in lowest eignevectors of $H = \sum_i H_i$.

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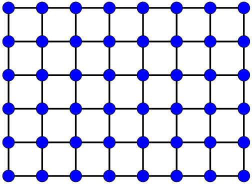

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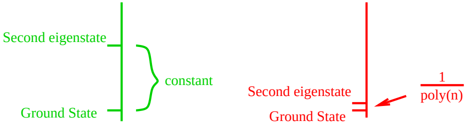

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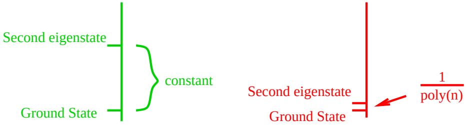

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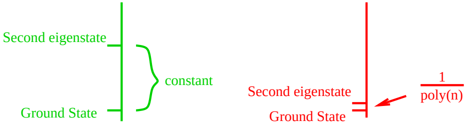

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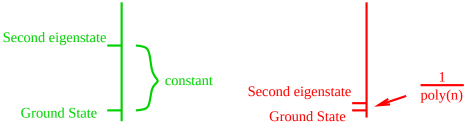

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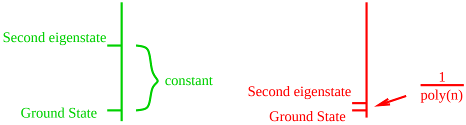

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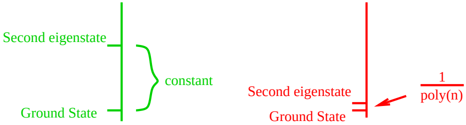

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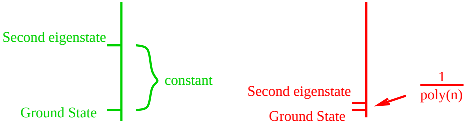

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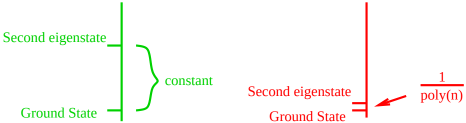

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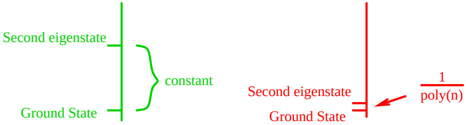

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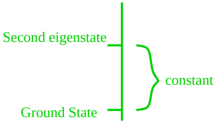

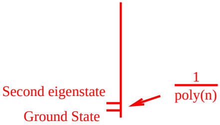

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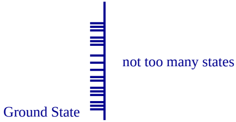
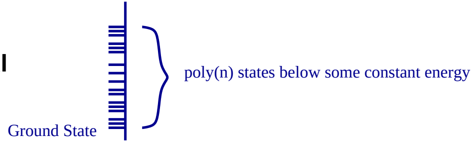


Result: Algorithm for critical systems

poly(n) eigenvalues in constant interval



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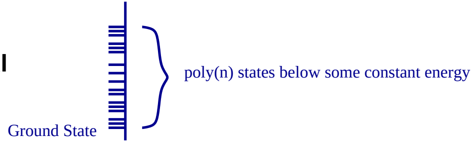

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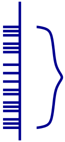

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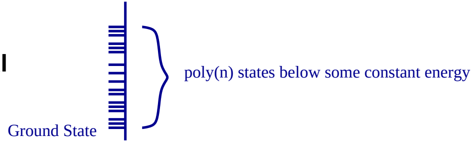

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- More physically plausible: No convex optimization.
- Hierarchical structure reminiscent of Renormalization Group.

Result: Algorithm for degenerate ground space

Degenerate ground space with gap:

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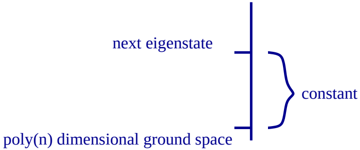

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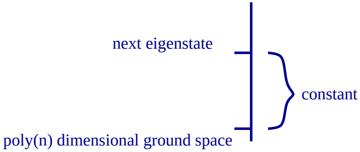

- Polynomial time algorithm for finding the ground space.

