Recovering communities in the general stochastic block model

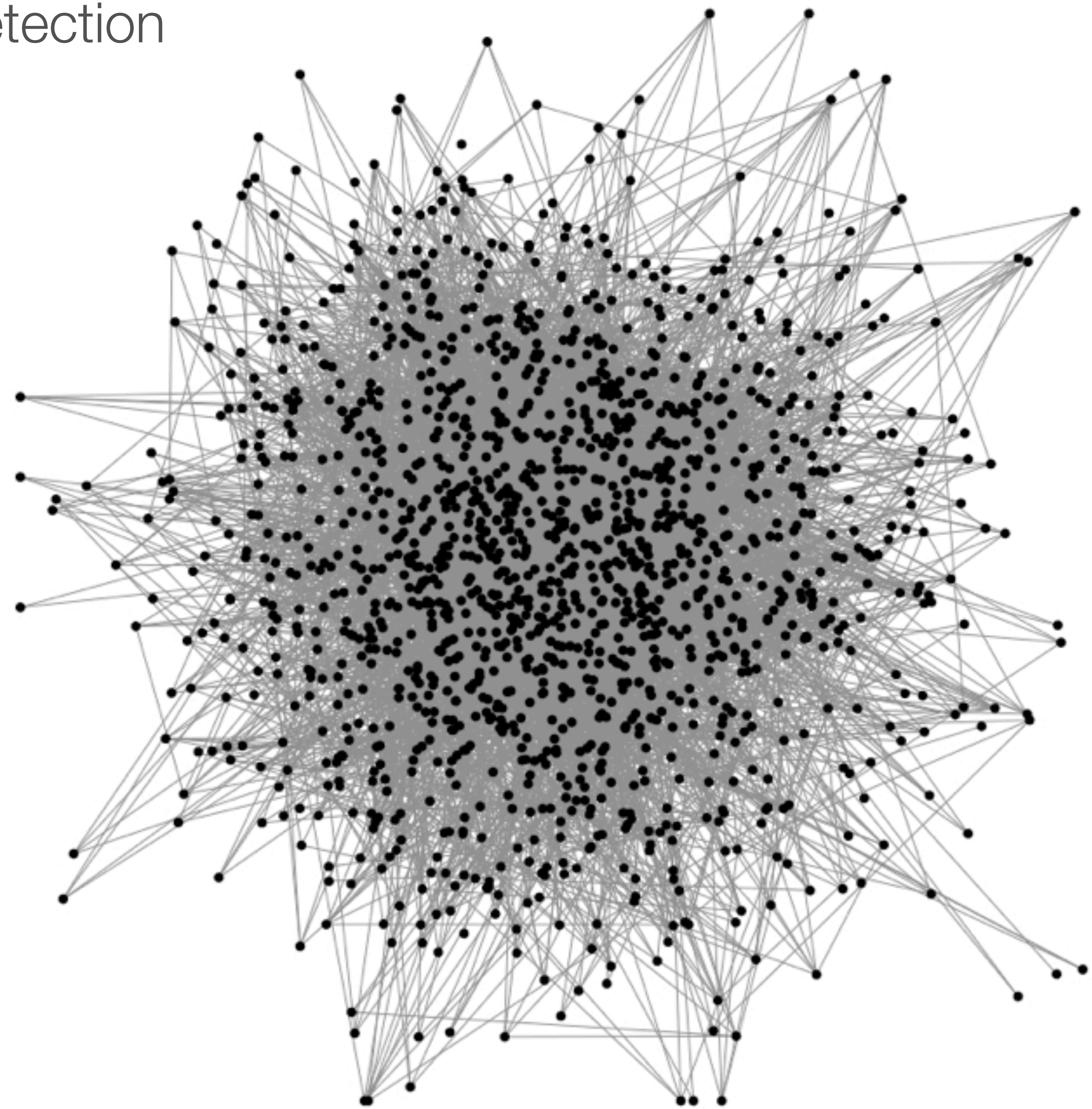
Emmanuel Abbe and Colin Sandon
Princeton University

http://arxiv.org/abs/1503.00609

Simons Institute, 03.16.15

community detection

community detection



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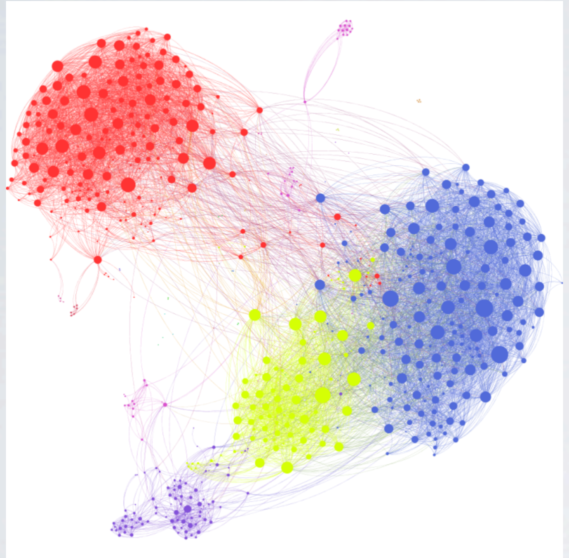
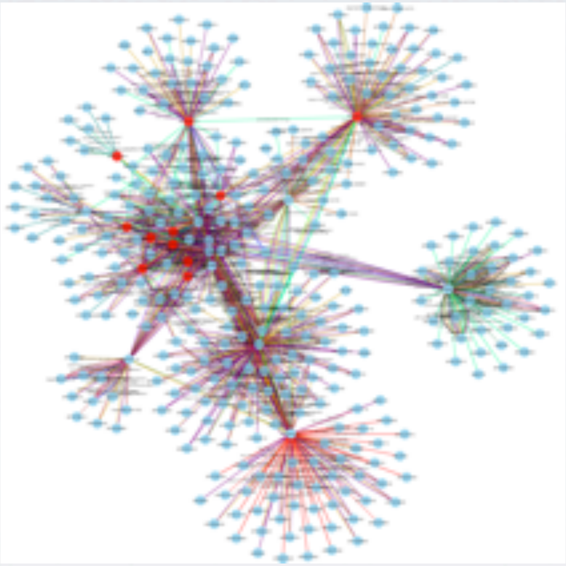
community detection: applications



social networks



biological networks

communication networks

Also: image segmentation, classification, recommendation systems, advertisement, information retrieval, ...

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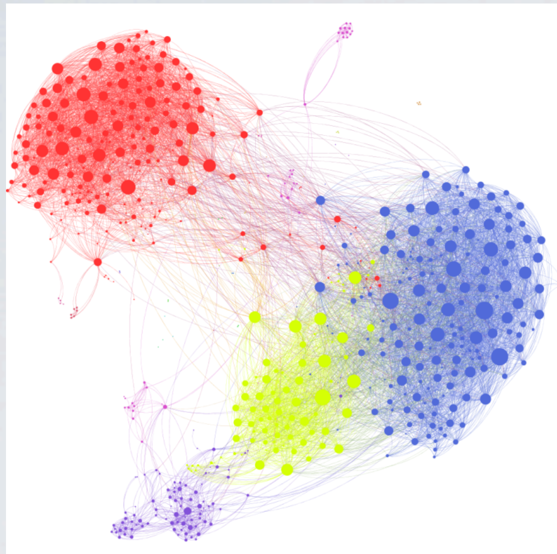

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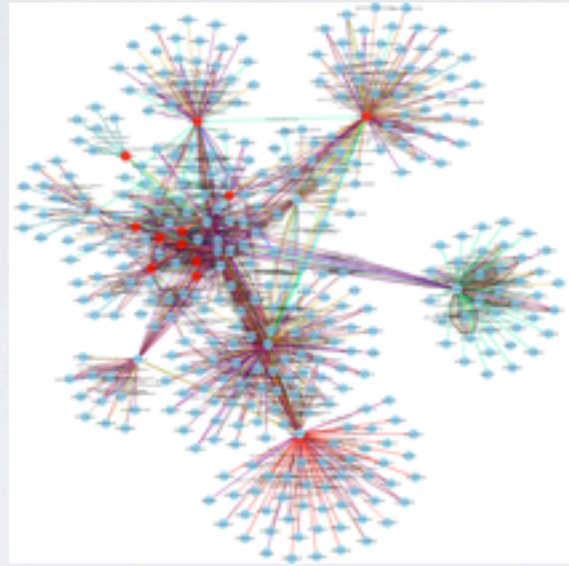

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Also: image segmentation, classification, recommendation systems, advertisement, information retrieval, ...

Identify groups that are alike from similarity relationships in data sets

The stochastic block model: a random graph model with communities

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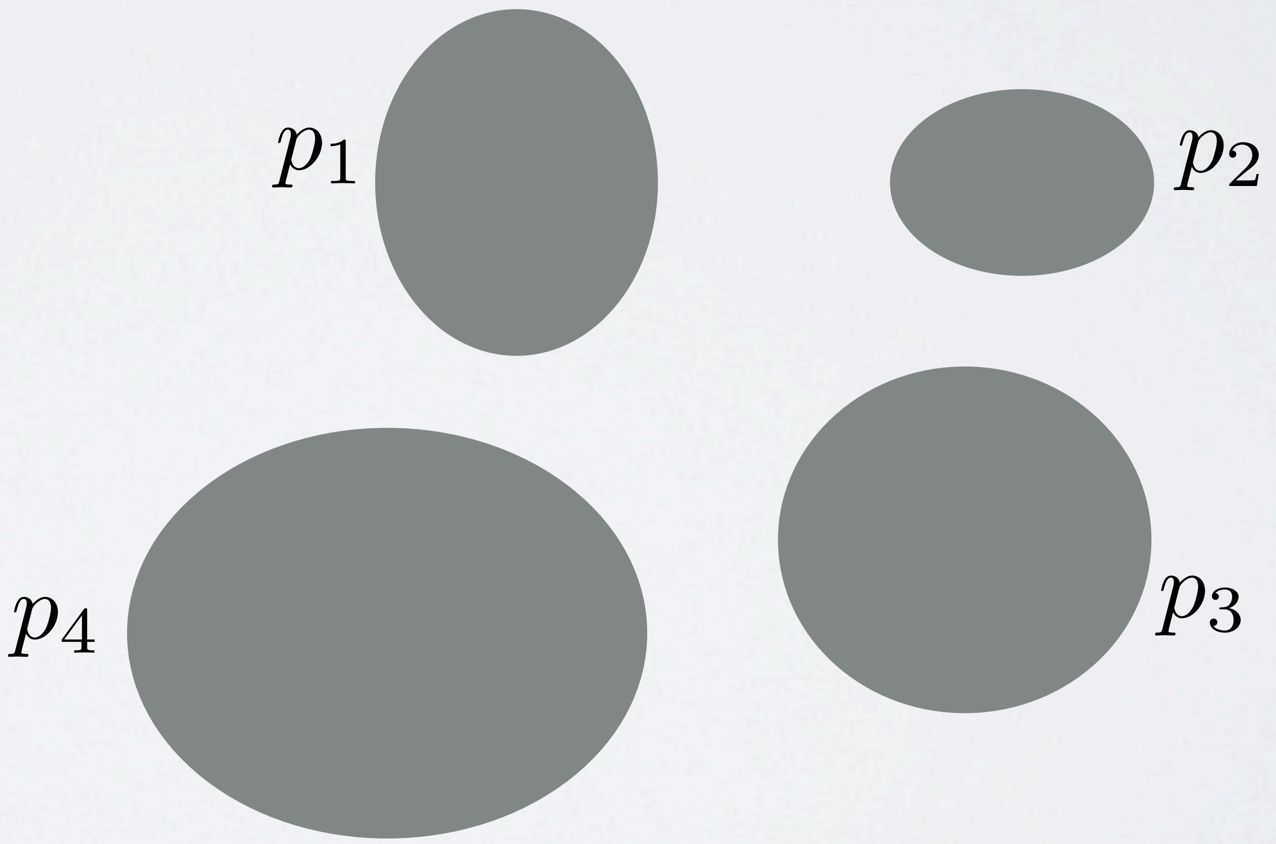

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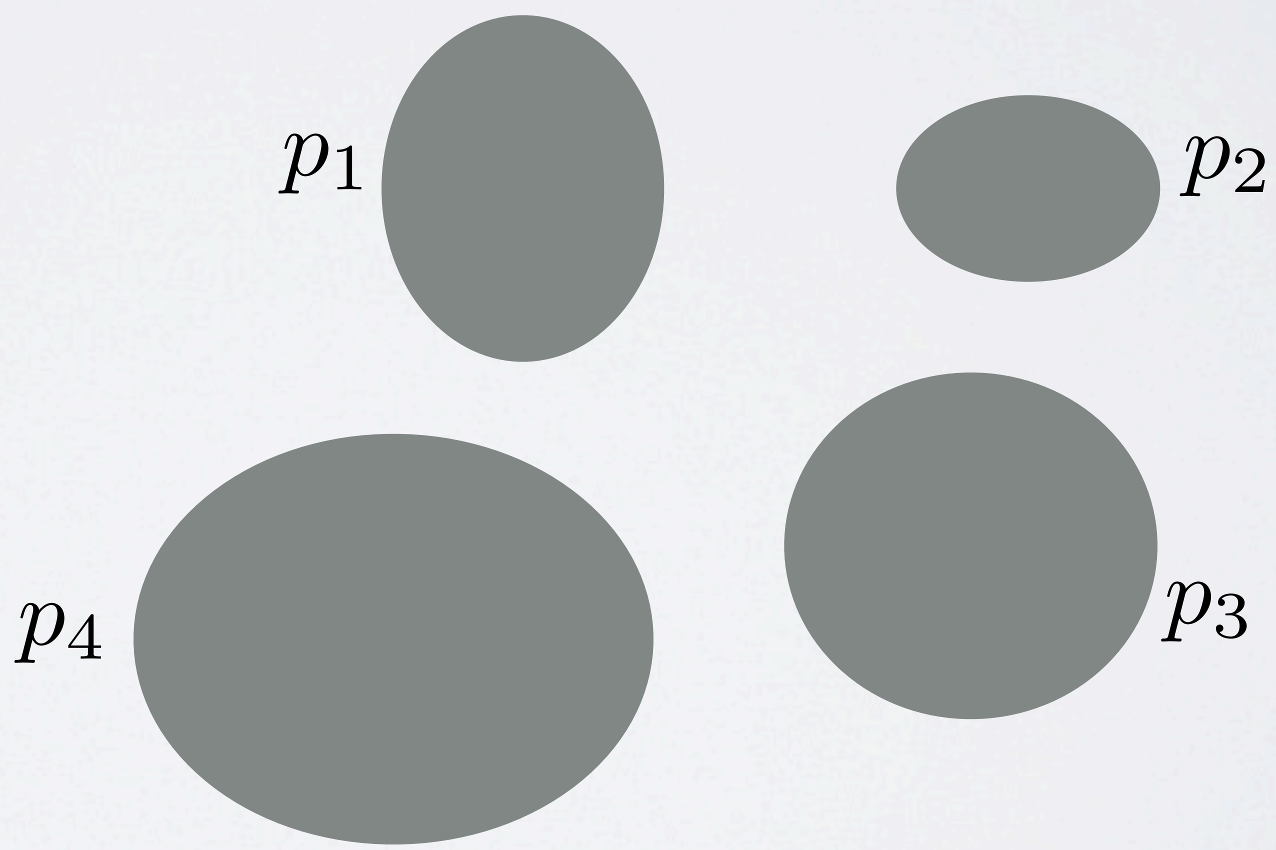

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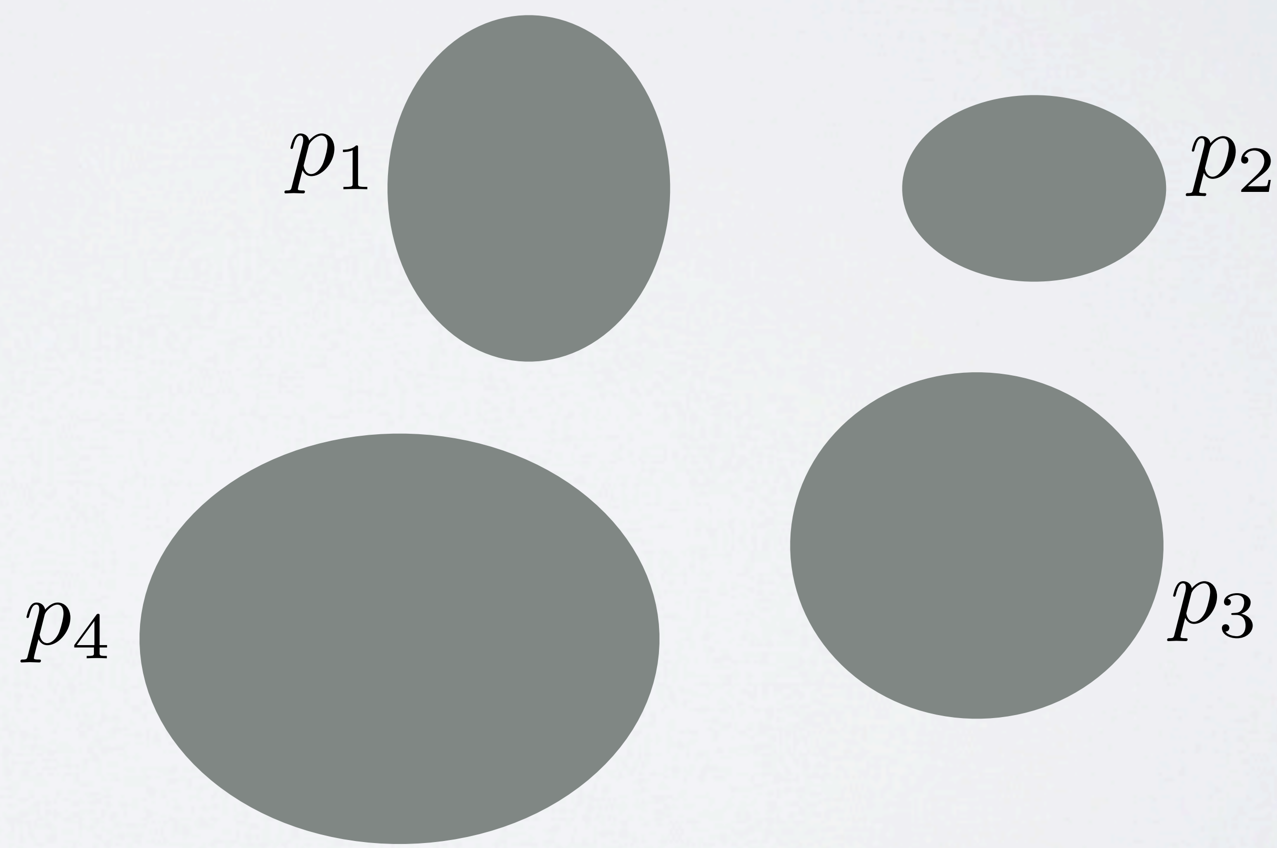
<- symmetric matrix with entries in [0,1]
= connectivity among communities



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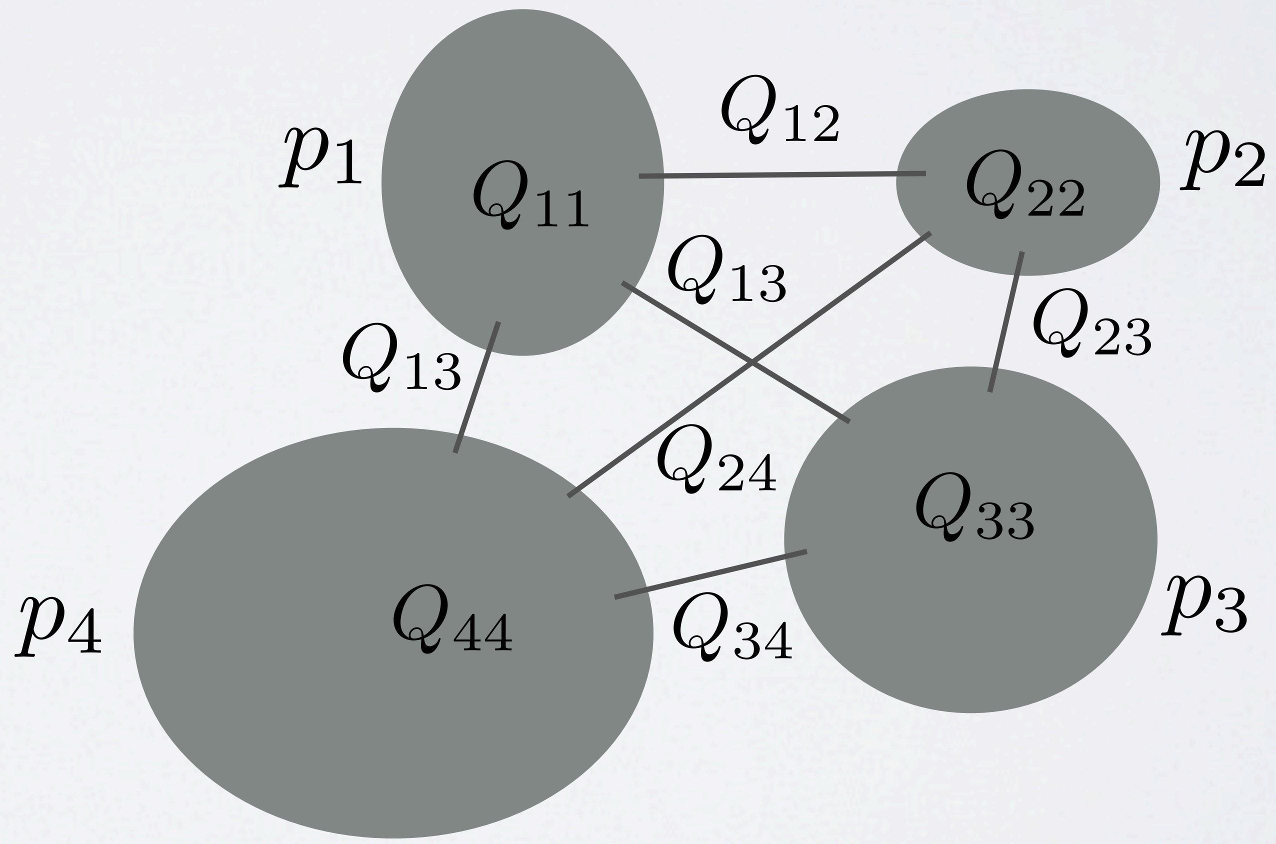

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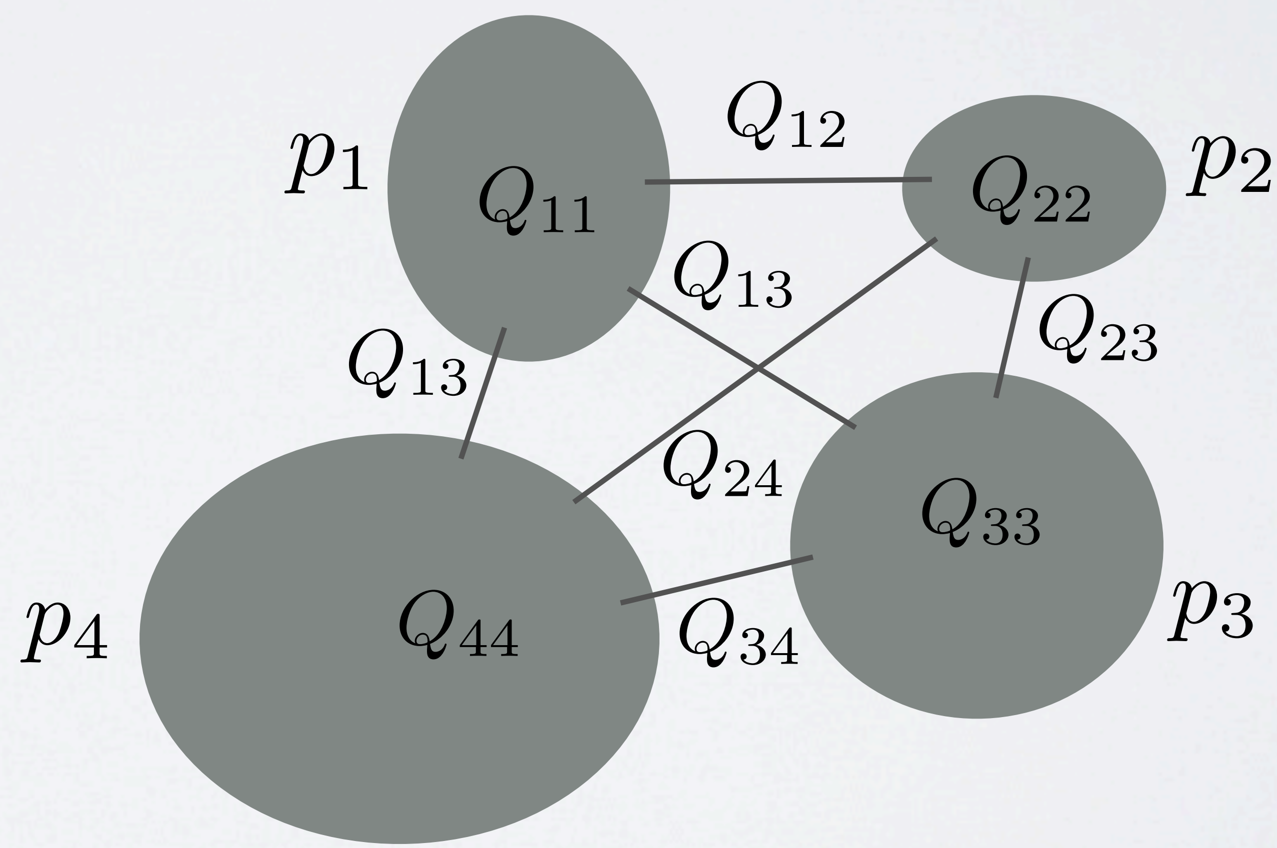

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The DMC of clustering..?

