Communication complexity of sparse set disjointness and exists-equal

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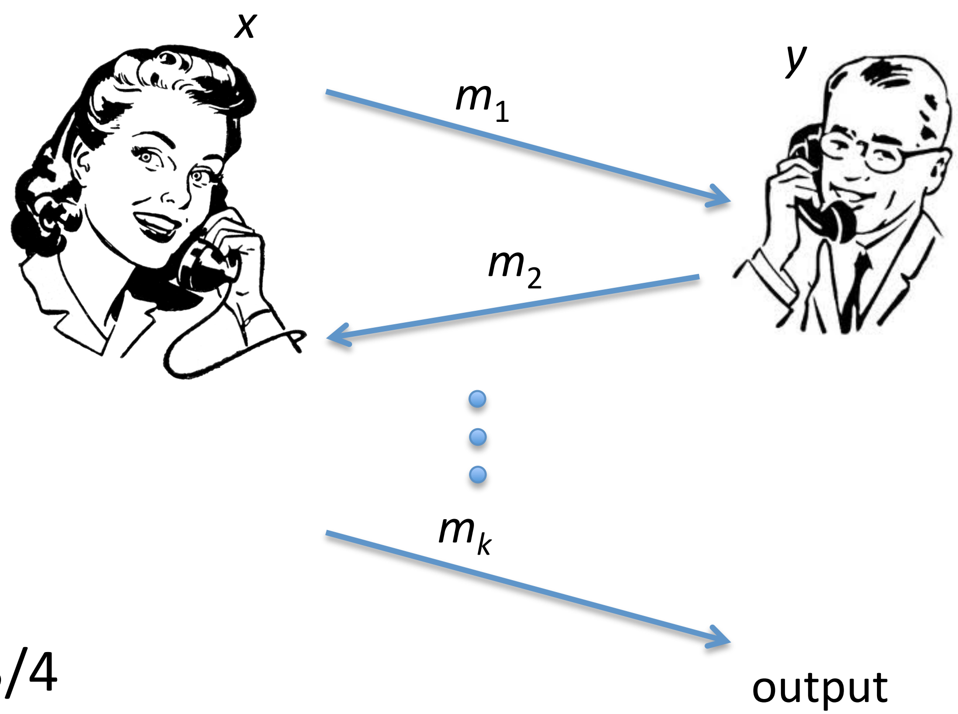
Model considered

- 2-party

  Alice and Bob
  computing $f(x,y)$

- randomized
  with joint random source

  $\forall x,y$: Pr[output = $f(x,y)$] > 3/4



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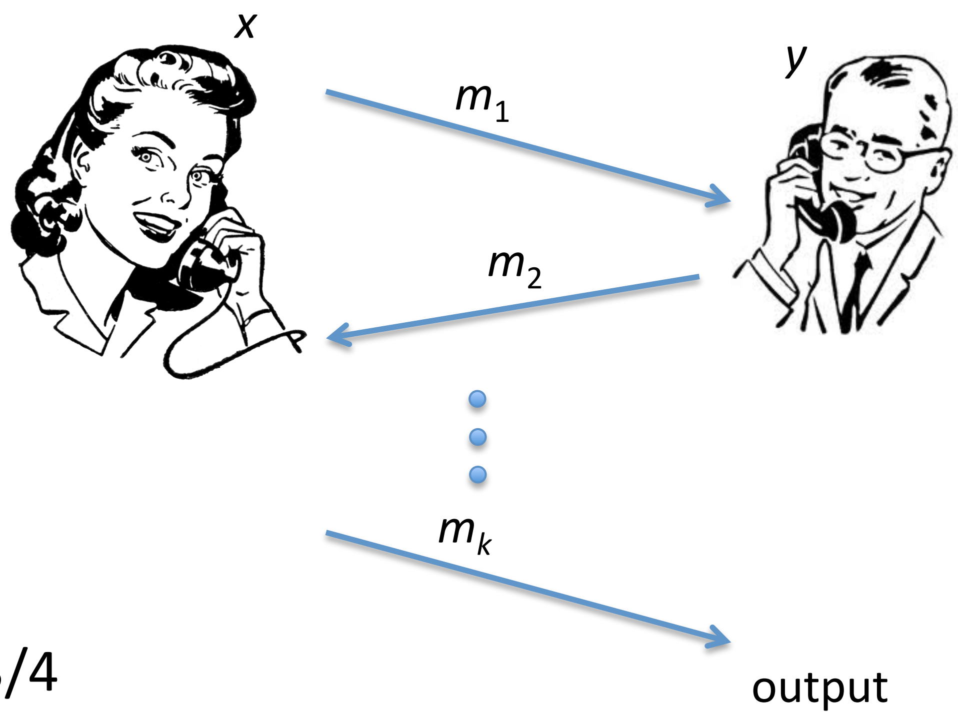

Goals

1. Minimize total communication: $|m_1|+|m_2|+\ldots+|m_k|$
2. Minimize # of rounds: k

tradeoff?

main examples

- Set disjointness D_n: $x,y \subseteq H$; $x \cap y = \emptyset$? $|H|=n$

Can we meet next week?