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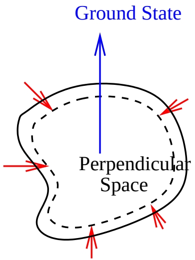

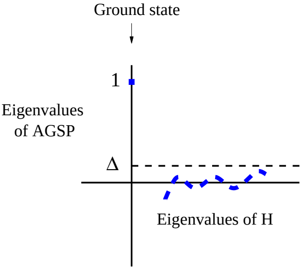

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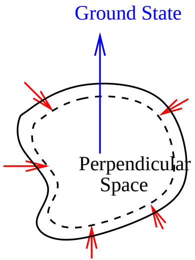
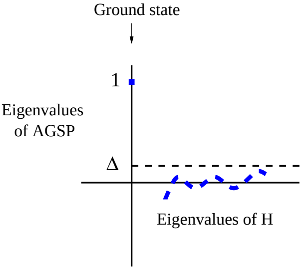
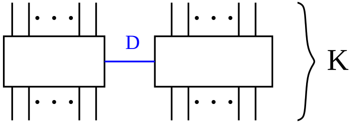




- It has small entanglement rank:



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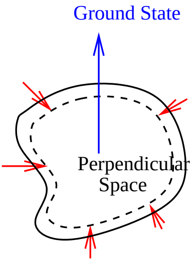

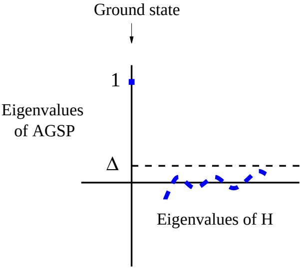

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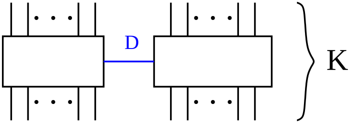
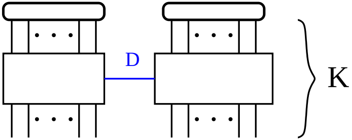


Critical threshold $D\Delta < 1$.

Steps towards an algorithm

A first pass.

1. Start with unentangled state.
2. Apply AGSP. Randomly pick one Schmidt vector. Repeat.



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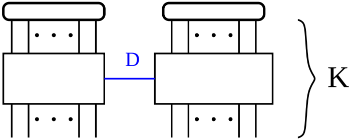

Each iteration moves closer $\rightarrow$ Area Law.

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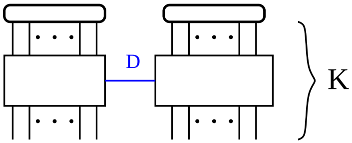

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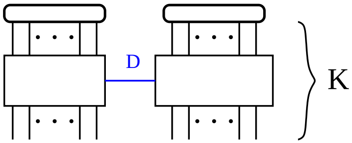

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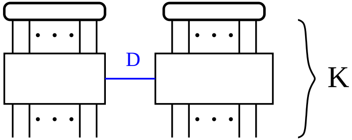

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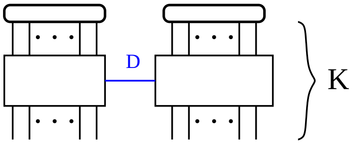

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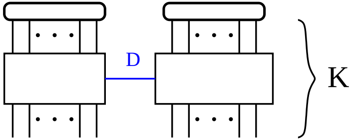

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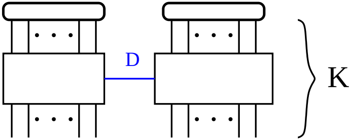

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1. Reduce complexity $\rightarrow$ original algorithm.
2. Reduce iterations $\rightarrow$ new results.

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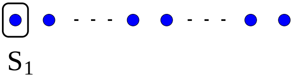

Viable set: subspace containing left Schmidt vectors of a good approximation to ground space

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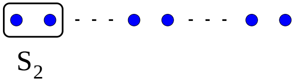

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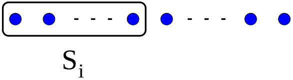

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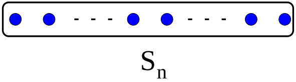

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Move more quickly via simultaneous local movement?