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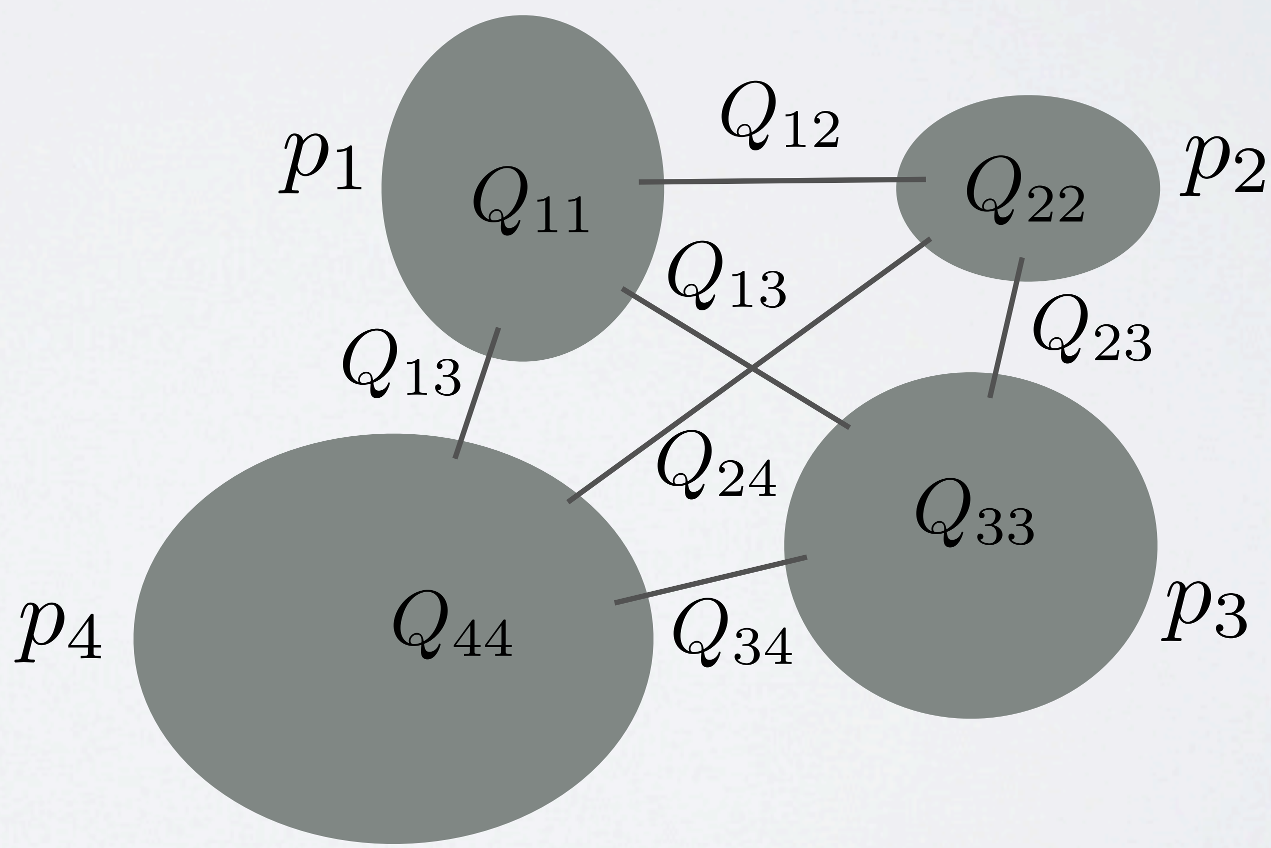
nice and reasonable model



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Next:

- warm up: two symmetric communities
- new results: partial and exact recovery in the general SBM
- analogy with the channel coding theorem
- some real data

SBM with two symmetric communities (planted bisection model)

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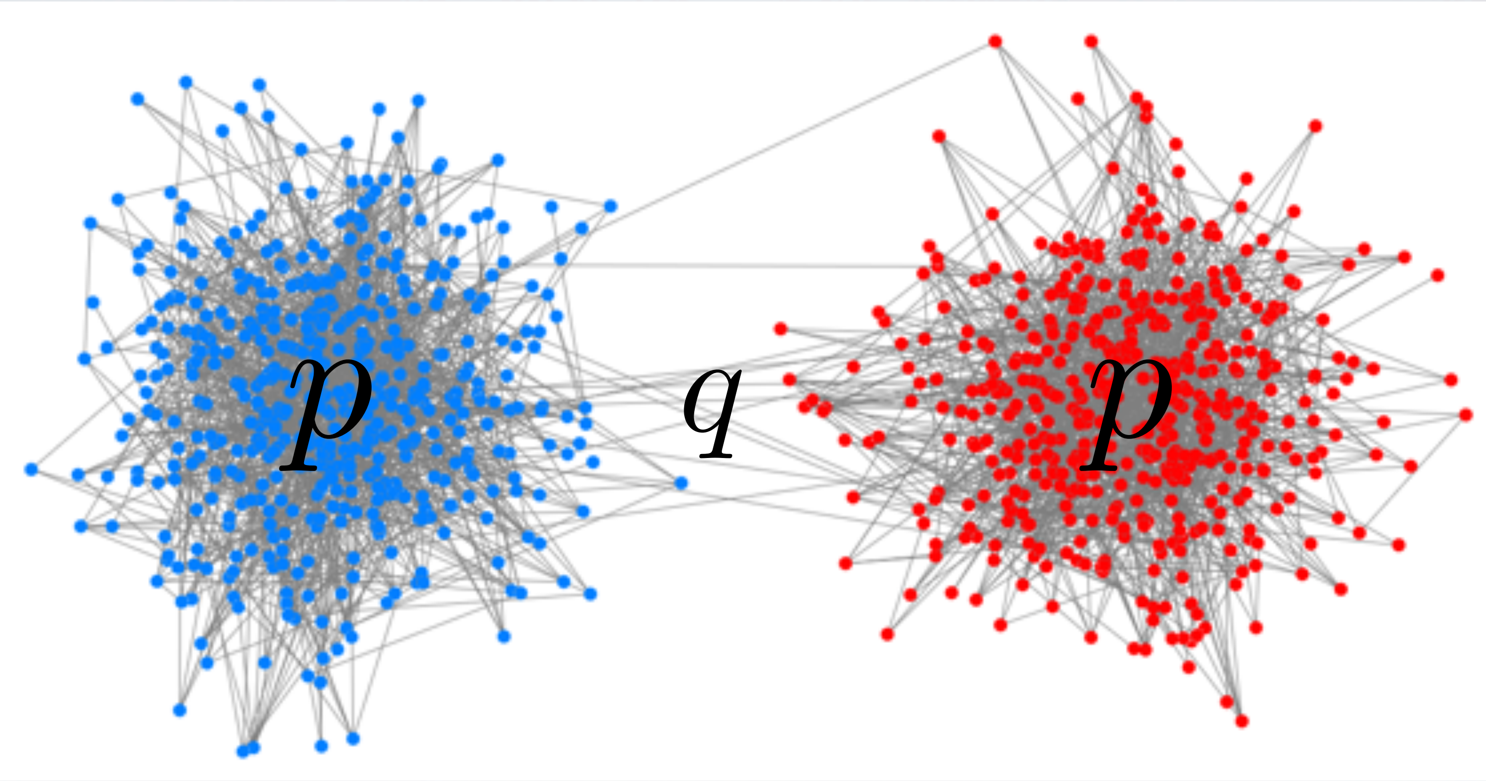


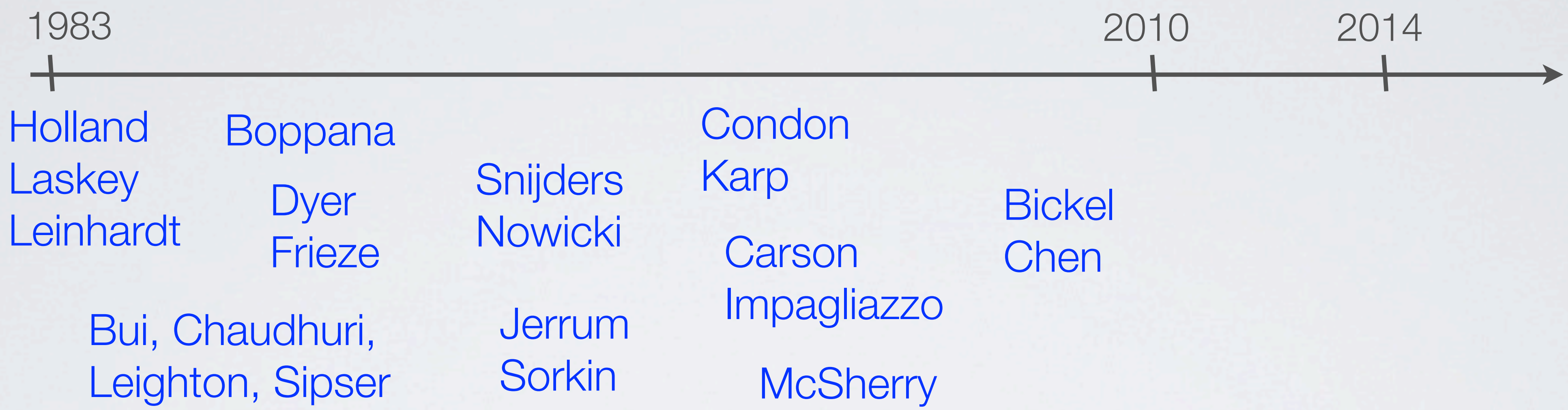
$\frac{n}{2}$ $\frac{n}{2}$

$$p_1 = p_2 = 1/2$$

$$Q_{11} = Q_{22} = p \qquad Q_{12} = q$$

Some history

Recovery



Some history

$$\mathbb{P}(\hat{X}^n = X^n) \to 1$$

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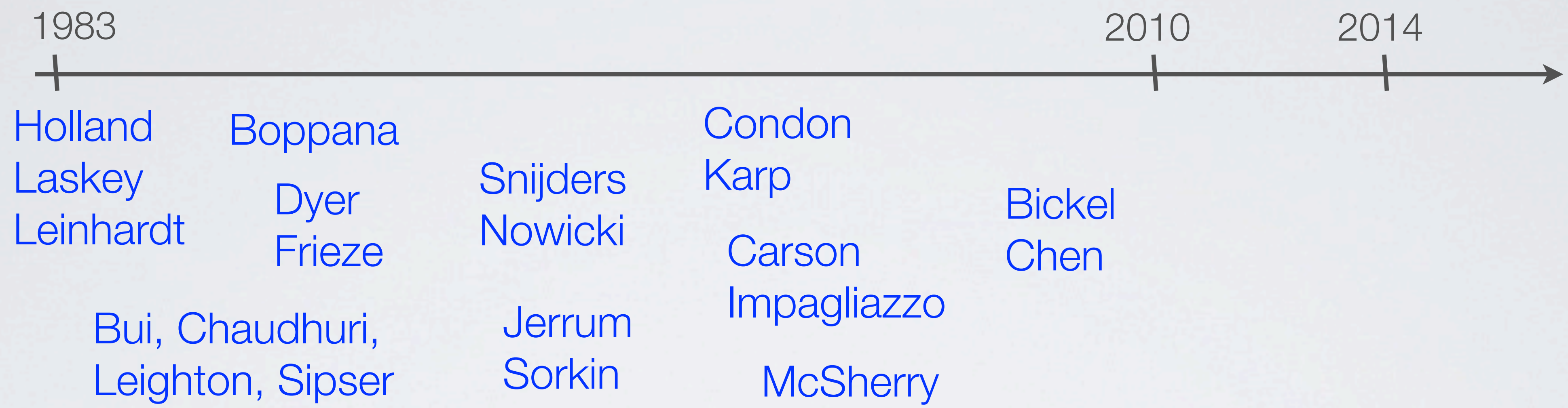

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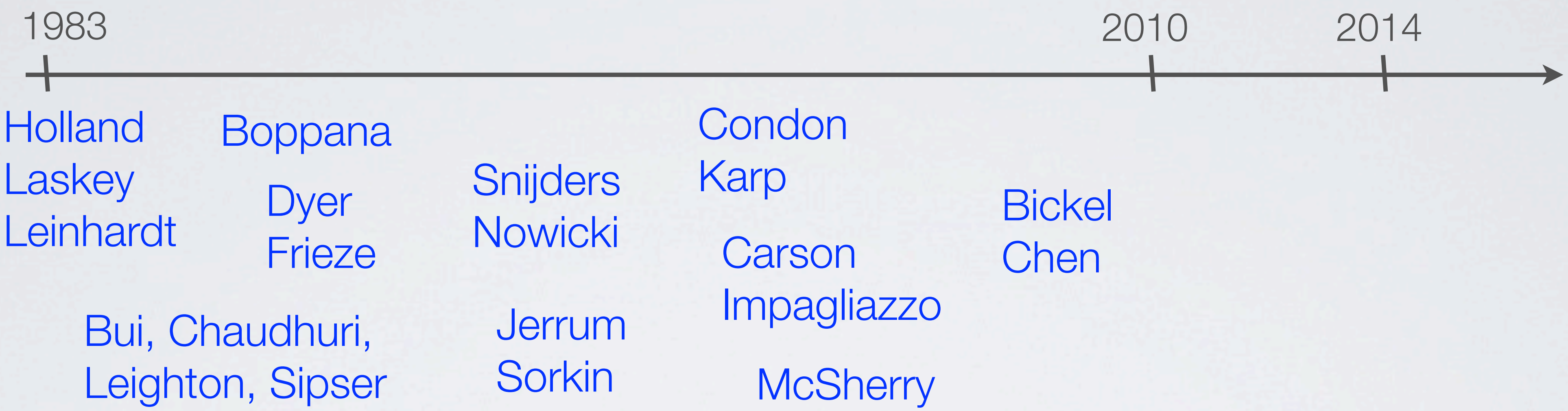

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Boppana '87	spectral meth.	$(p-q)/\sqrt{p+q} = \Omega(\sqrt{\log(n)/n})$
Dyer, Frieze '89	min-cut via degrees	$p - q = \Omega(1)$
Snijders, Nowicki '97	EM algo.	$p - q = \Omega(1)$
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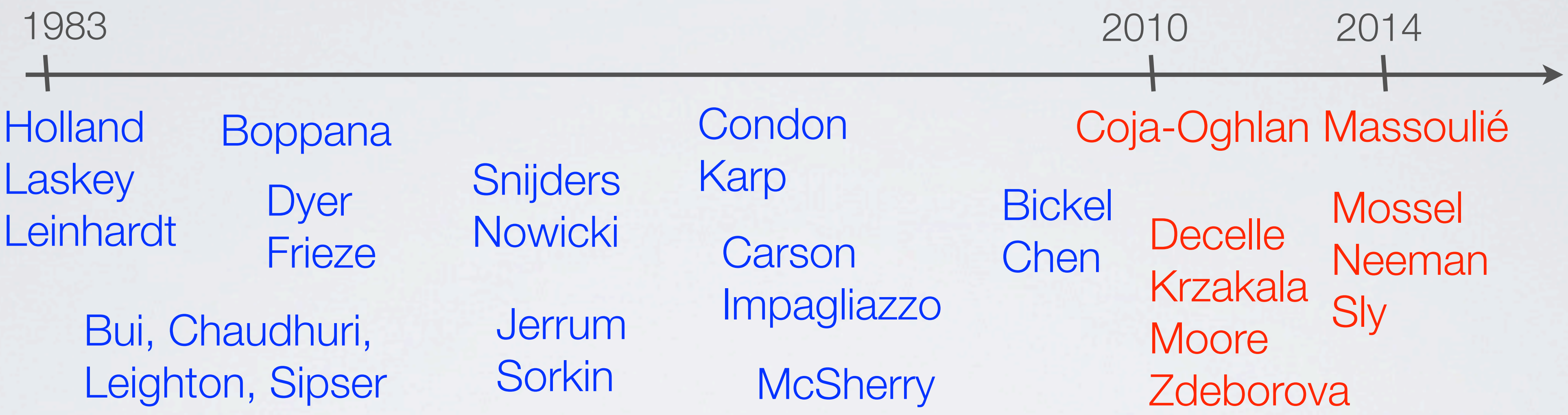
Recovery | **Detection**



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Abbe-Bandeira-Hall '14

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→ not clear for multiple communities

Recovery in the general SBM

Recovery in the general SBM -> an information theoretic motivation

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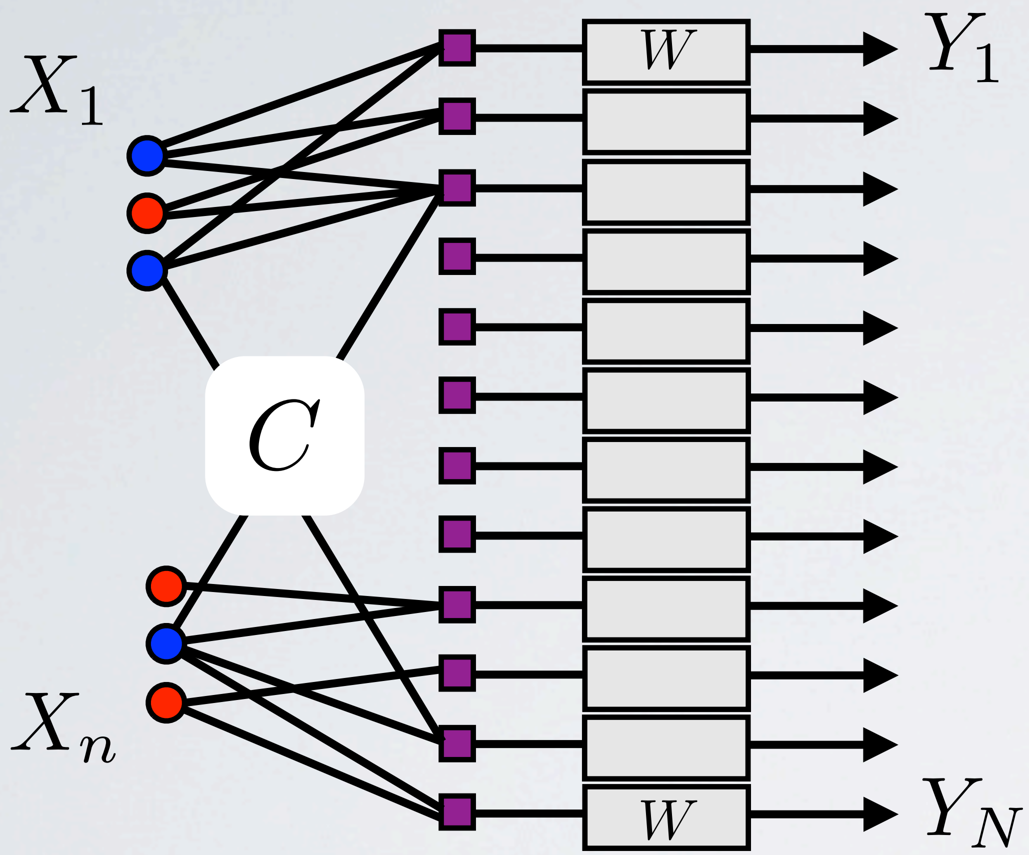

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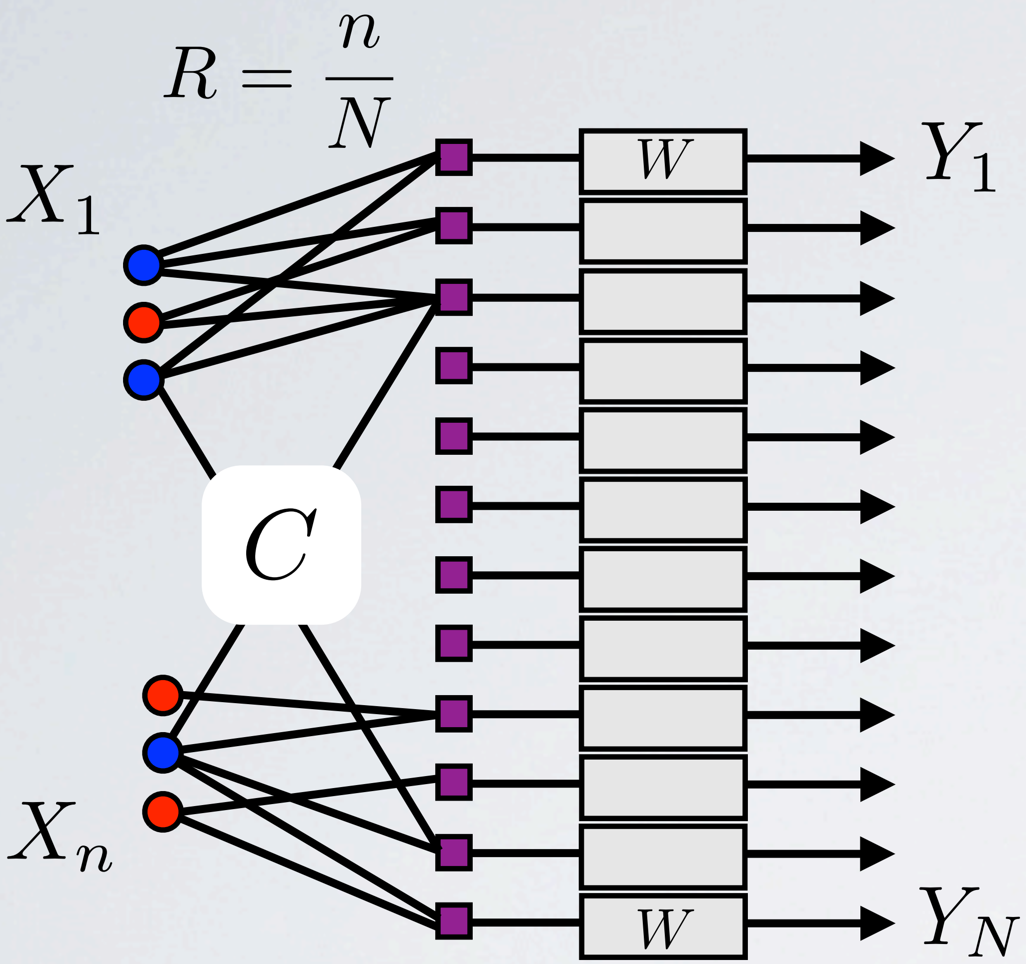
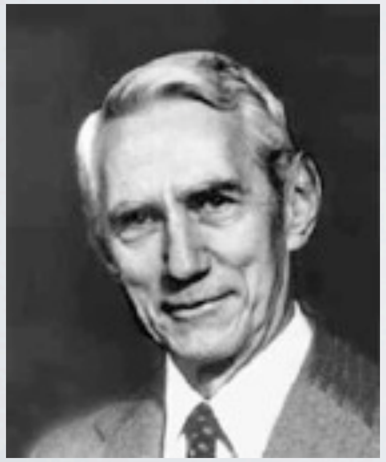
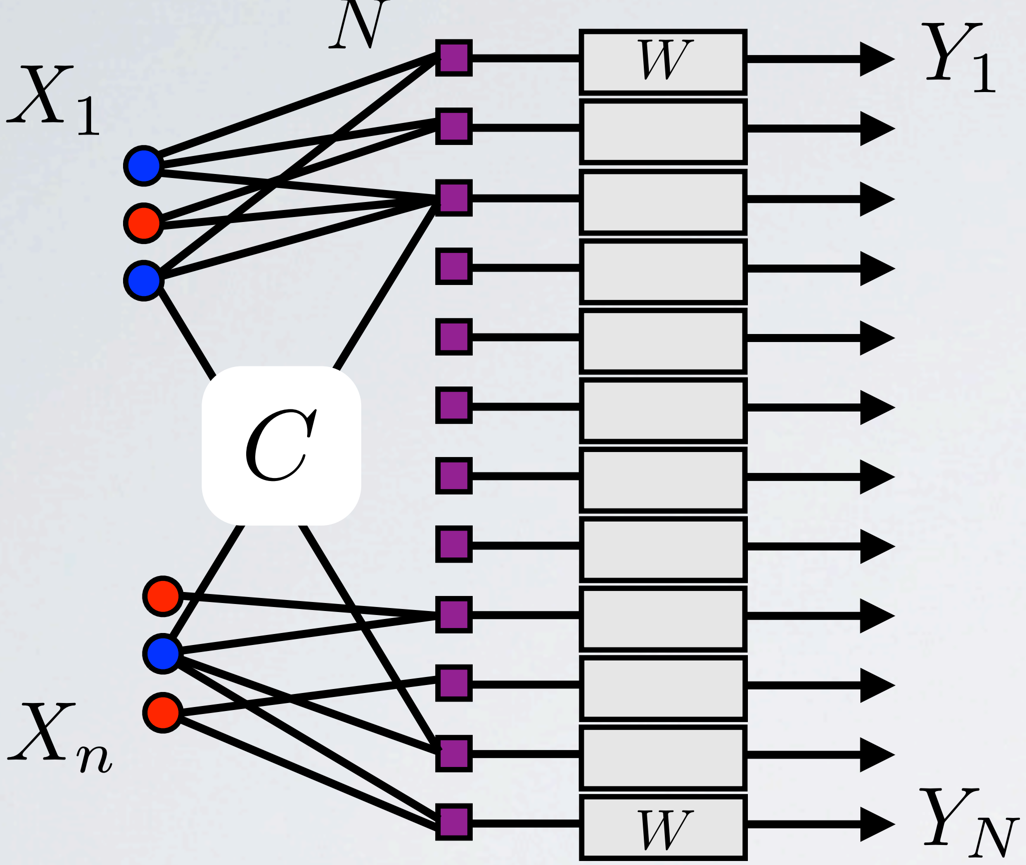
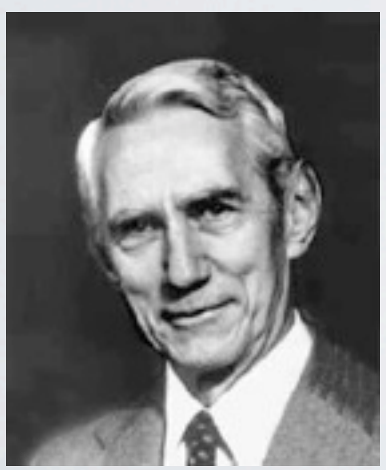