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- Equality E_n: $x,y \in \{0,1\}^n$; $x = y$?

Are our copies of
Harry Potter and the
Sourcerer’s Stone
identical?

main examples

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Well understood:

- D_n requires $\Omega(n)$ bits of communication (Kalyanasundaram-Schnitger)
- 2 bits / 1 round enough for E_n

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We consider variants of these problems

set disjointness

$D_n \; : x, y \subseteq H; \quad x \cap y = \emptyset \, ? \quad |H| = n$

sparse set disjointness

$SD_{k,n}$: $x,y \subseteq H$; $x \cap y = \emptyset$? $|H|=n$; $|x|,|y| \leq k$; $k << n$

sparse set disjointness

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containments:

$$D_k \leq SD_{k,n} \leq D_n$$

Complexity of $SD_{k,n}$ is $\Omega(k)$,

$O(n)$, $O(k \log n)$, $O(k \log k)$.

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Thm [Håstad-Wigderson (2007)]: It is $\Theta(k)$.

Håstad-Wigderson protocol for $SD_{k,n}$

- O(k) bits (optimal)
- O(log k) rounds
- constant error proved

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• 0/1 output	outputs the actual intersection of the input sets
	+ r-round O(k log$^{(r)}k$) bit protocol for r < log*k
	+ optimality proof for all r

r times iterated log

Håstad-Wigderson protocol

$x \subseteq H$ **A.** picks random S_0 with $x \subseteq S_0 \subseteq H$

Alice sends S_0

“$x \cap y \subseteq S_0$”

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$y \subseteq H$

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B. picks random S_1 with $y \cap S_0 \subseteq S_1 \subseteq H$

Bob sends S_1

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$x \cap y \subseteq y \cap S_0 \cap S_2$

Alice sends S_2

etc.

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e.g. $x \cap \bigcap S_i$

Stop and output "disjoint" if current set is empty,

otherwise output "intersect" when
the O(log k) rounds or O(k) bits are used up.

Analysis of the Håstad-Wigderson protocol

- How to send a random set containing x?

  w_1, w_2,… random sets from joint random source

  Send index $\min\{j \mid w_j \supseteq x\}$.

  $$E[\text{\# of bits sent}] \approx |x|$$

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  Current set will be empty in $O(\log k)$ expected rounds.

  Expected total # of bits sent: $O(k)$.

protocol with fewer rounds

- biased random sets: contains each element independently with probability p << 1/2
- small p: quicker decrease in set size: $k \to pk$

  longer message for a random

  set containing x: $|x|\log(1/p)$

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  longer message for a random set containing x: $|x|\log(1/p)$
- optimal tradeoff:

$$1/p_{i+1} = 2^{1/p_i}$$

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Problem: “current set” x_i contains the true intersection $x \cap y$ — remains large throughout protocol

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Solution: no need for Bob to give full message S_{i+1}, enough to send a few bits — Alice chooses S_{i+1} to minimize $|x_i$-$S_{i+1}|$

Equality problem

x y

Is $x = y$?

O(1) bits enough in single round

(with joint random source)

Exists-equal problem

$x_1, x_2, \ldots, x_k$

Is $x_1 = y_1$ or

$x_2 = y_2$ or

…

$x_k = y_k$?

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Special case of Sparse Set Disjointness:

$\{(1,x_1),(2,x_2),\ldots,(k,x_k)\} \cap \{(1,y_1),(2,y_2),\ldots,(k,y_k)\} \neq \emptyset$?

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- $O(k)$ bits in $\log^* k$ rounds suffice.
- Or $O(k \log^{(r)} k)$ bits in r rounds.

This is optimal for any r.

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OR of k equalities

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best is to solve each
equality separately with $O(1/k)$ error

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- for Exists-equal   (sparse set disjointness follows)

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Hamming ball around x

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Obtained through "shifting"

expectation for random x

Hamming ball around x

Thank you