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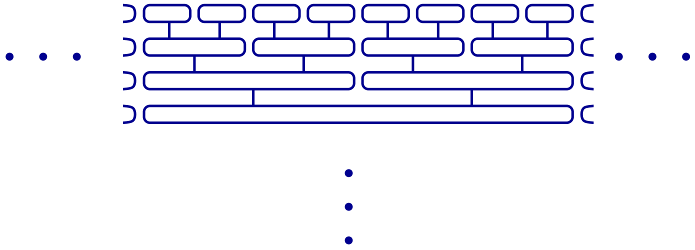

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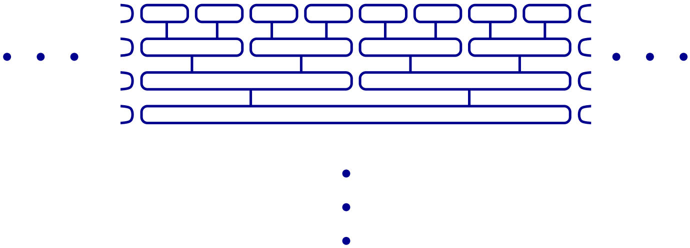

Need: a way to merge two viable sets.

New Algorithm Ingredients

- *Merge process*
- Modified AGSP
- Viable set for subspaces

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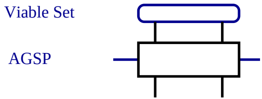
   1. Choose a small random subspace of $V' \subset V$,
   2. "Apply AGSP"
      write $K = \sum_{i,j=1}^{D} A_i \otimes B_{i,j} \otimes C_j$, and choose $V'' = span\{B_{i,j}V'\}$.



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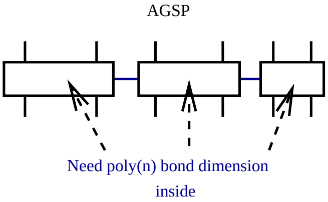

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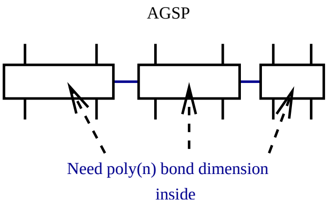

Requires soft truncation

- Uses [Molnar, Schuch, Verstraete, Cirac]: $e^{-\beta H}$ has small bond dimension MPO approximation.

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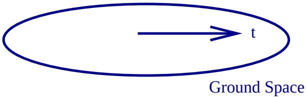

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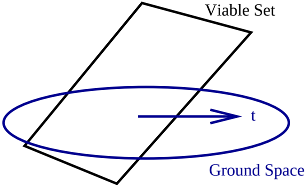

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- Modified AGSP
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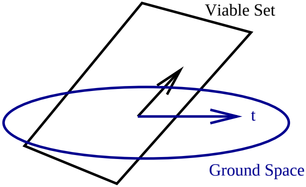

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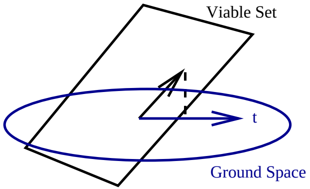
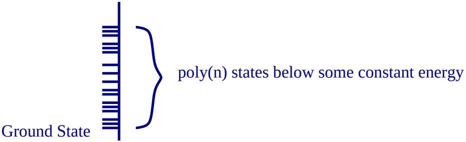
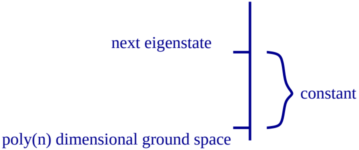


Discussion





- Physics viewpoint: when do density of states conditions occur? Examples when they don't? Proofs?
- Could this be a skeleton for a practical algorithm? Are there interesting questions in this regime to answer? What kinds of challenges exist?
- What kinds of questions/connections does the tree-like structure of the algorithm suggest?