Recovery in the general SBM -> an information theoretic motivation

$$R = \frac{n}{N}$$





$$W = \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}$$

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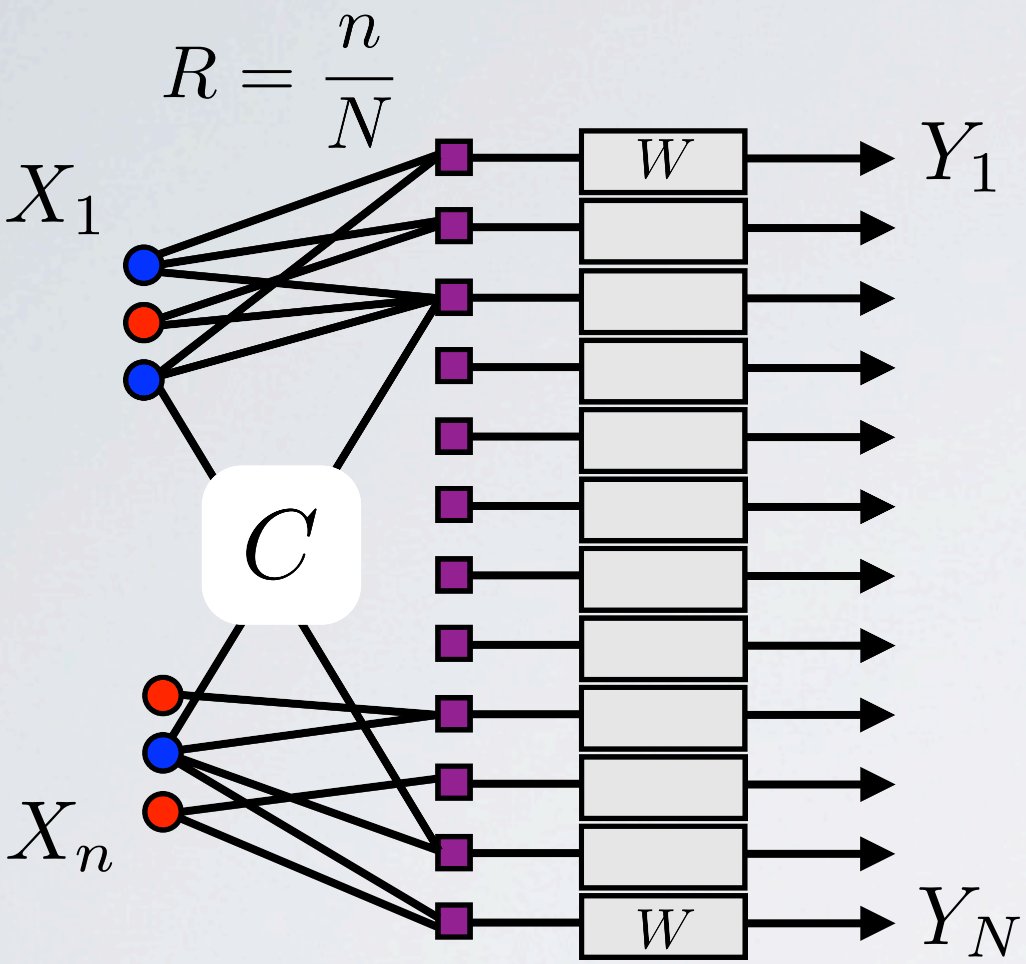

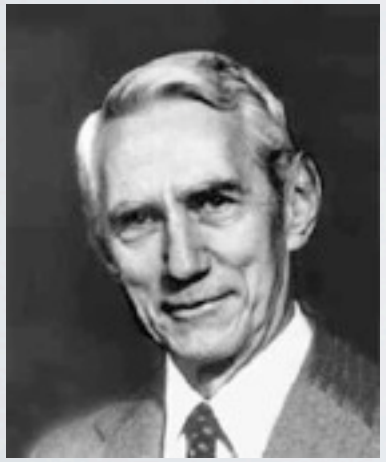

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reliable comm. iff R < 1-H($\boldsymbol{\varepsilon}$)

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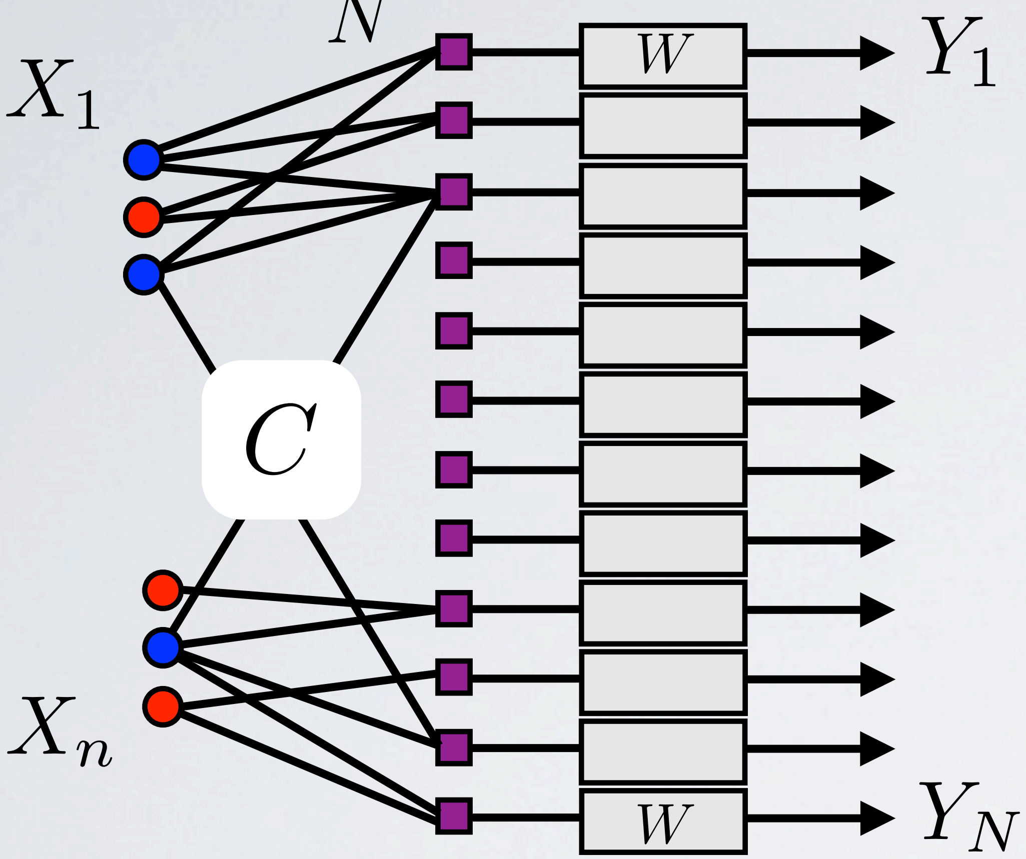

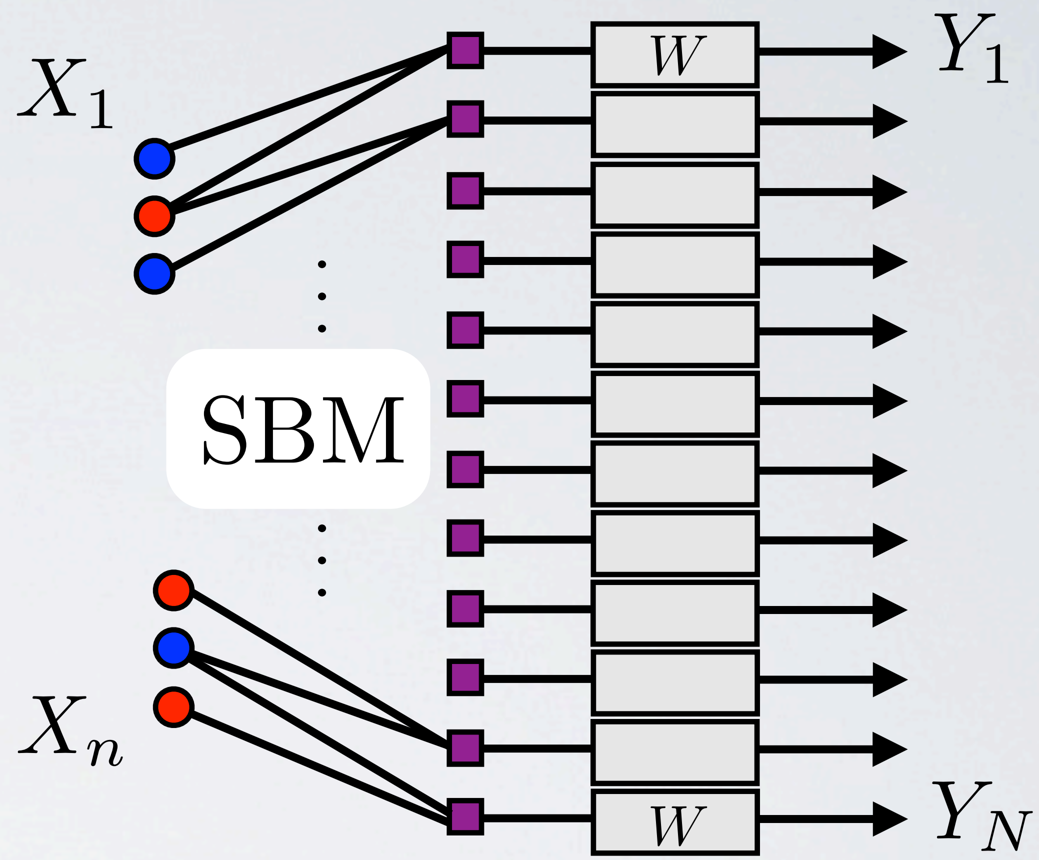

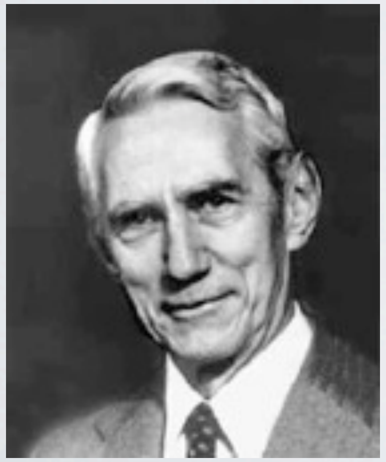

$$W = \begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}$$

reliable comm. iff R < 1-H(**ε**)

Recovery in the general SBM -> an information theoretic motivation

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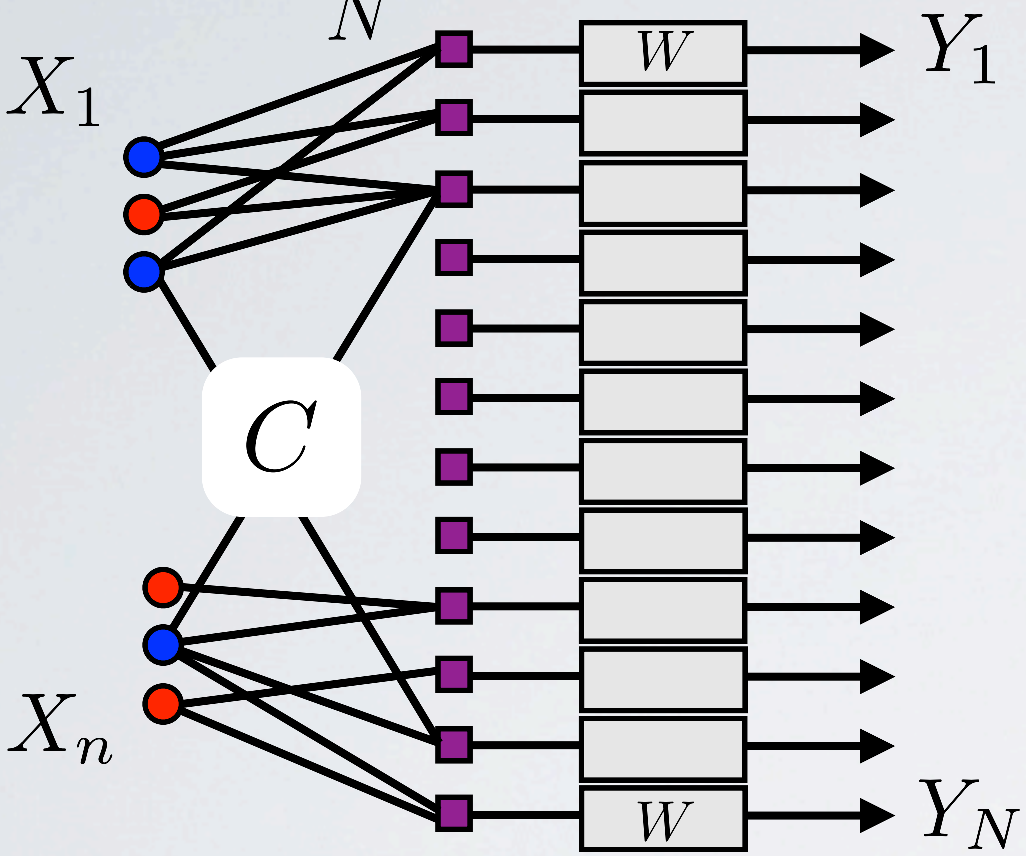

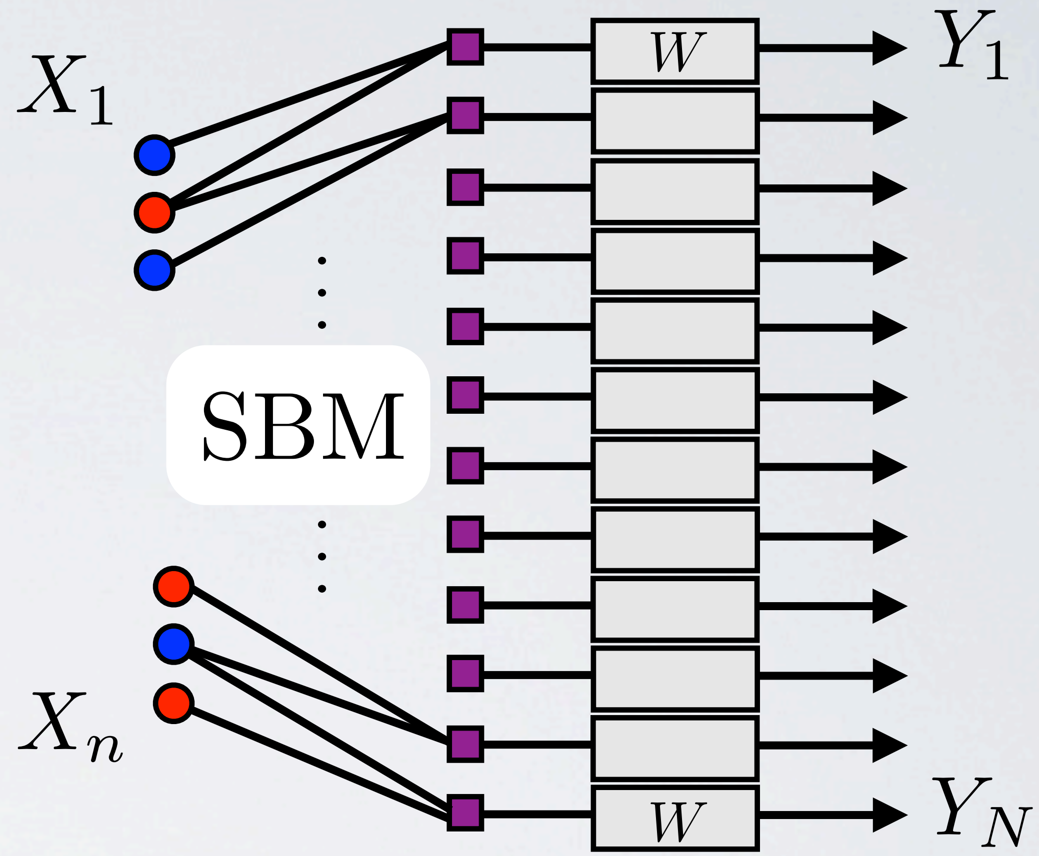

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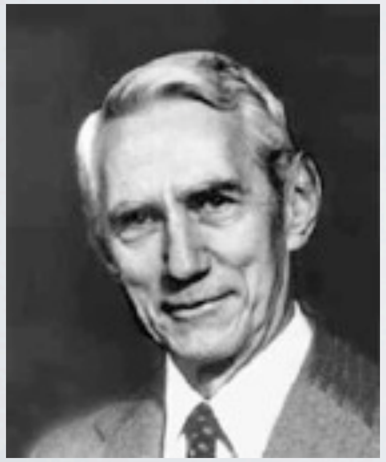

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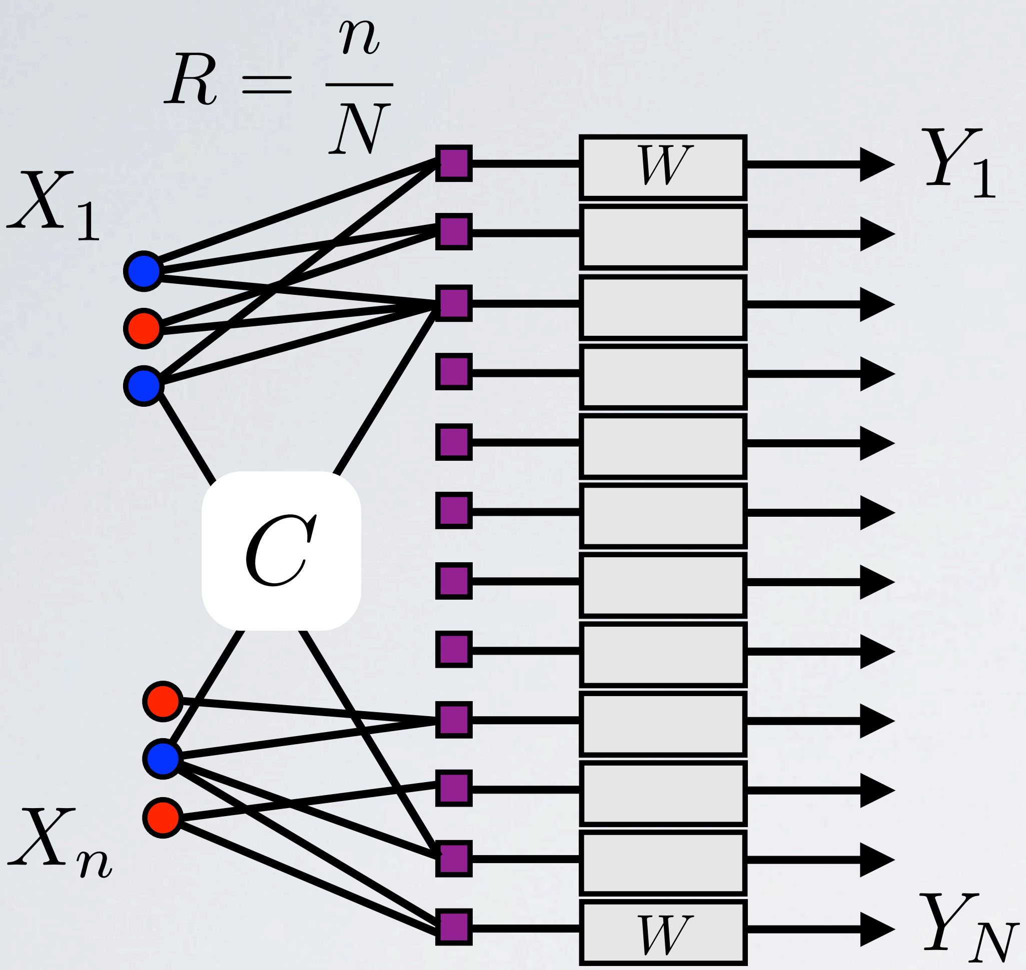

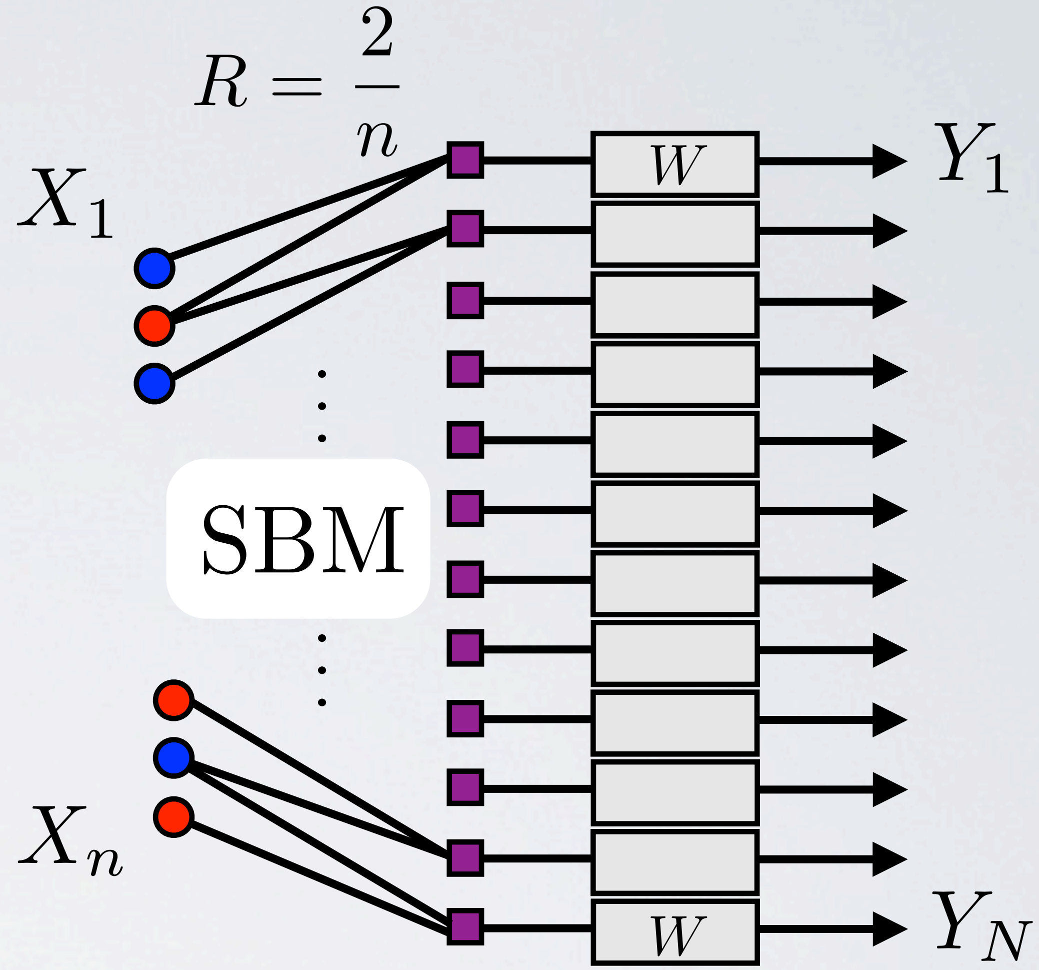

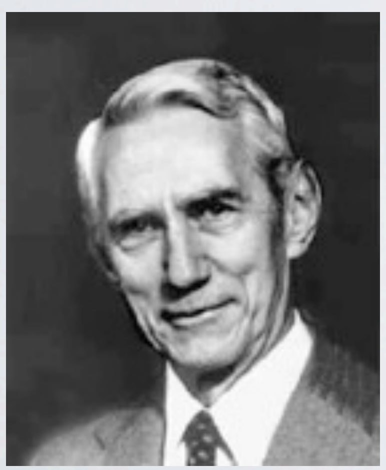

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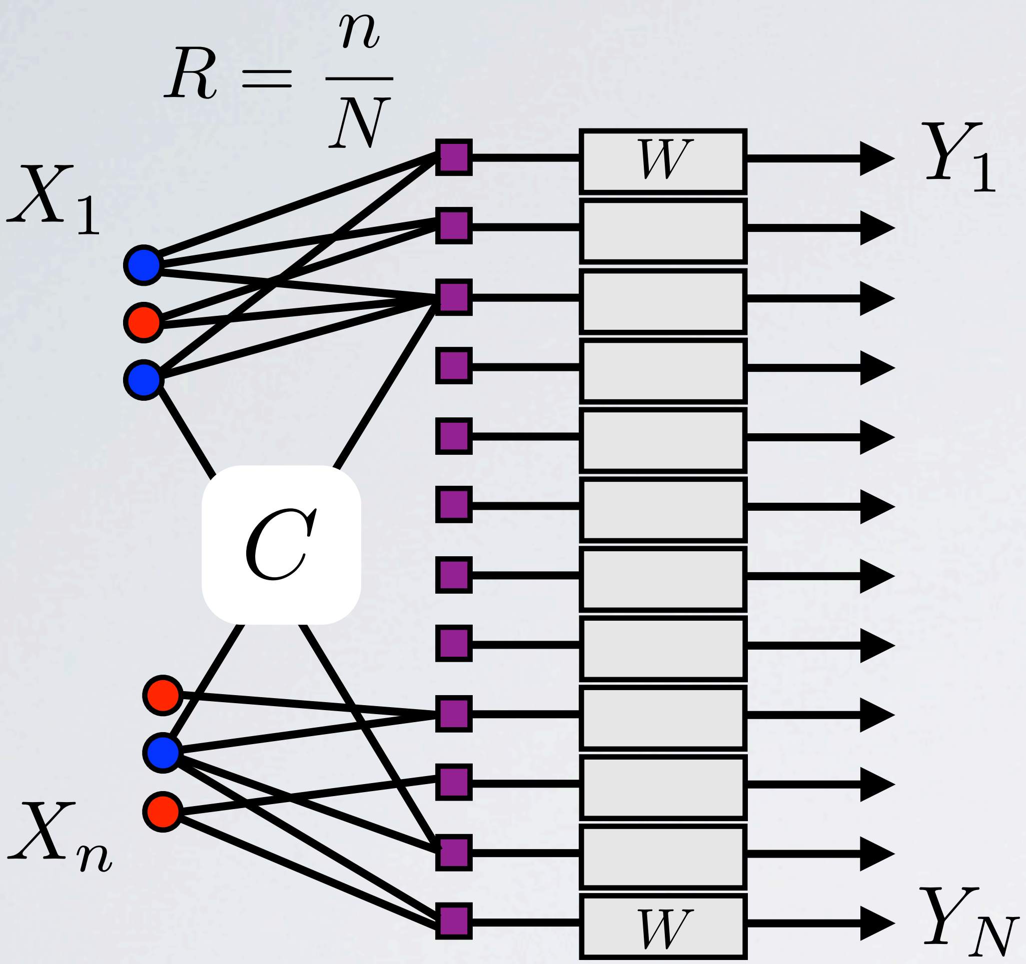

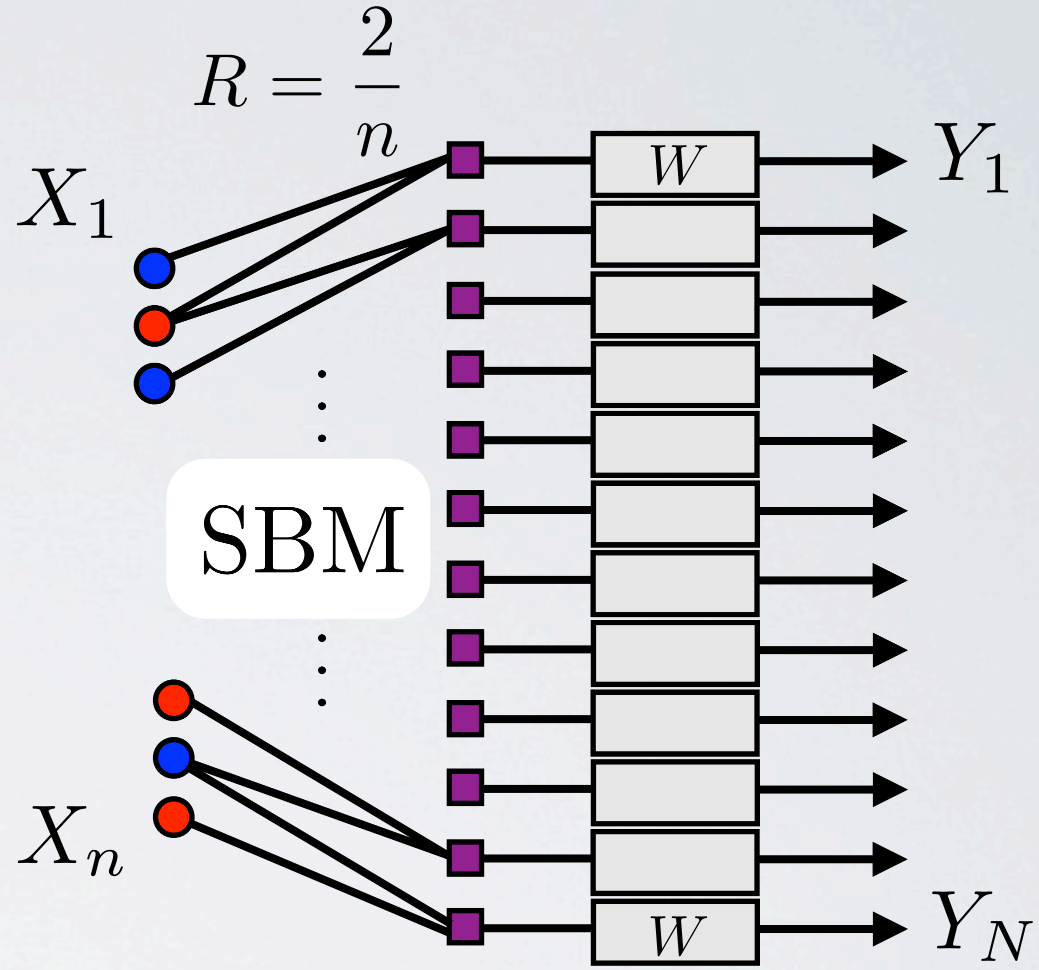

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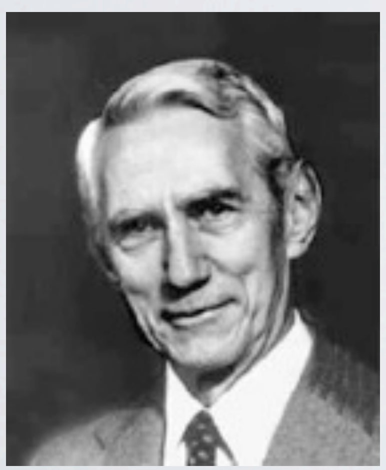

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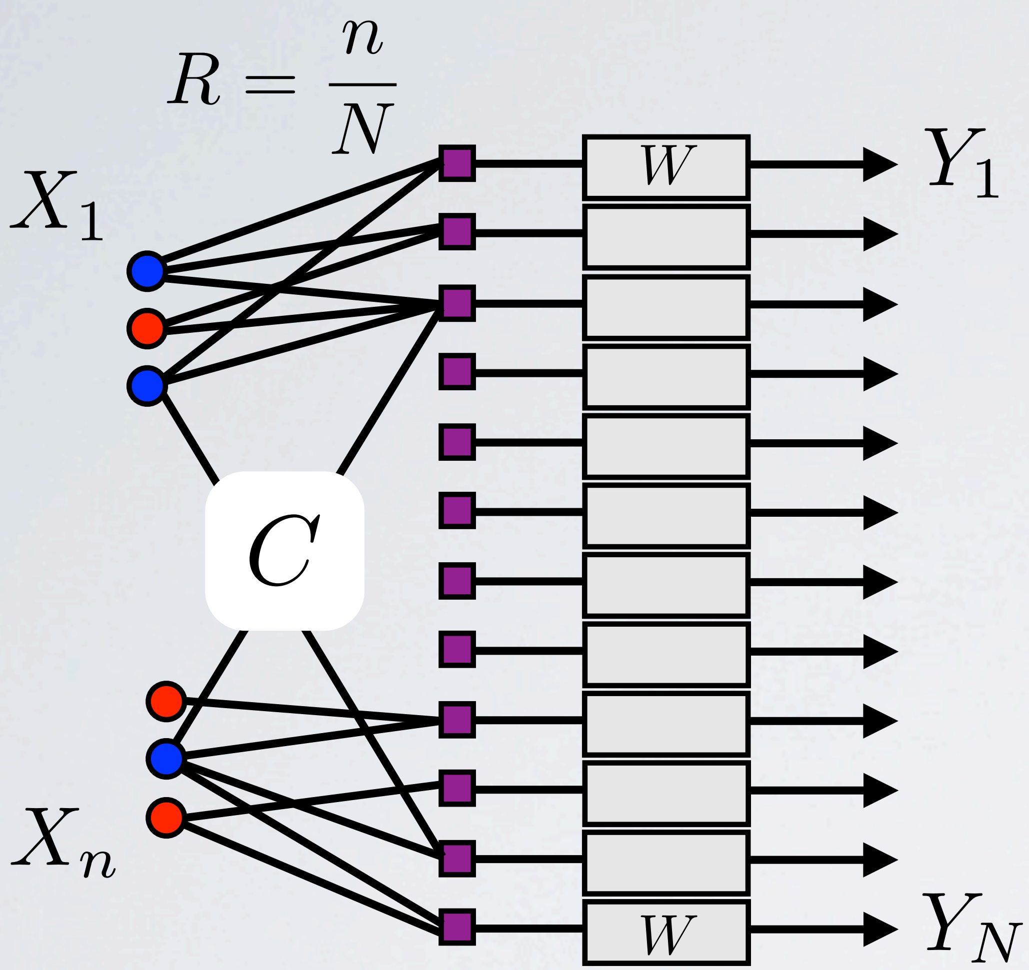

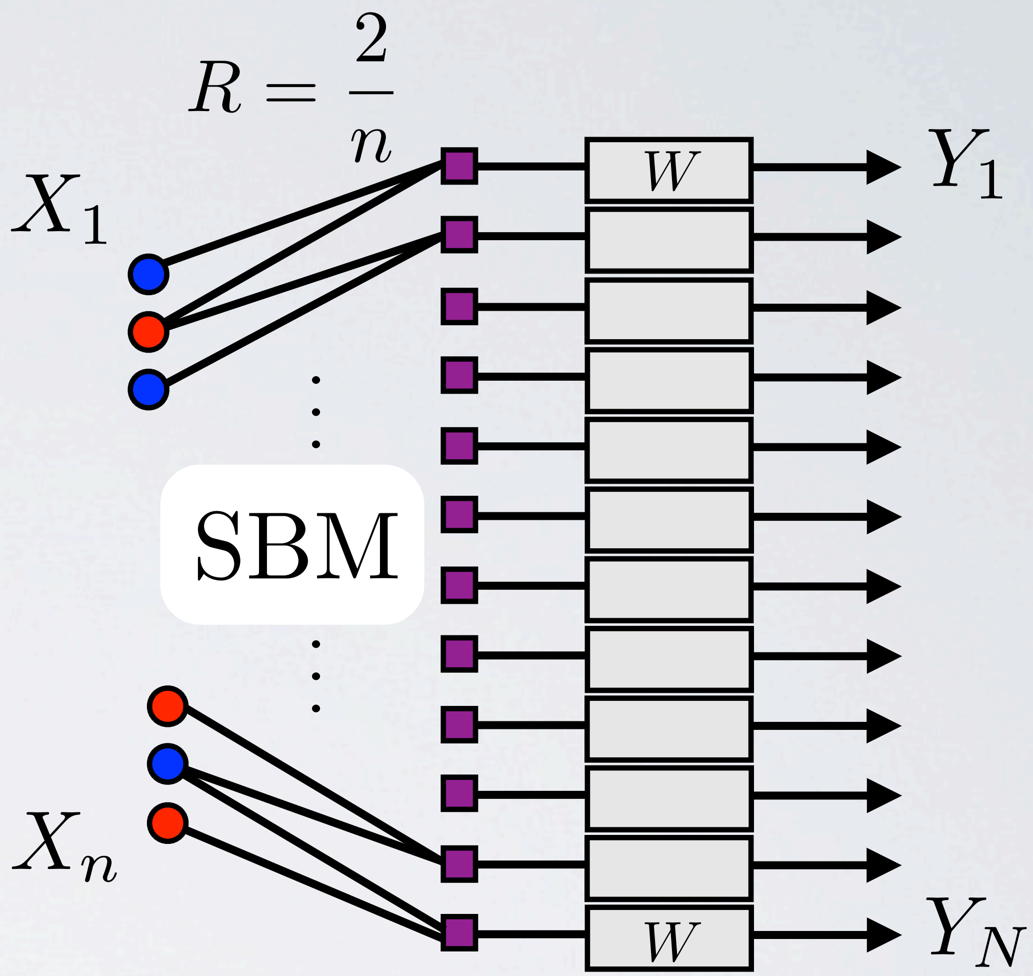

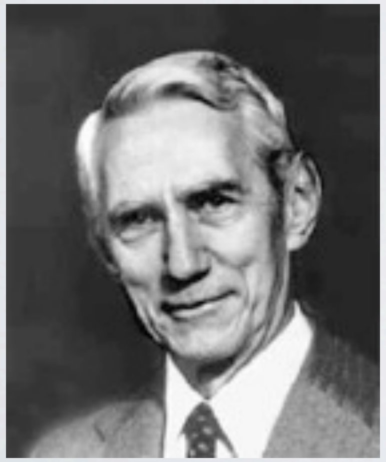

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Recovery in the general SBM -> an information theoretic motivation

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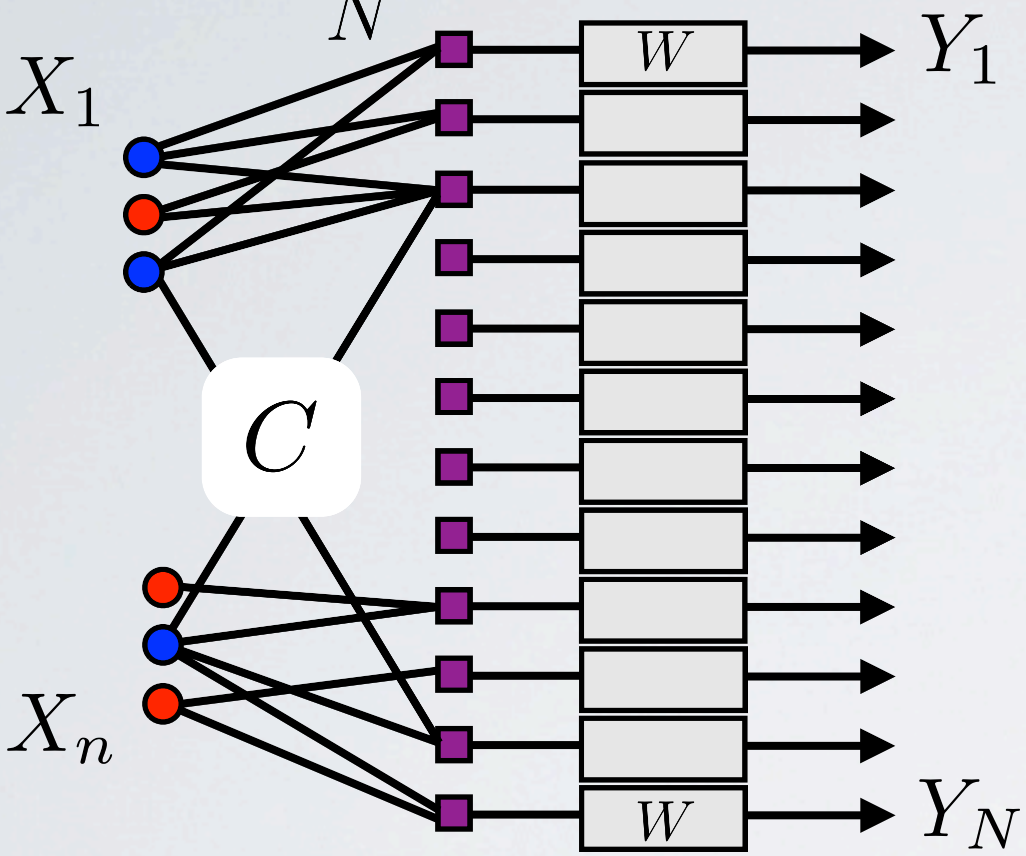

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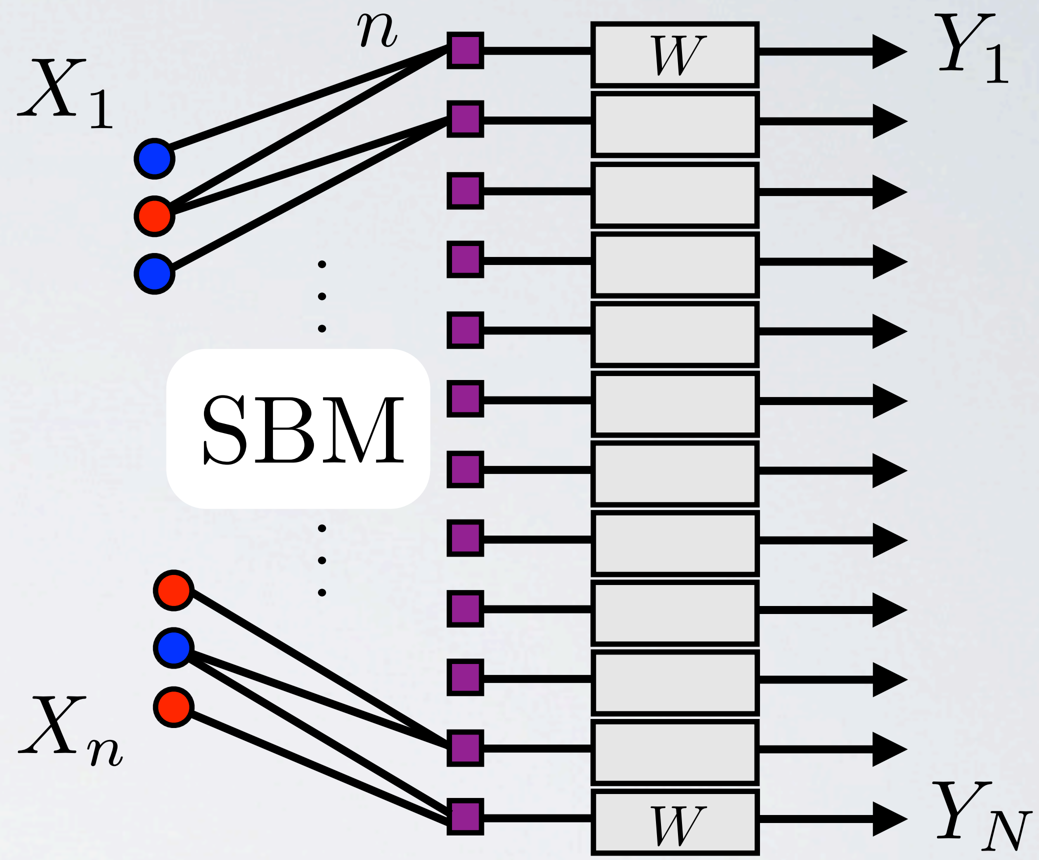

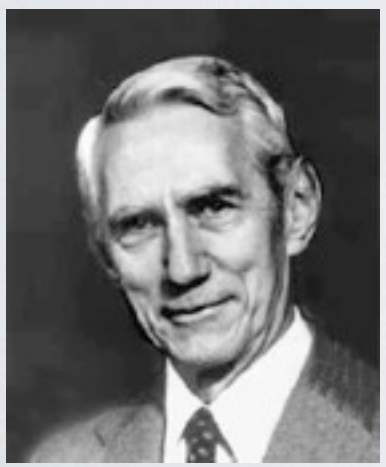

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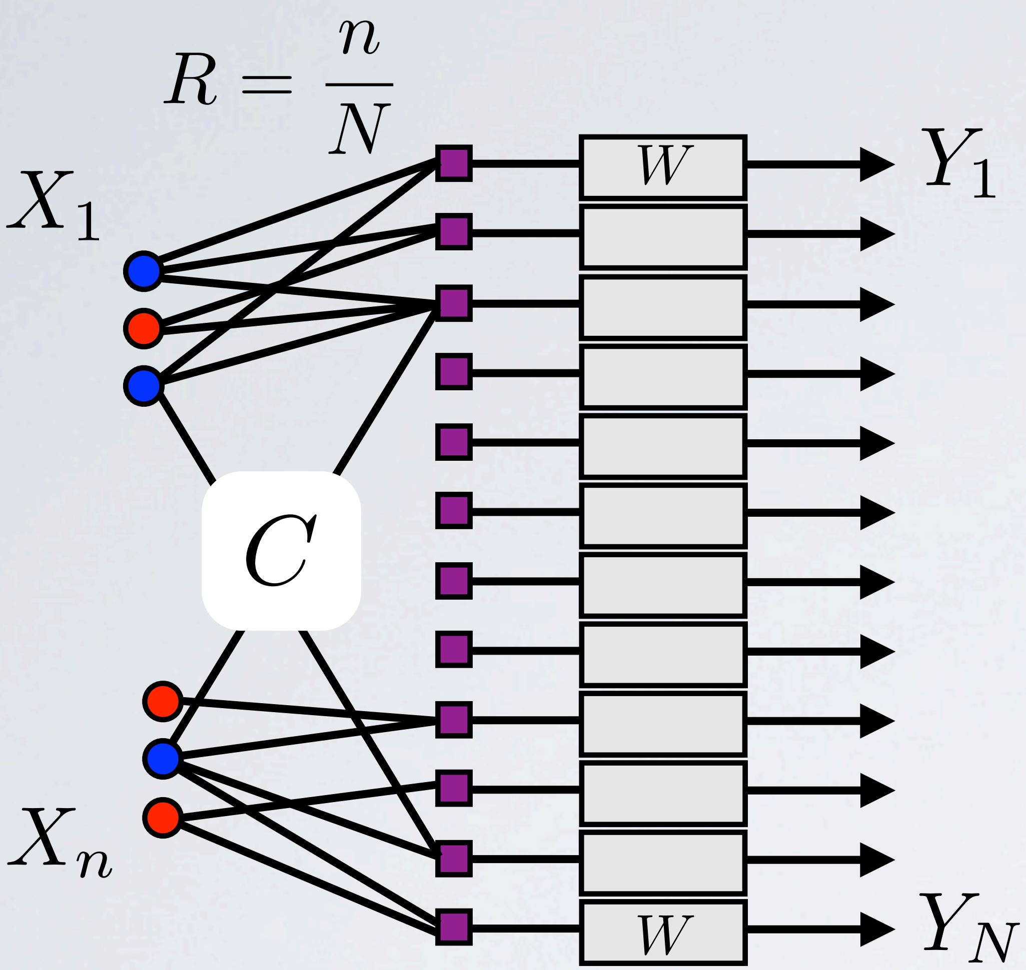

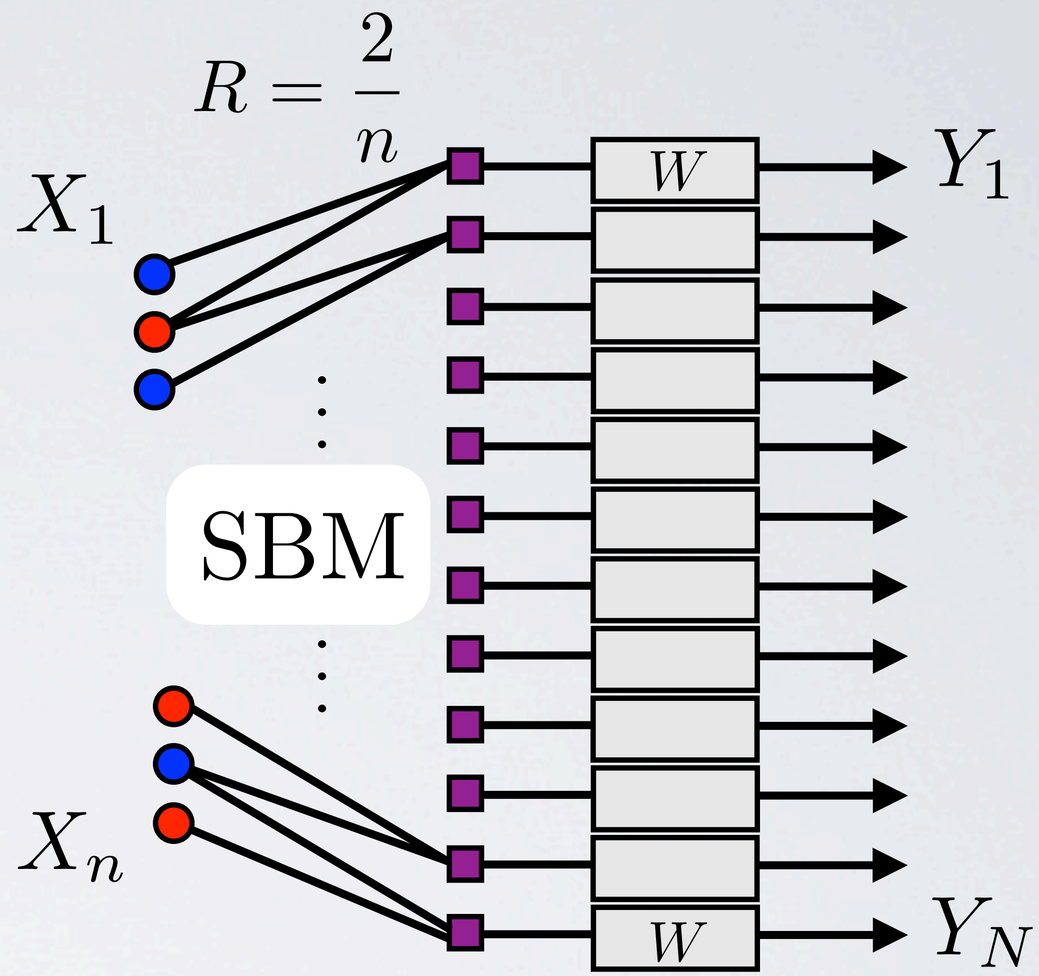

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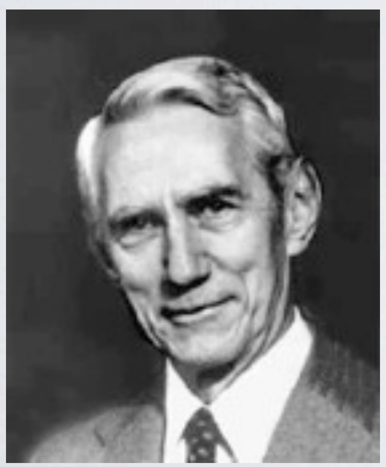

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reliable comm. iff 1 < J(p,W) ???

Results

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Theorem 1. Recovery is solvable in $\mathrm{SBM}(n, p, Q\log(n)/n)$ if and only if

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Abbe-Bandeira-Hall '14

Mossel-Neeman-Sly '14

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We call D_+ the CH-divergence.

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Exact recovery in the general SBM is solvable efficiently whenever it is solvable information theoretically

What about recovering a subgroup of the communities?

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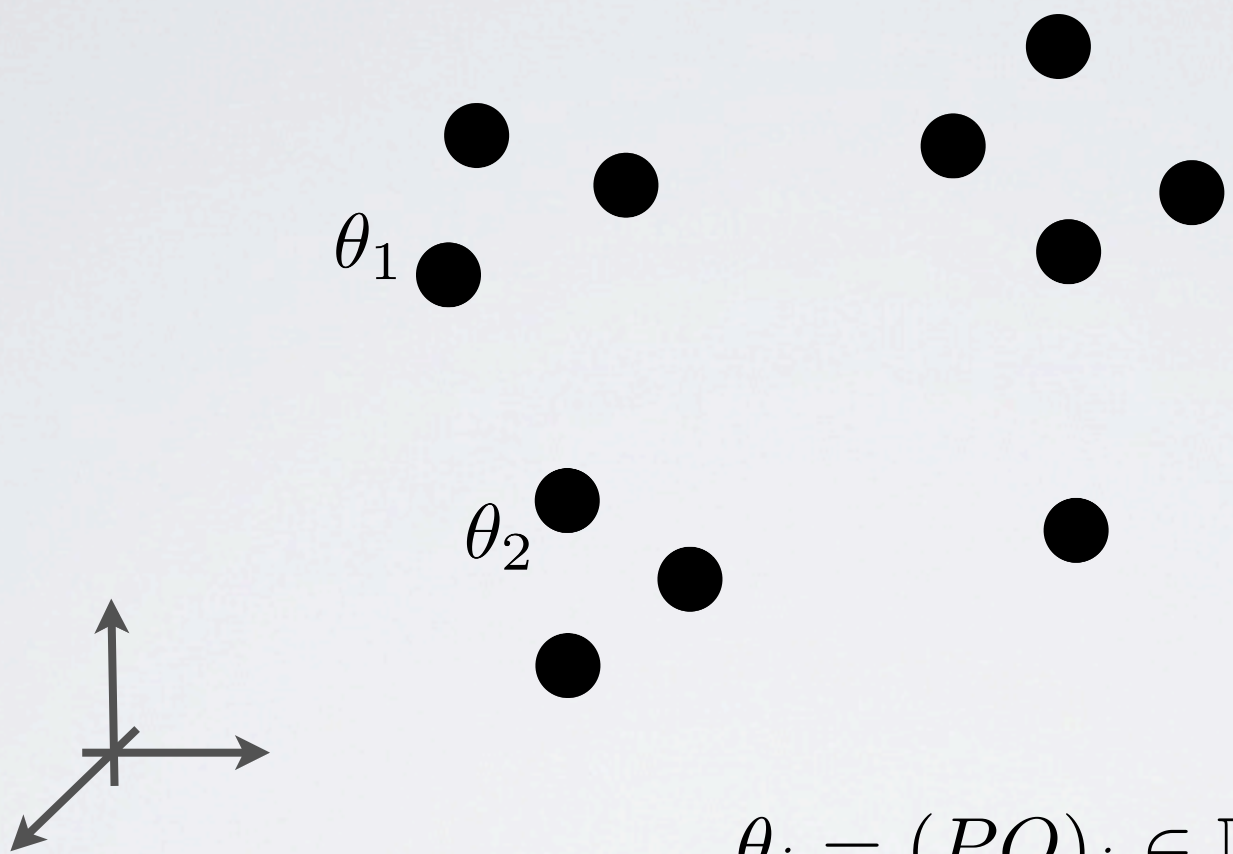

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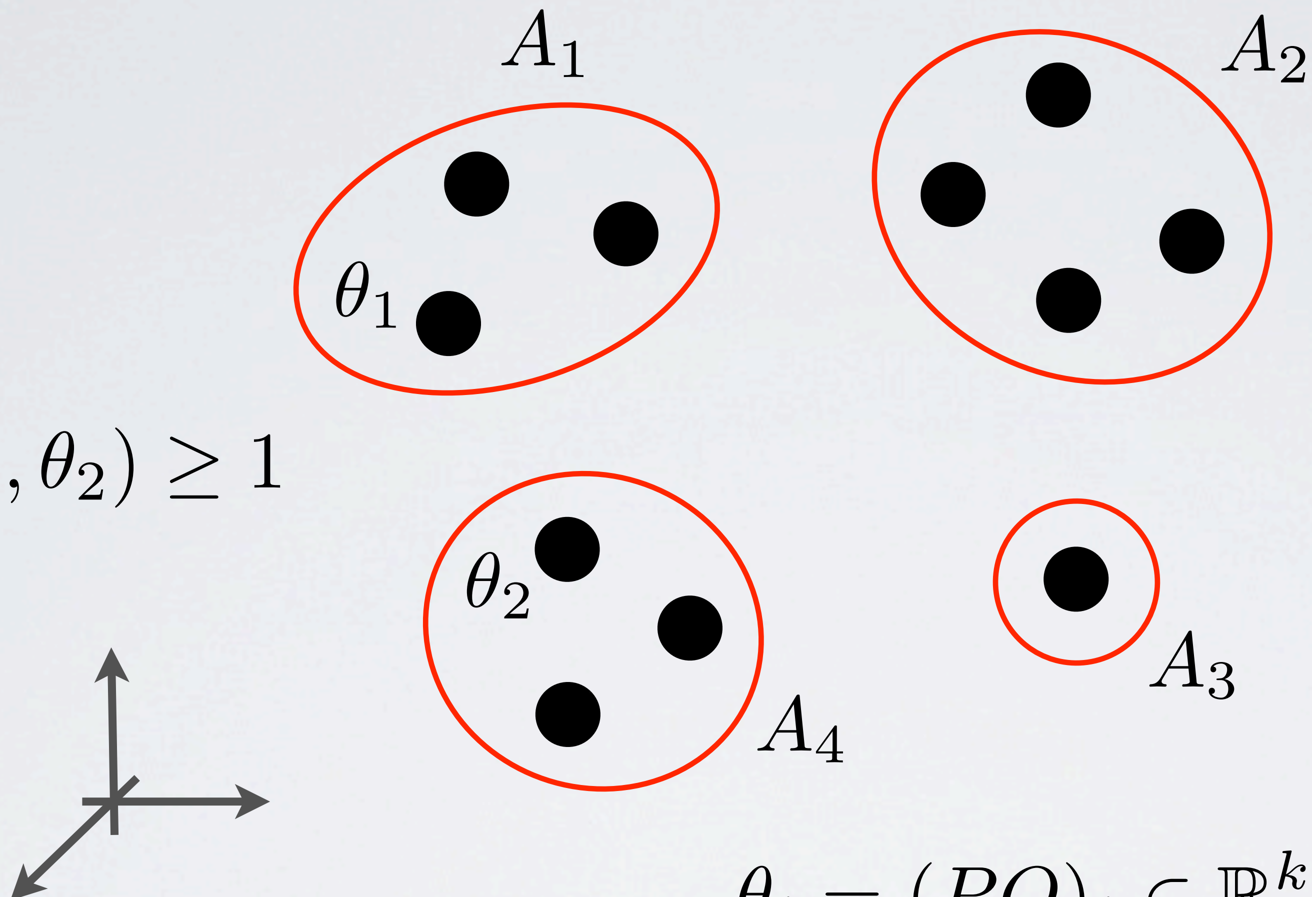

"finest partition"

$$D_+(\theta_1, \theta_2) \geq 1$$

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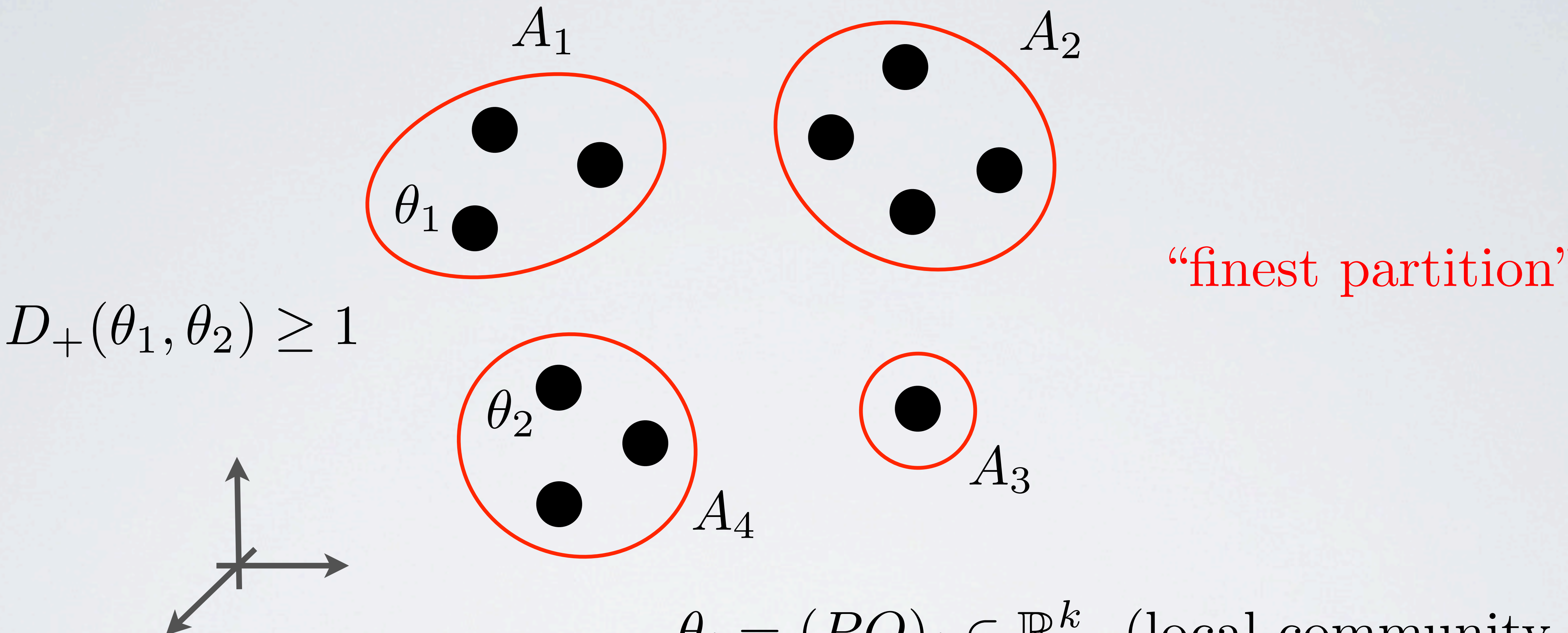

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Theorem 3. Exact recovery for a partition $[k] = \sqcup_{i=1}^{s} A_i$ is solvable in $\mathrm{SBM}(n, p, Q\log(n)/n)$ if and only if

$$\min_{x<y} \underbrace{D_+(A_x, A_y)}_{\min\limits_{i \in A_x, j \in A_y} D_+((PQ)_i, (PQ)_j)} \geq 1$$

$$\min_{i \in A_x, j \in A_y} D_+((PQ)_i, (PQ)_j) \geq 1$$

Proof idea and partial recovery

Message: recover first most of the nodes and then finish differently

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How to recover a fraction of the nodes (partial recovery)?

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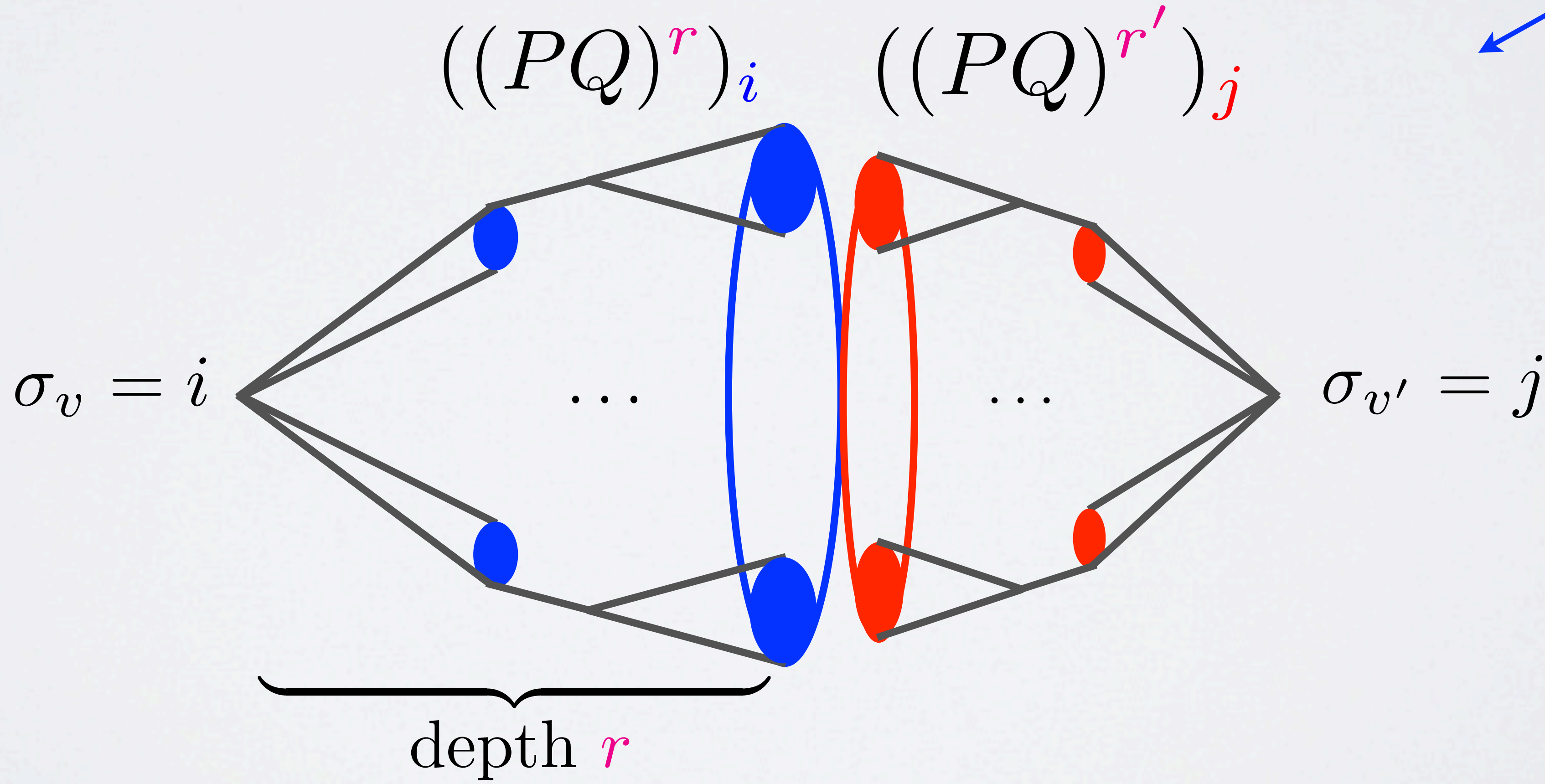

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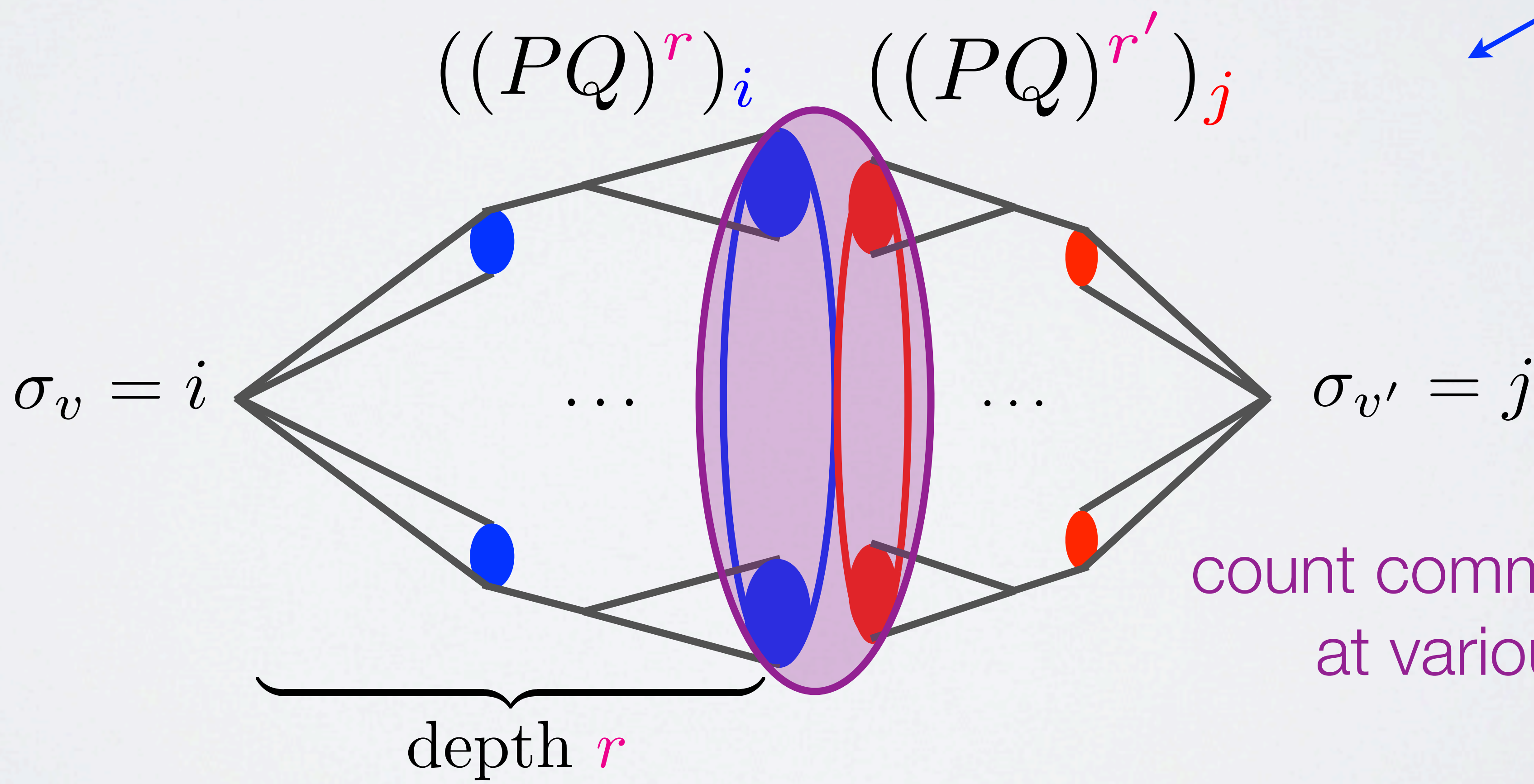

count common neighbors at various depths

What is a vertex neighborhood like? $\mathrm{SBM}(n, p, Q/n)$

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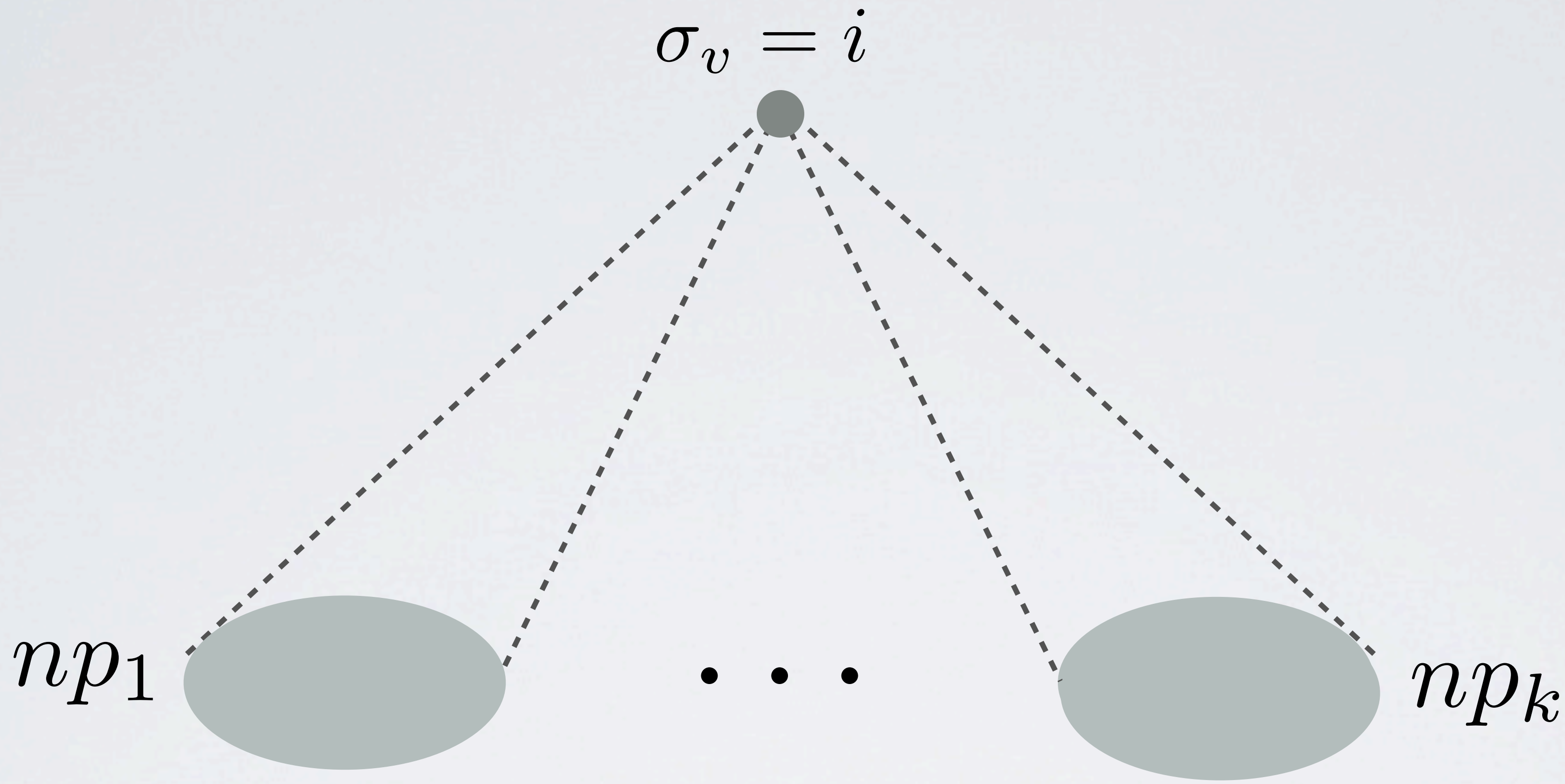

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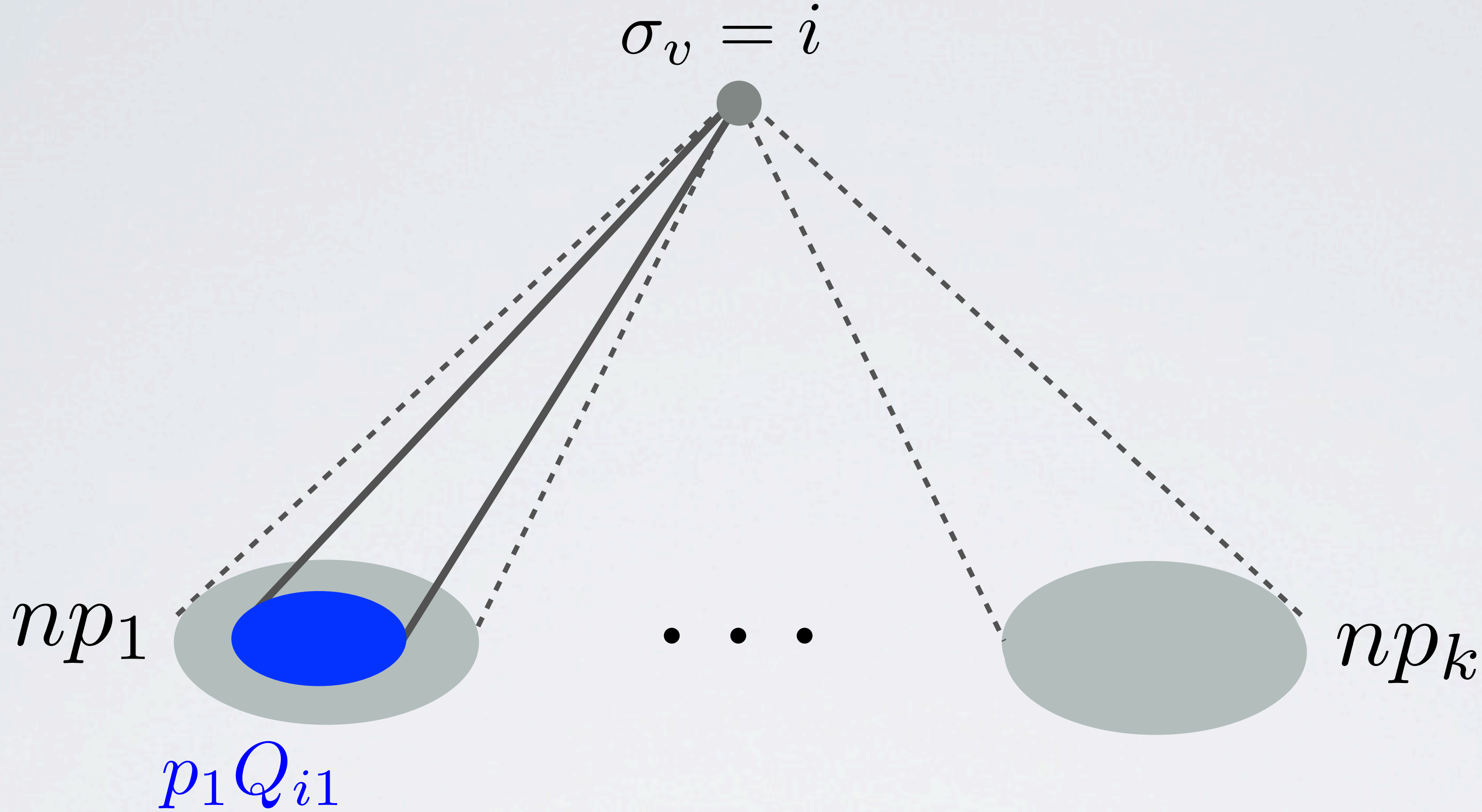

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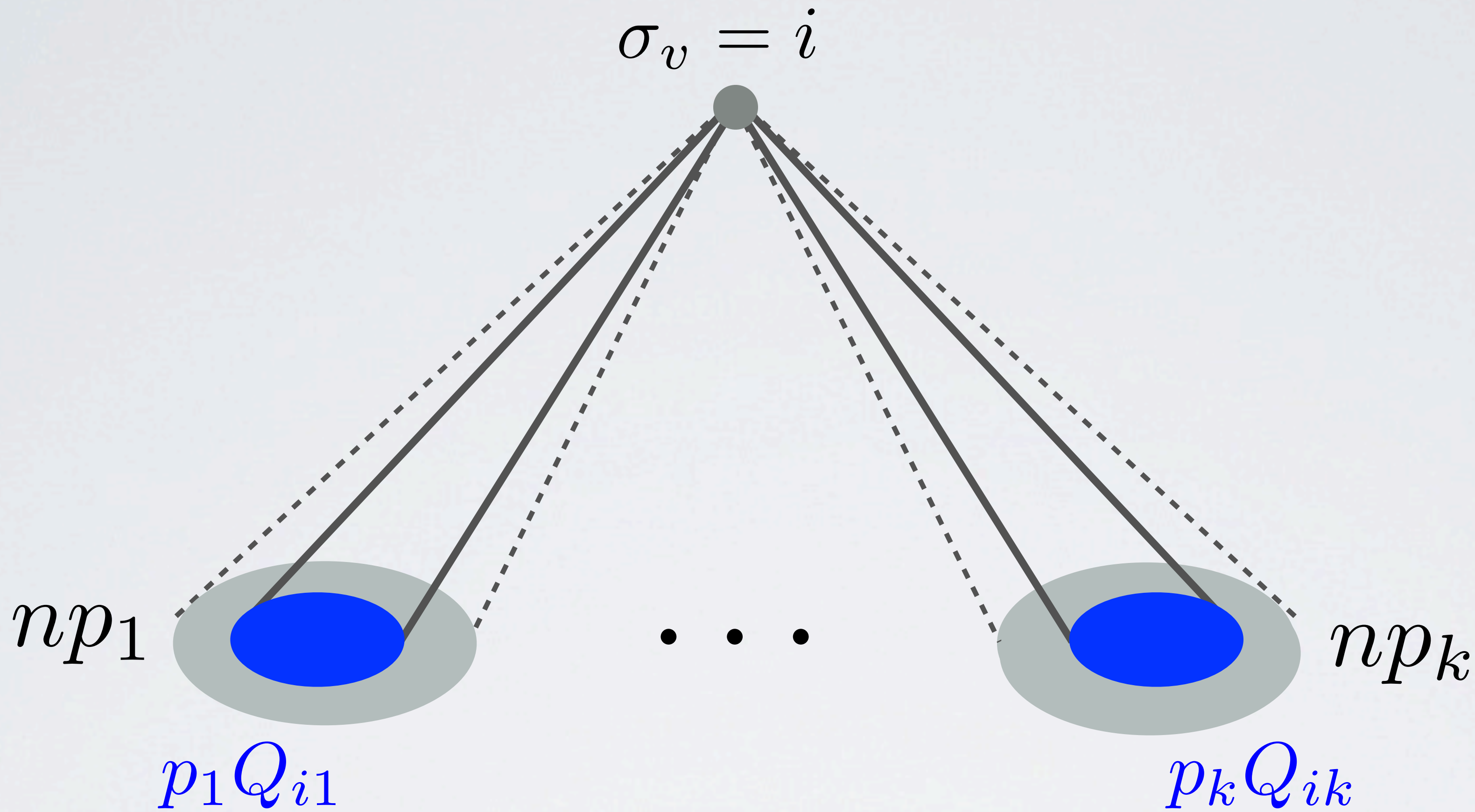

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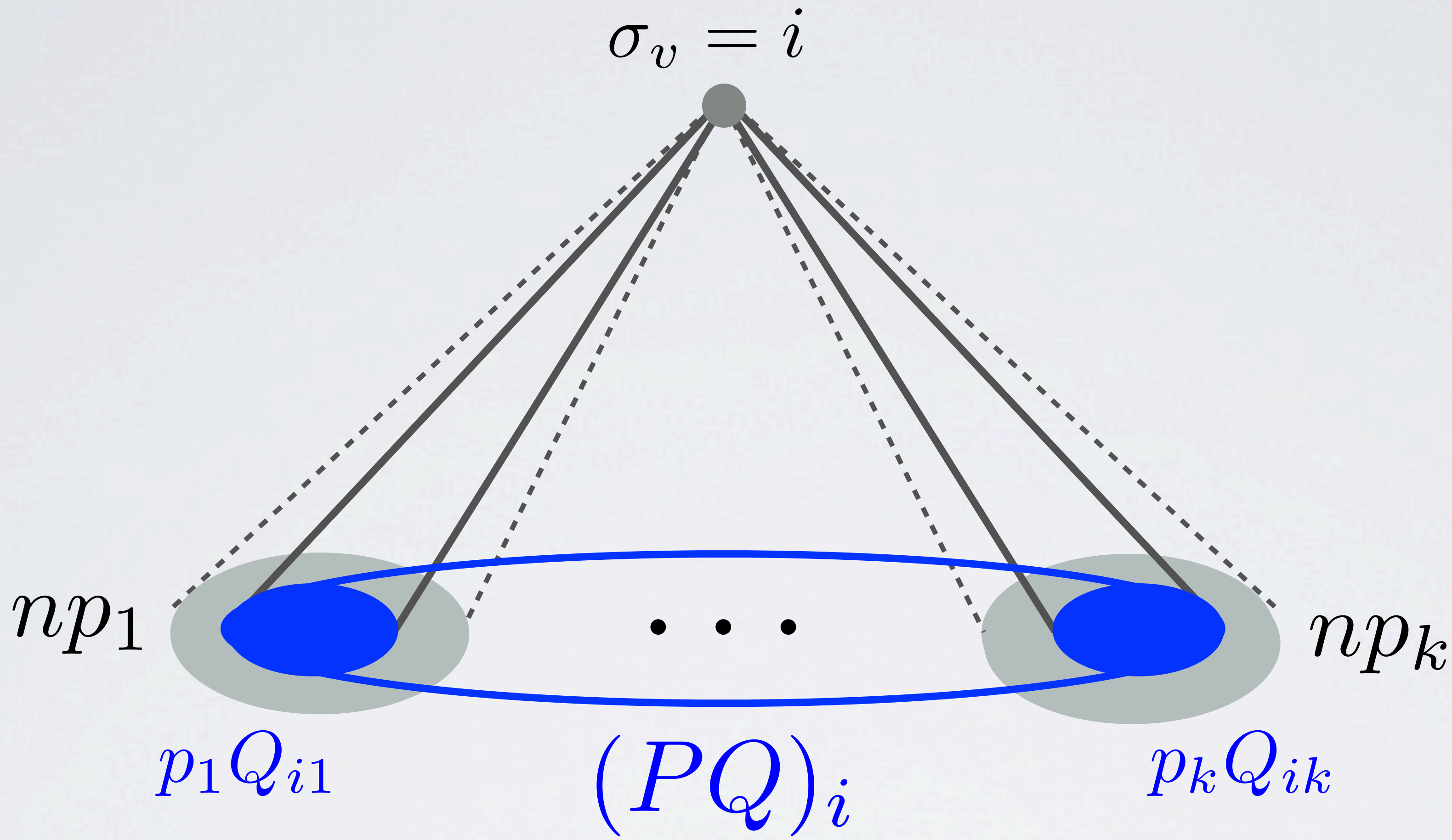

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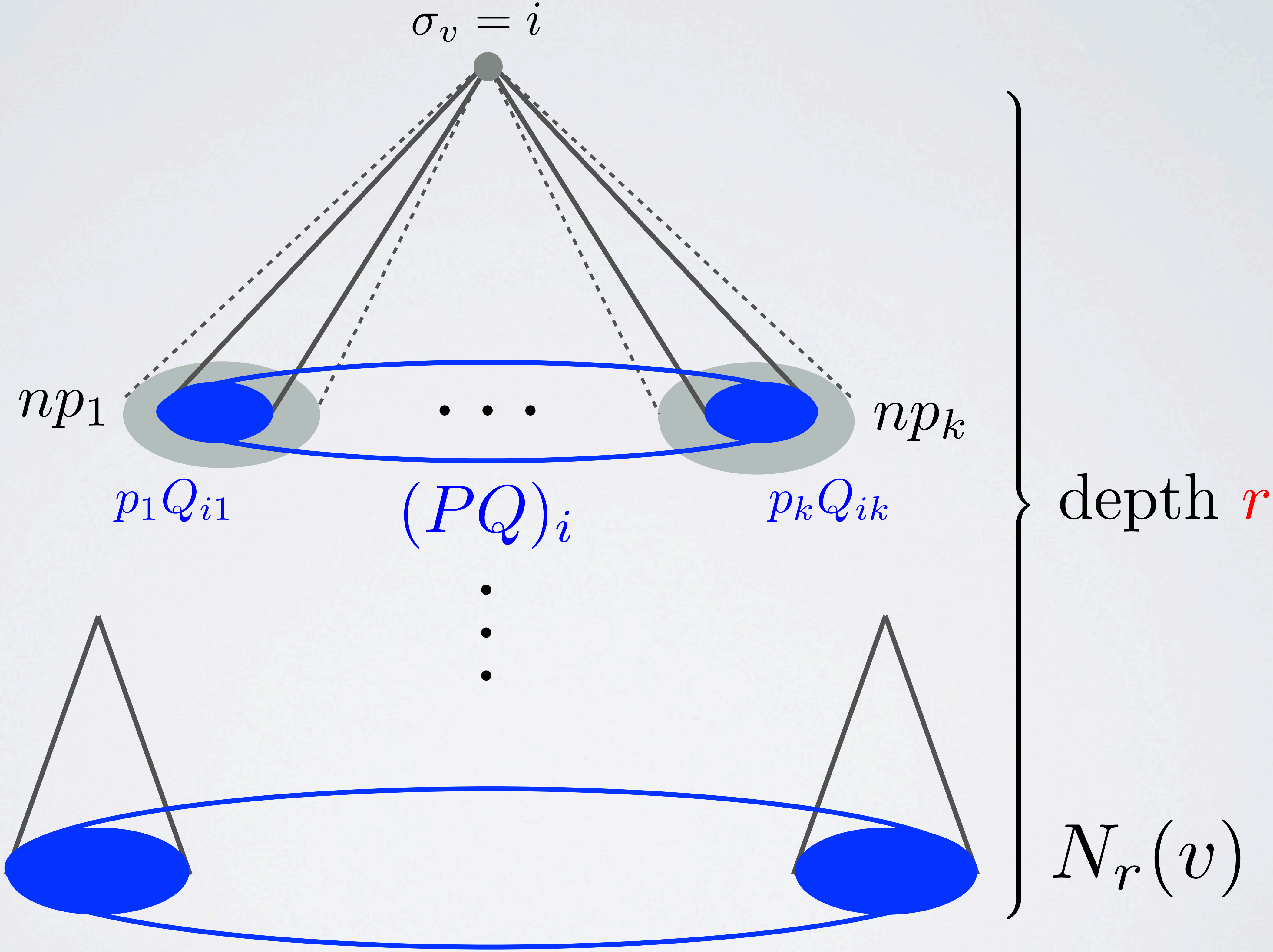

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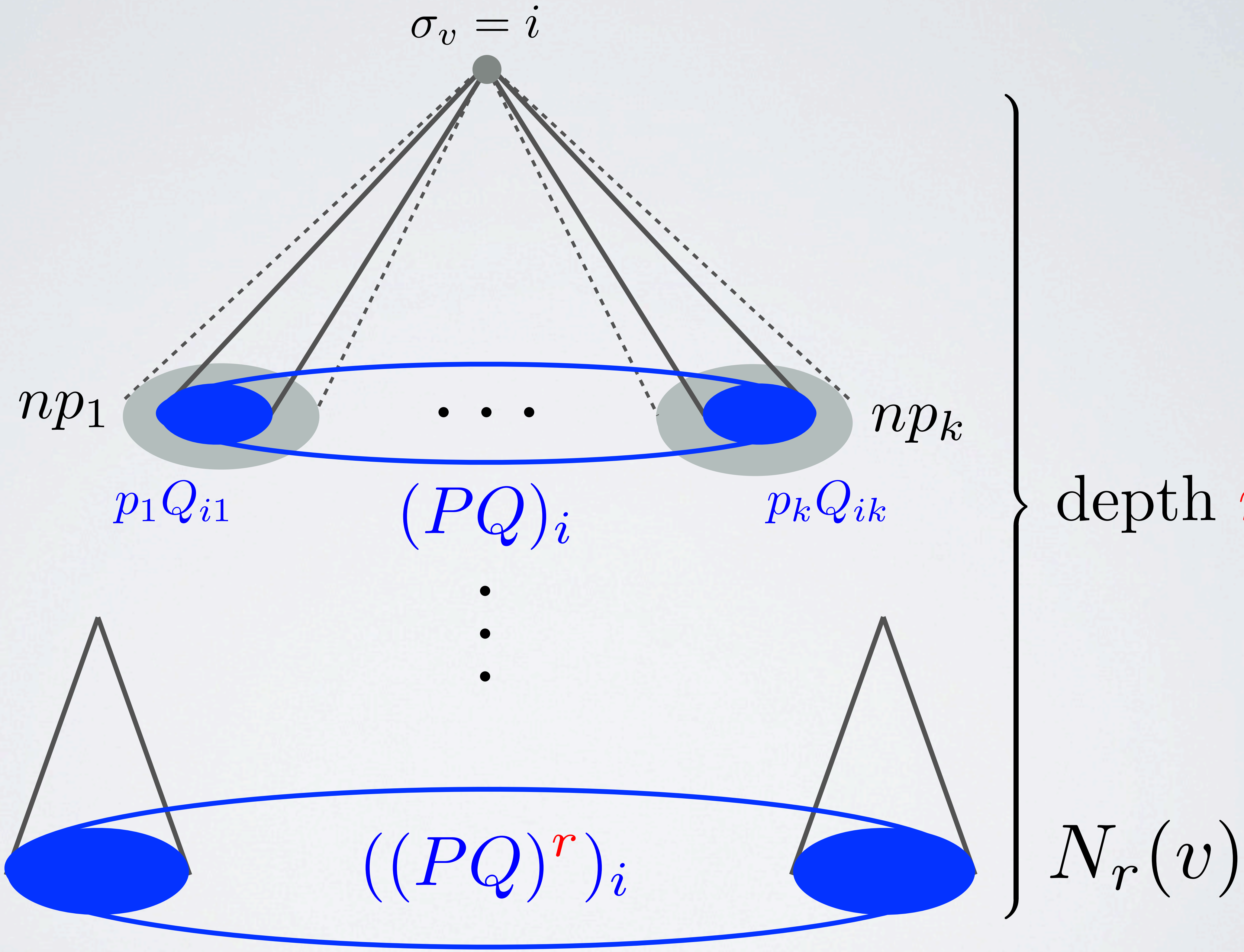

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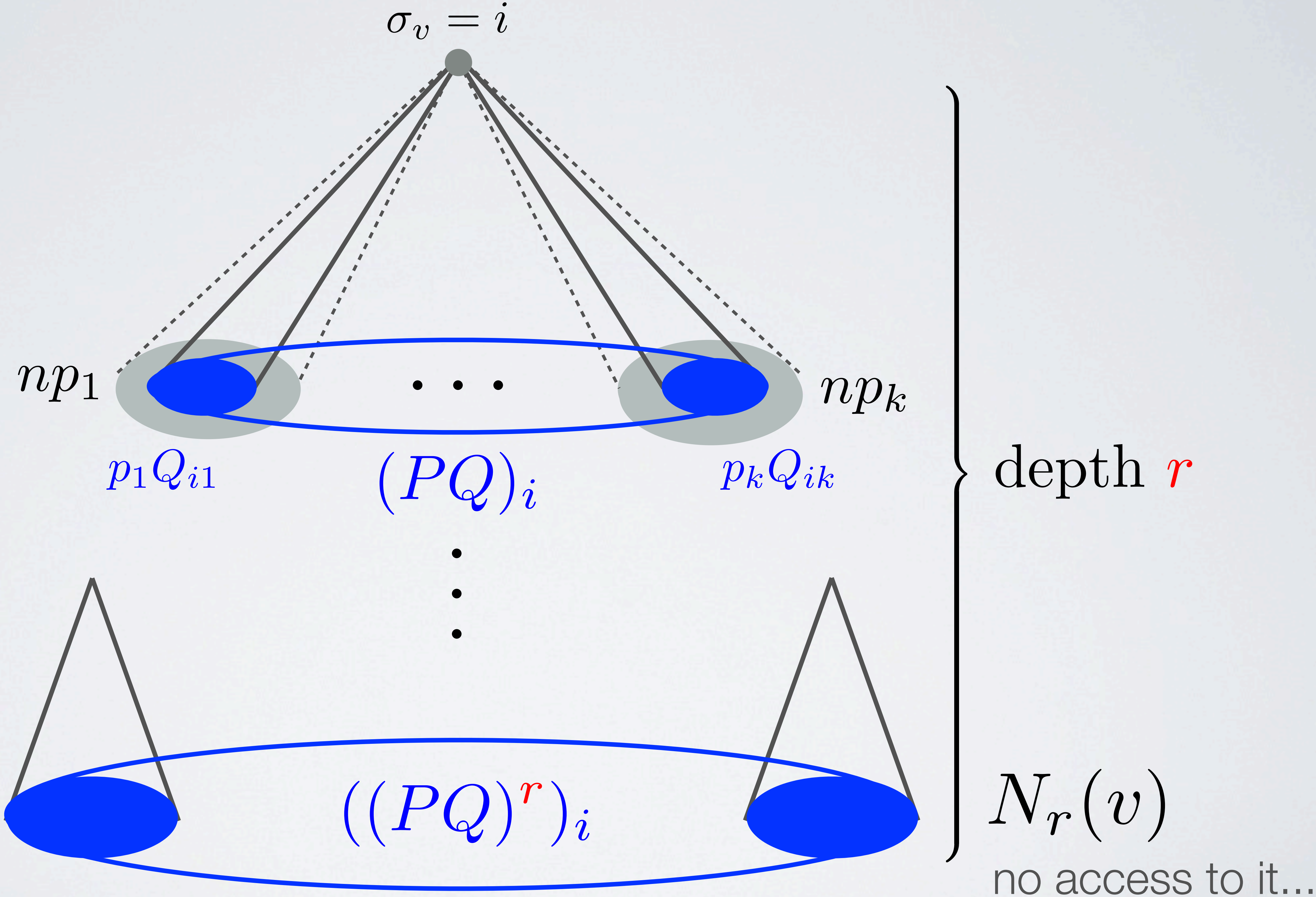
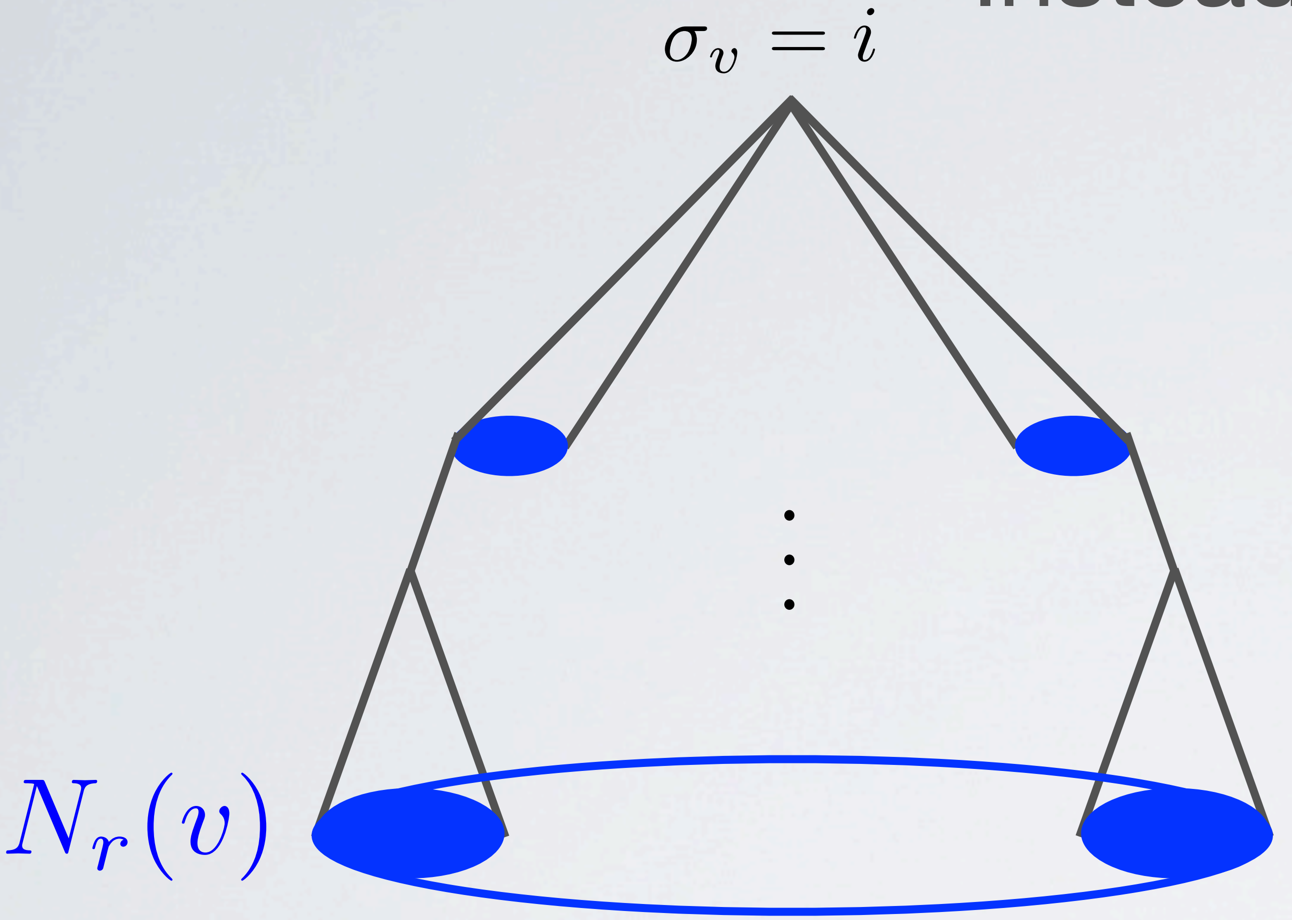


Instead: compare



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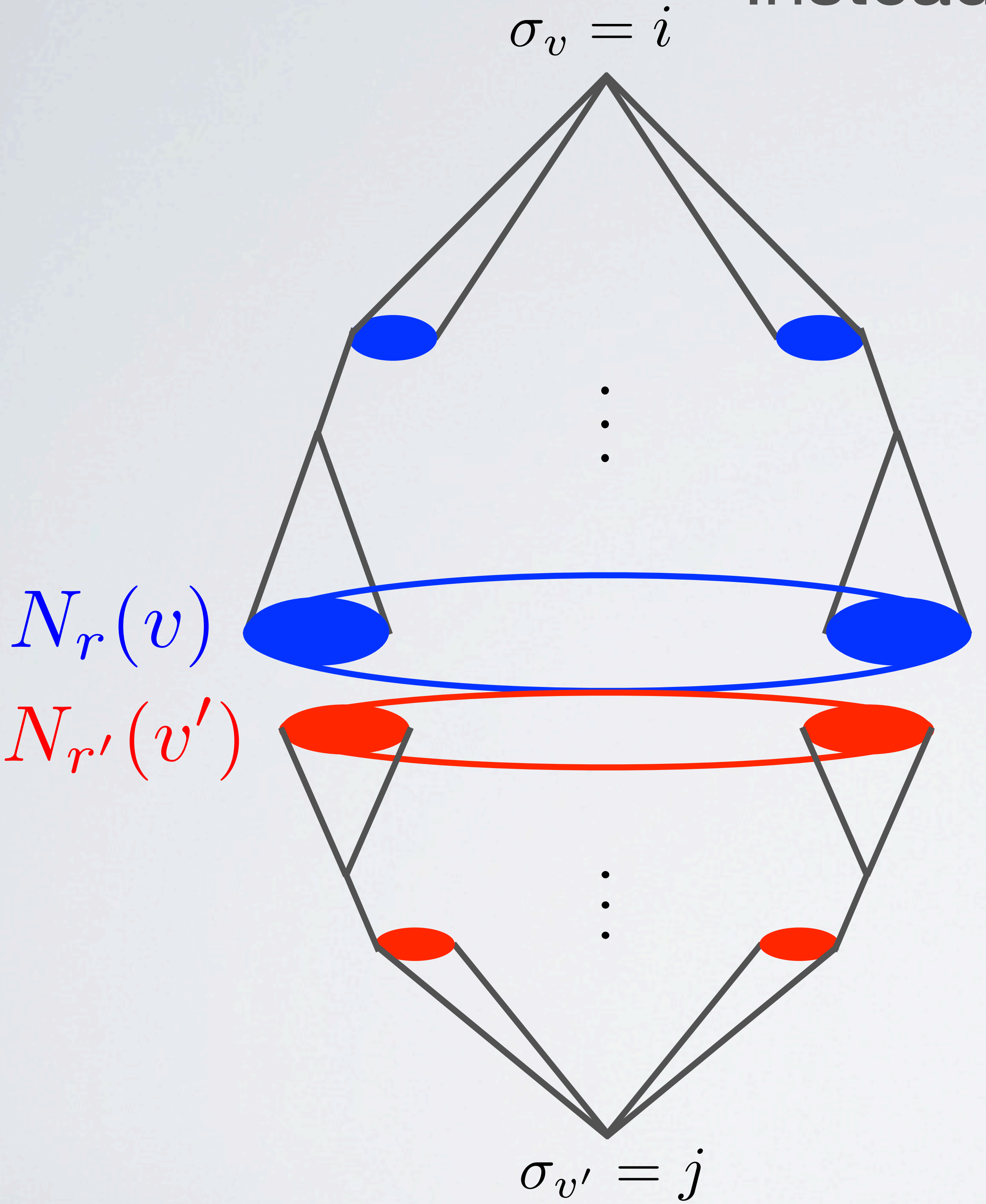

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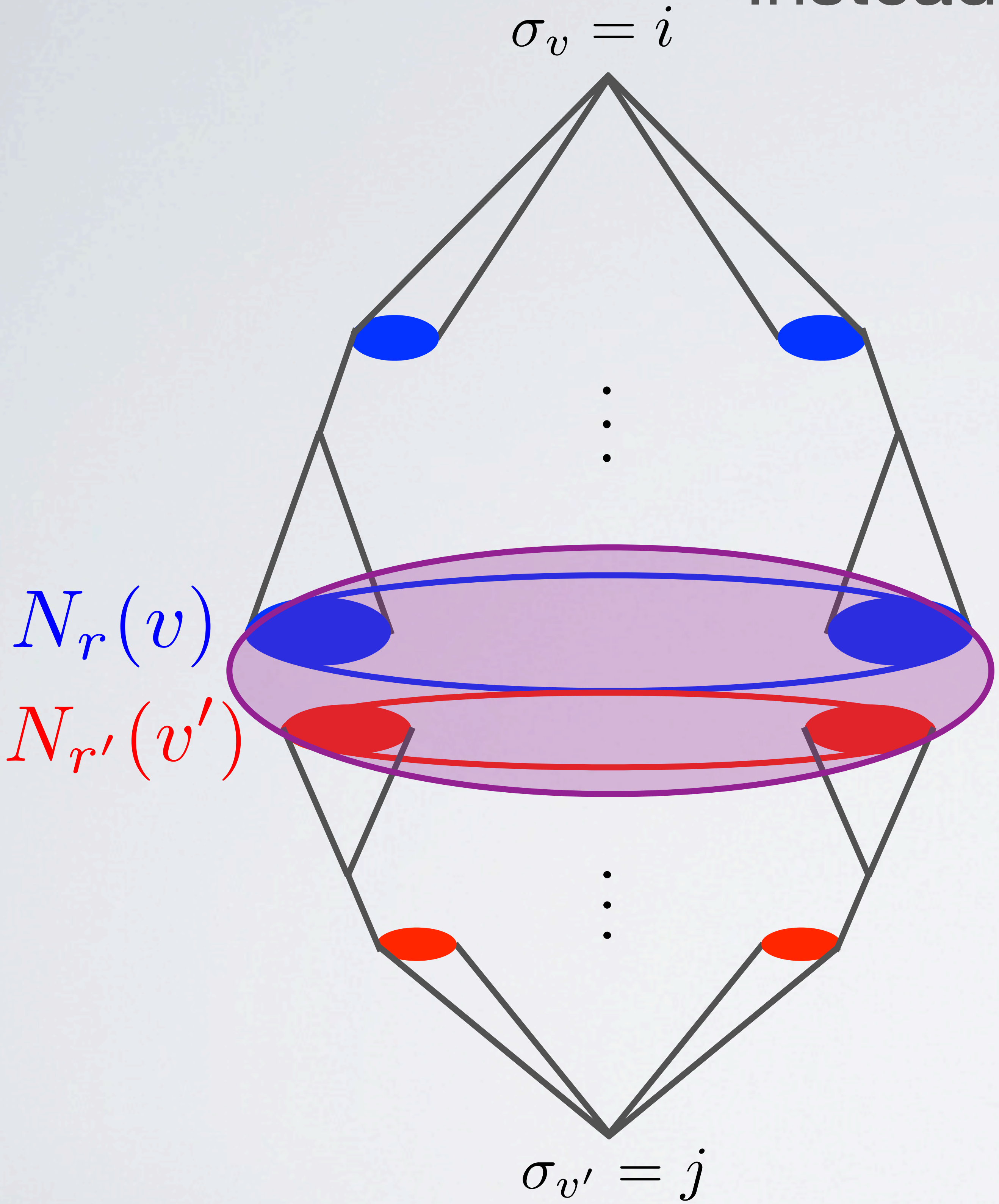

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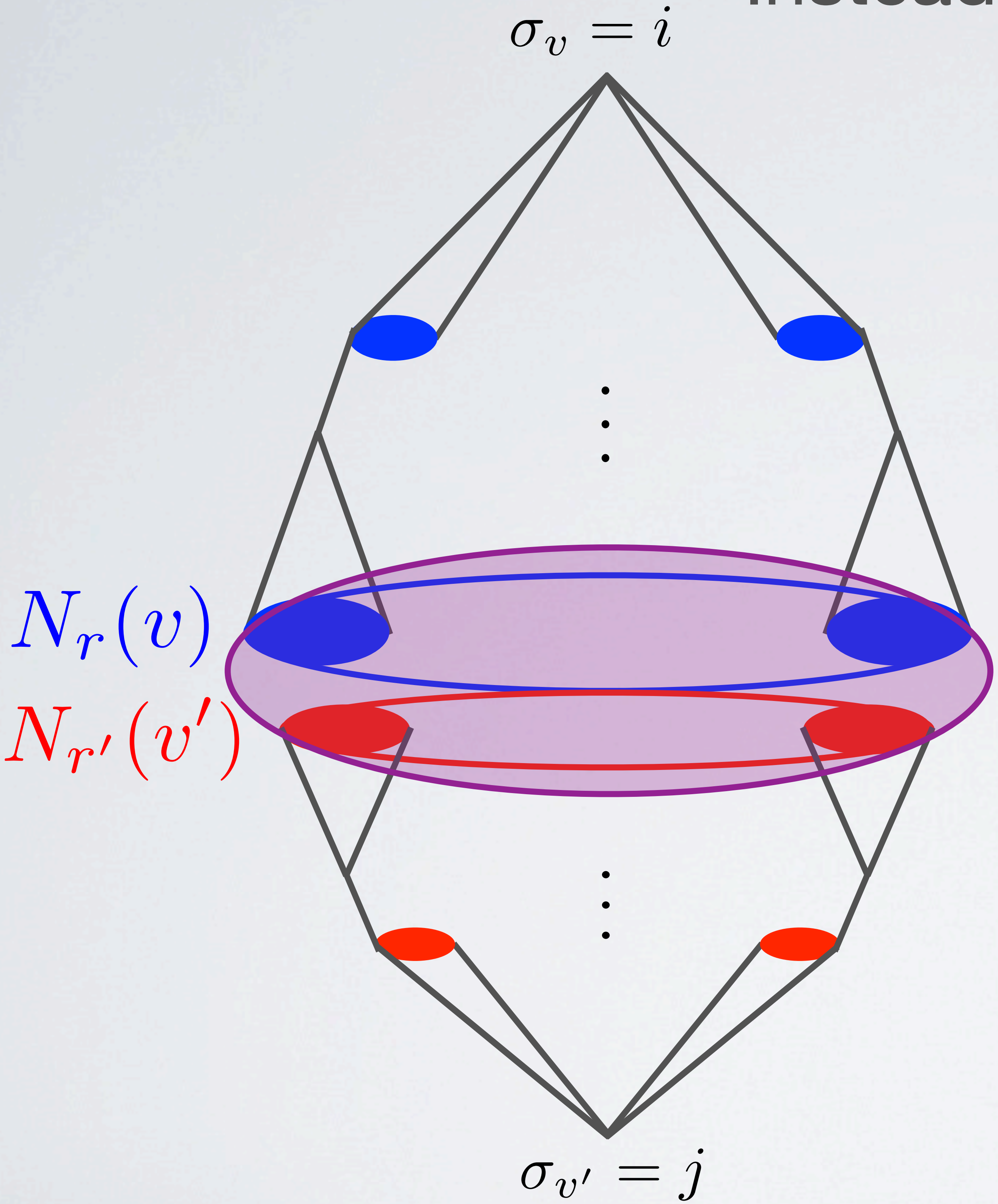

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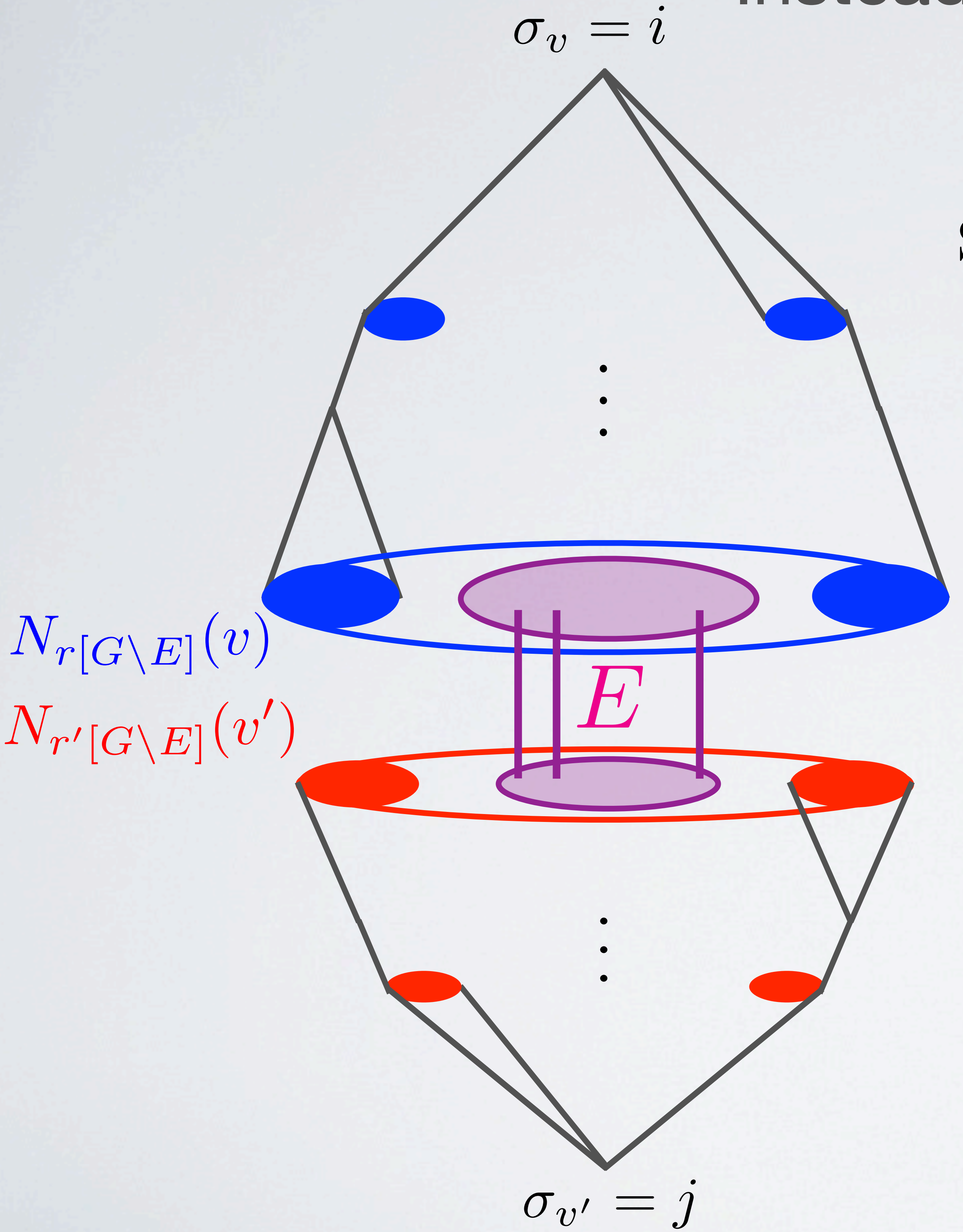
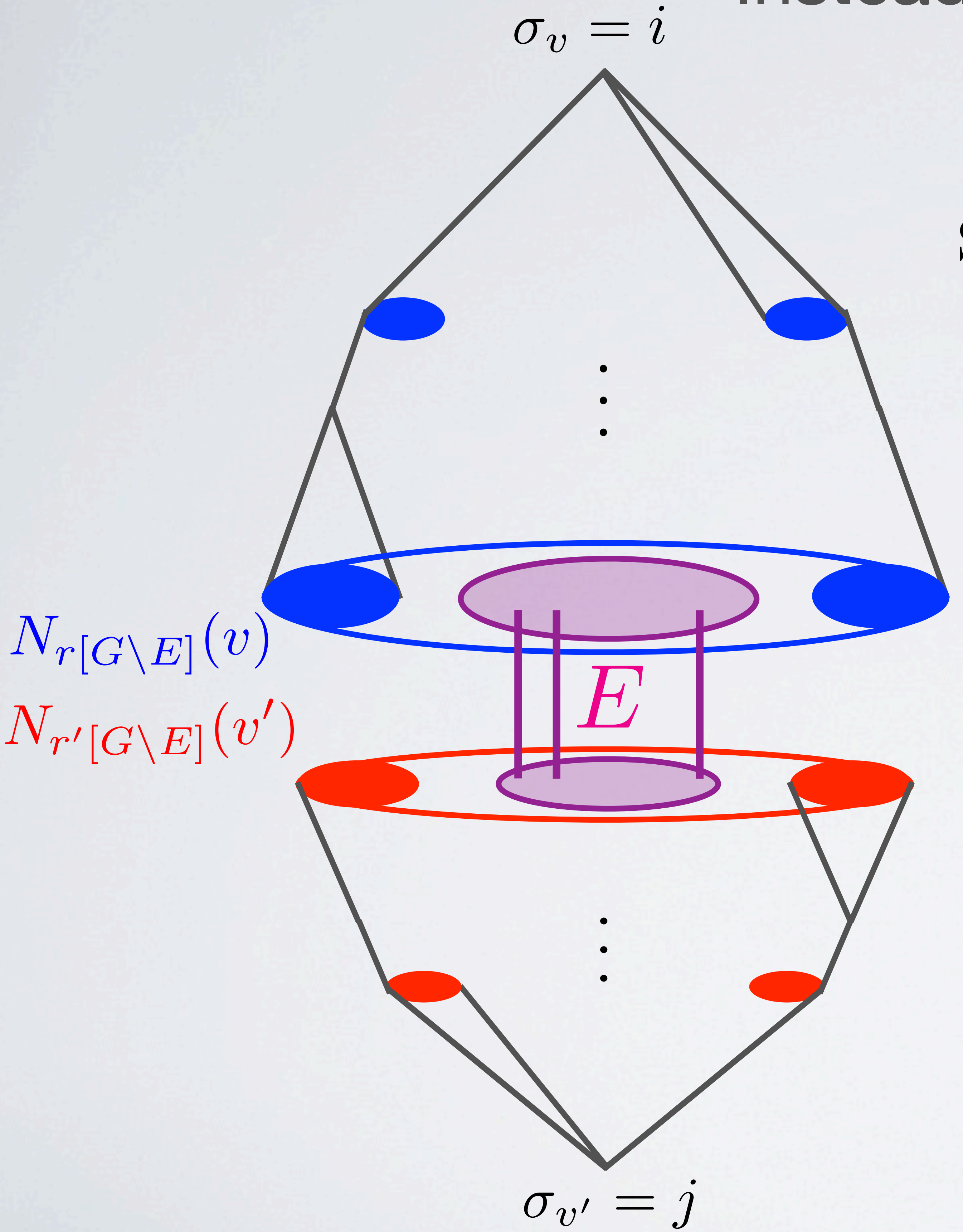
Not enough independence...

Instead: compare



Subsample G with prob. c to get E

Instead: compare



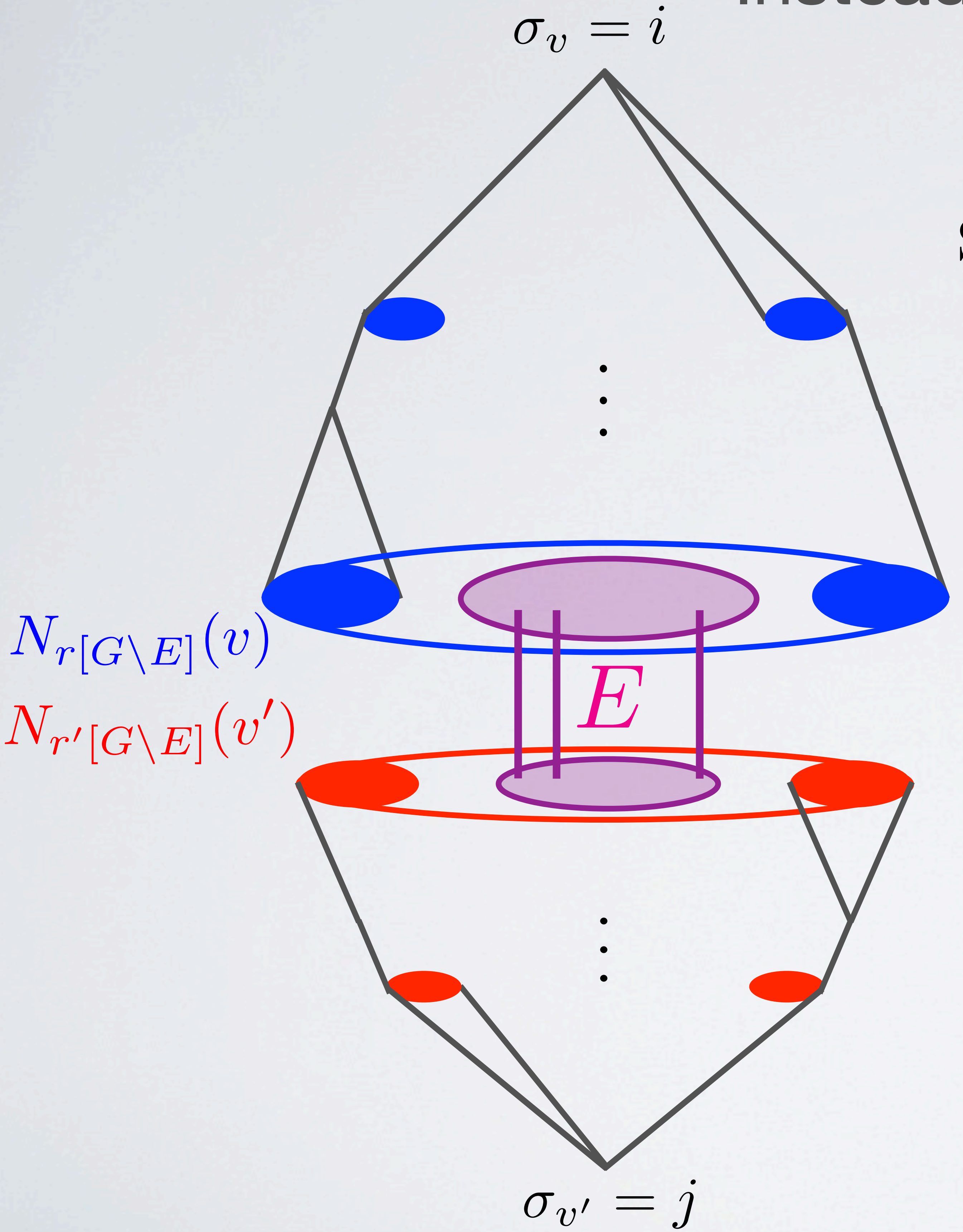
Subsample G with prob. c to get E

Compare v and v' from:

$$N_{r,r'[E]}(v \cdot v')$$

= number of such pairs of vertices

Instead: compare



Subsample G with prob. c to get E

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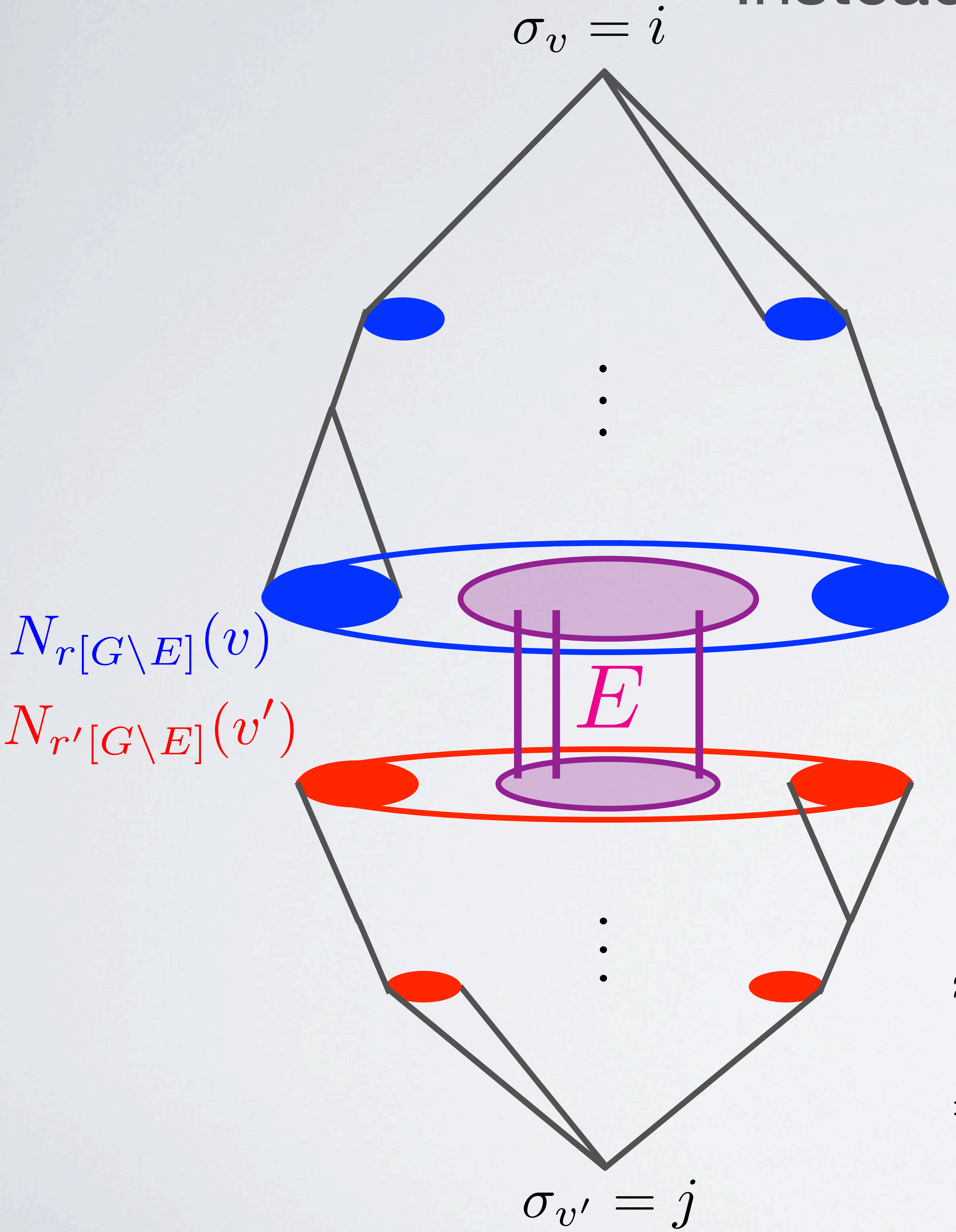
Instead: compare



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$$\approx N_{r[G\setminus E]}(v) \cdot \frac{cQ}{n} N_{r'[G\setminus E]}(v')$$

$$\approx ((1-c)PQ)^r e_{\sigma_v} \cdot \frac{cQ}{n}((1-c)PQ)^{r'} e_{\sigma_{v'}}$$

$$= c(1-c)^{r+r'} e_{\sigma_v} \cdot Q(PQ)^{r+r'} e_{\sigma_{v'}}/n$$

The `degree-profiling` algorithm

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(1) Split G into two graphs $\to$ G' sparse but large degree; G'' log-degree

The `degree-profiling` algorithm

(1) Split G into two graphs

- G' sparse but large degree
- G'' log-degree

(2) Run `Sphere-comparison` on G'

-> gets a fraction 1-o(1) with quasi-linear complexity

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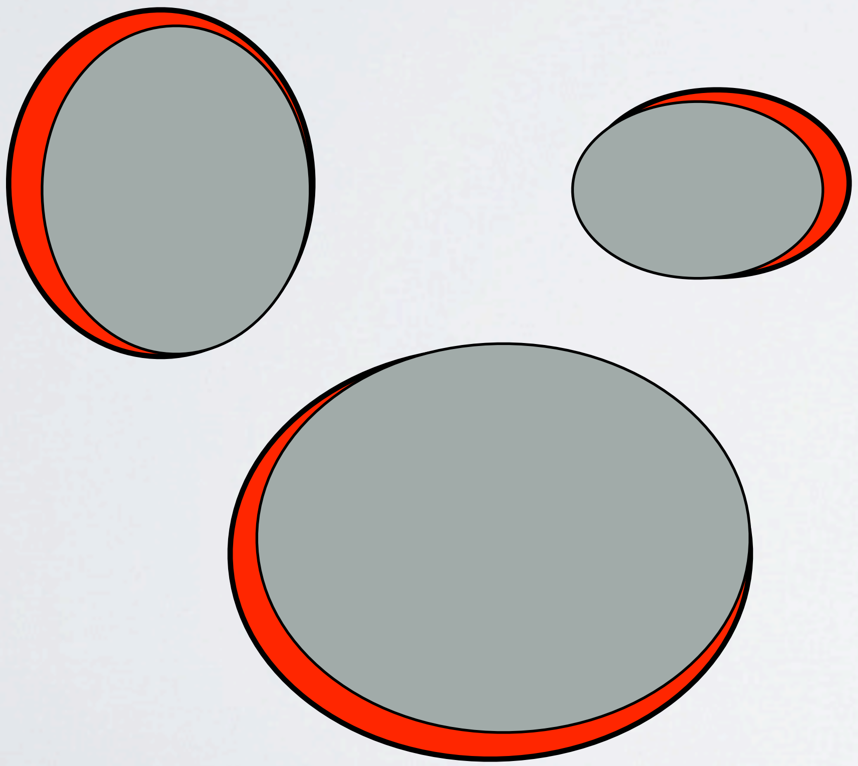

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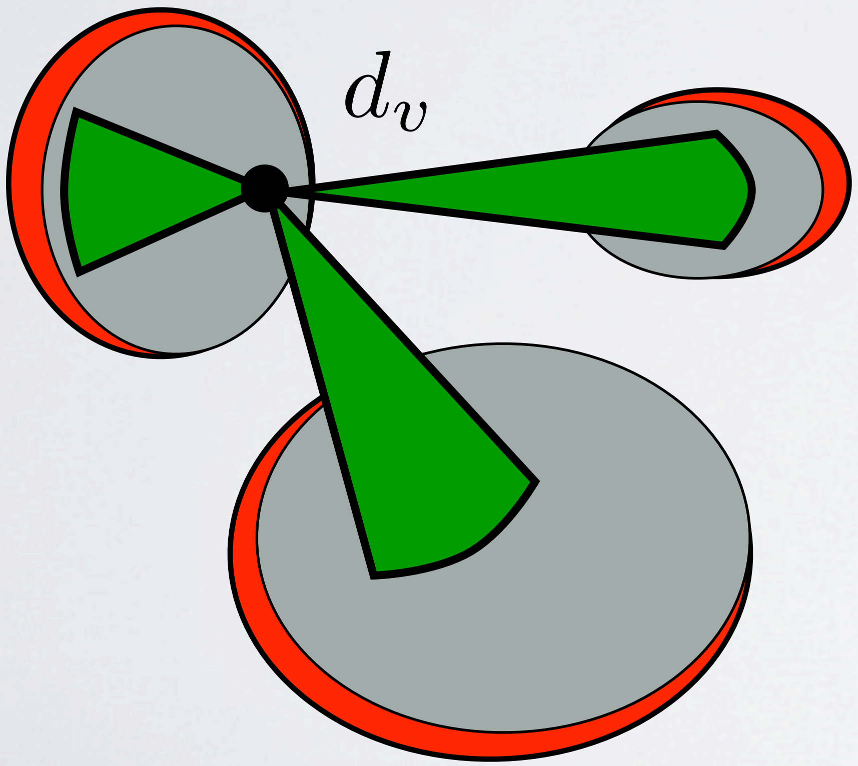

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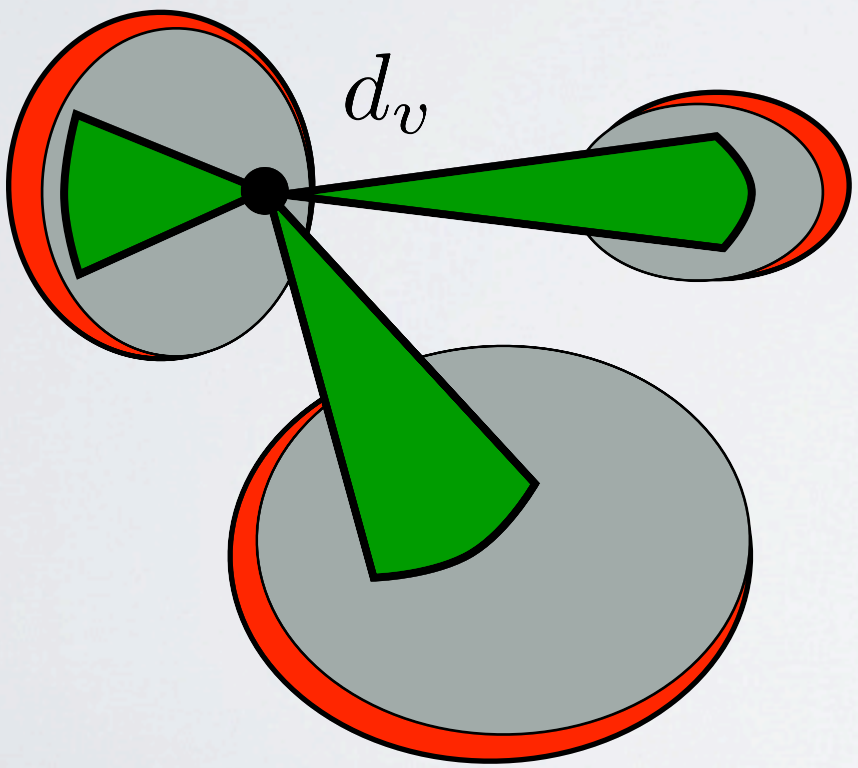

$$d_v \dddot{\sim} \mathcal{P}(\log(n)(PQ)_1)$$

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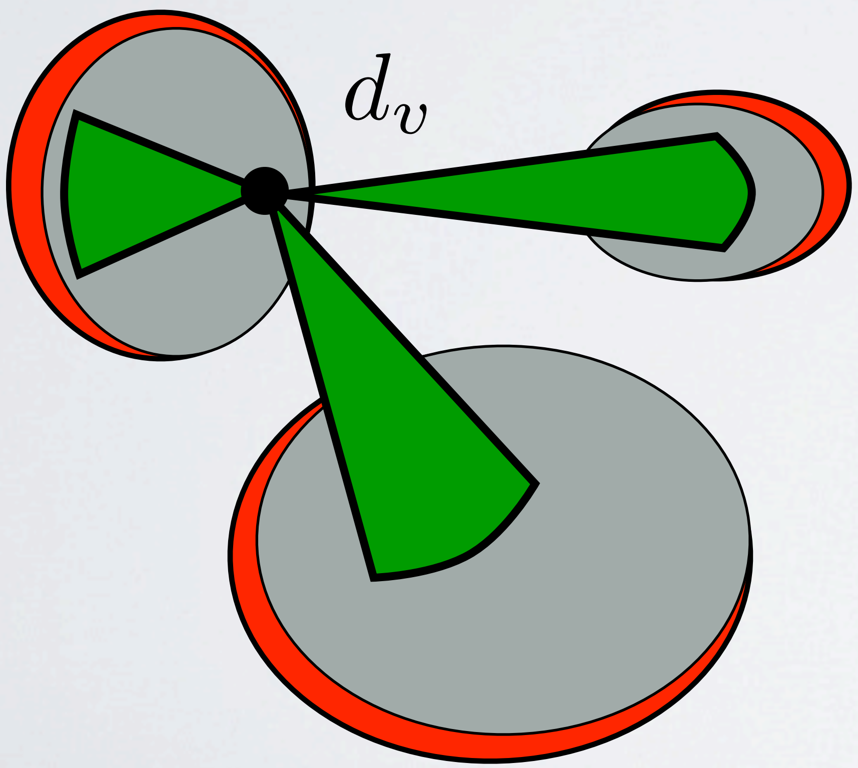

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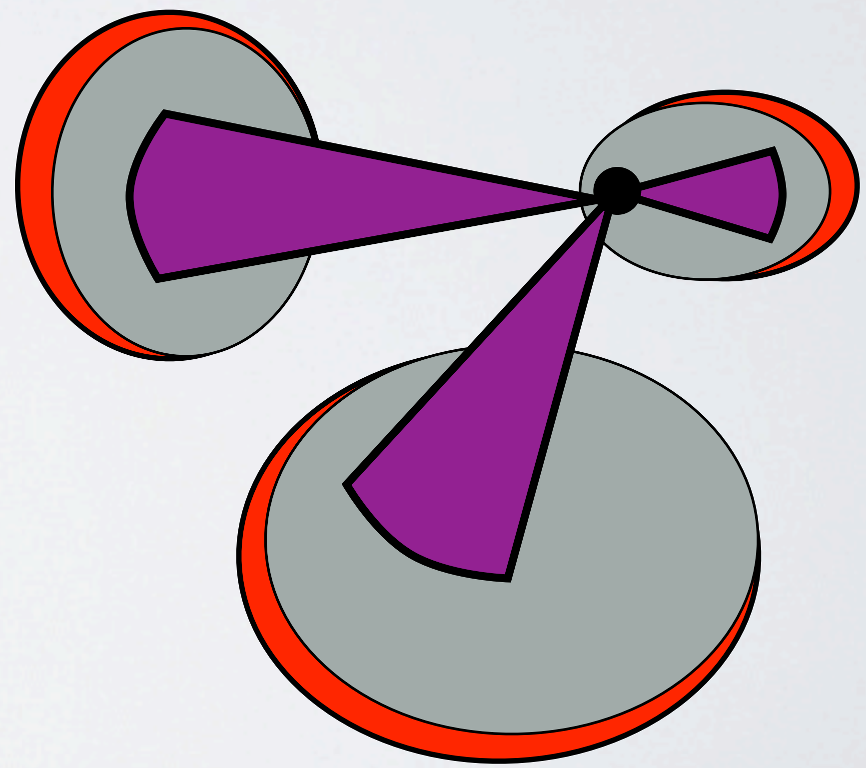

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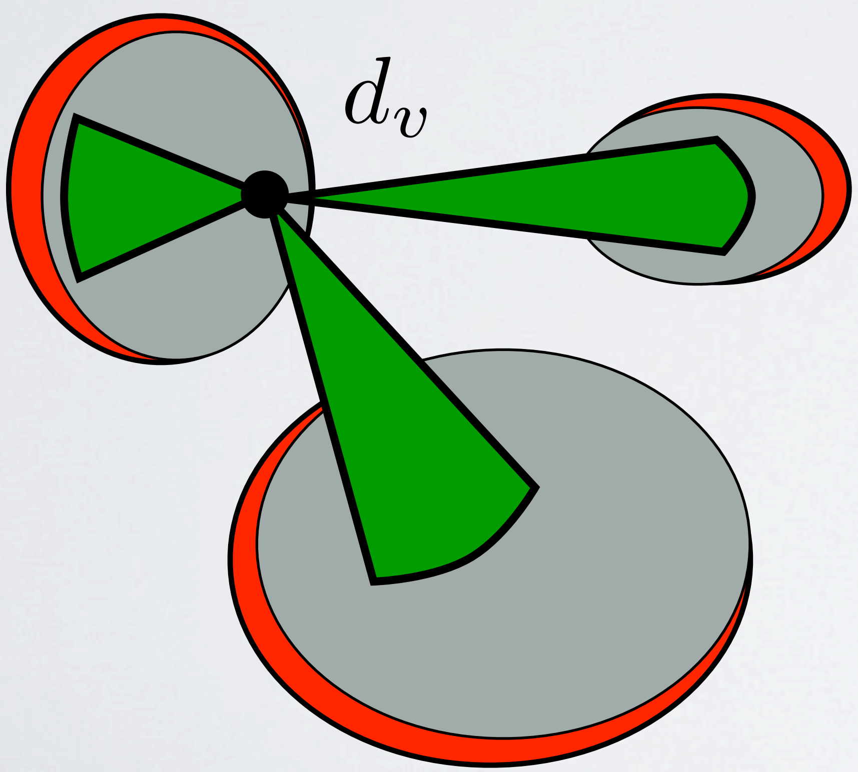
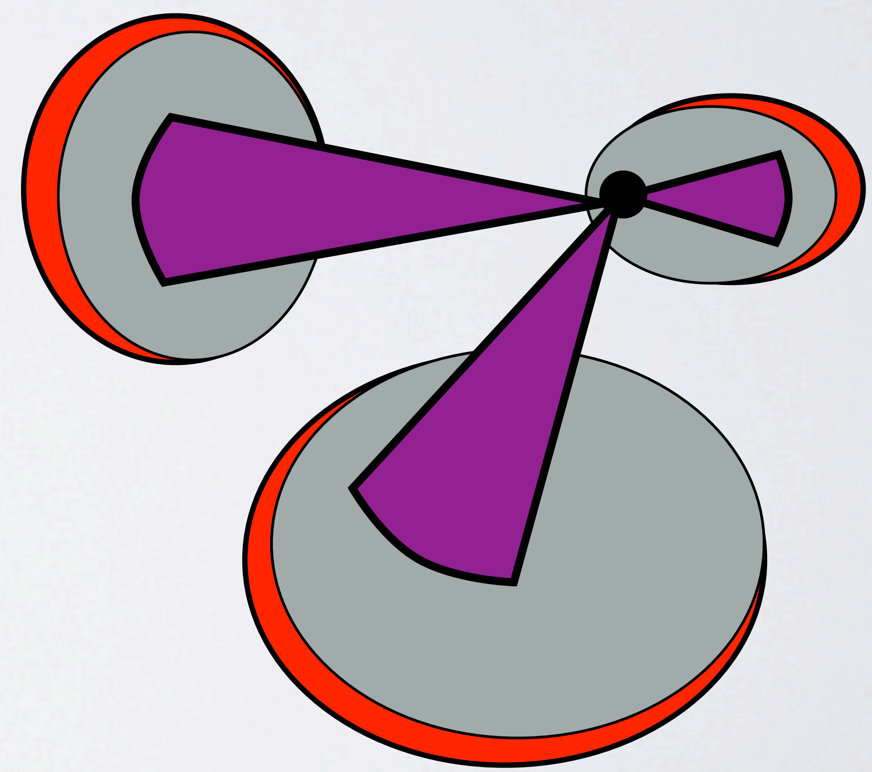


$$d_v \overset{..}{\sim} \mathcal{P}(\log(n)(PQ)_1)$$

$$d_v \overset{..}{\sim} \mathcal{P}(\log(n)(PQ)_2)$$



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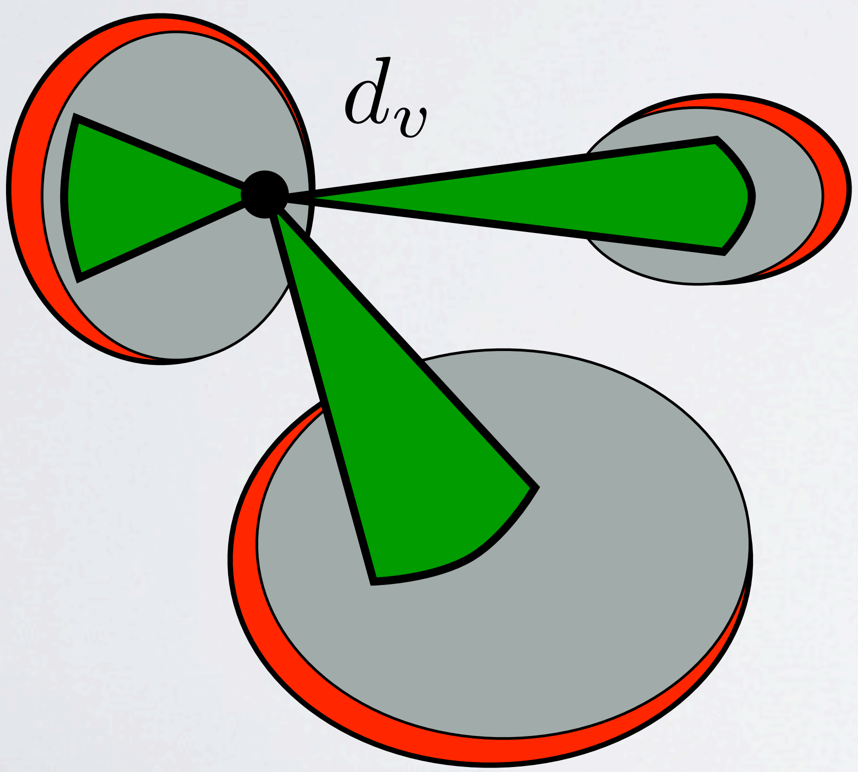
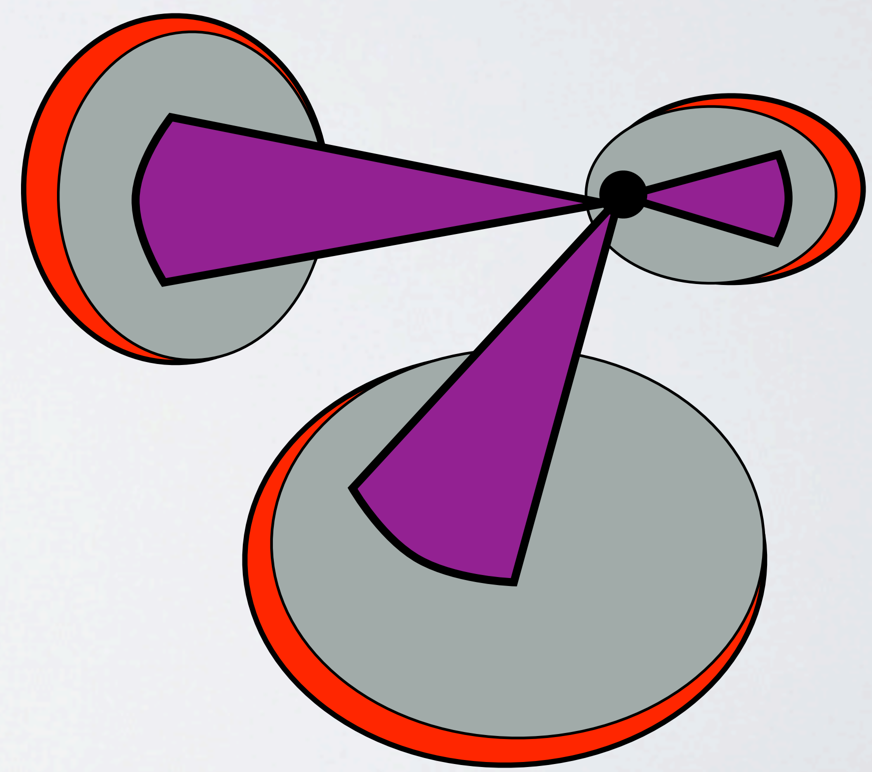


Hypothesis 1

$$d_v \ddot{\sim} \mathcal{P}(\log(n)(PQ)_1)$$

Hypothesis 2

$$d_v \ddot{\sim} \mathcal{P}(\log(n)(PQ)_2)$$



The degree-profiling algorithm

(1) Split G into two graphs

$\mathrm{G'}$ sparse but large degree

$\mathrm{G''}$ log-degree

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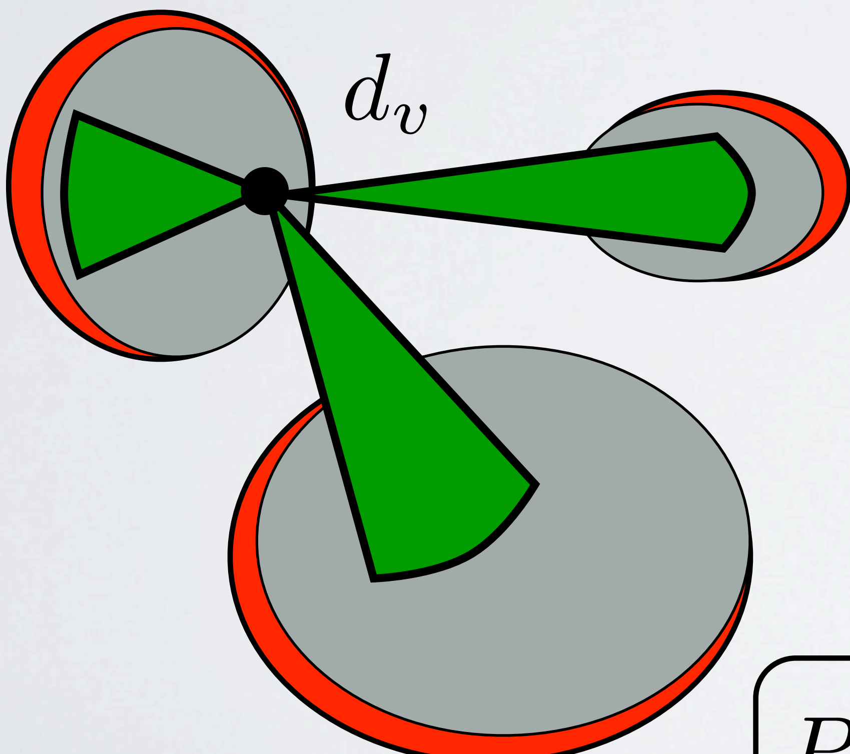

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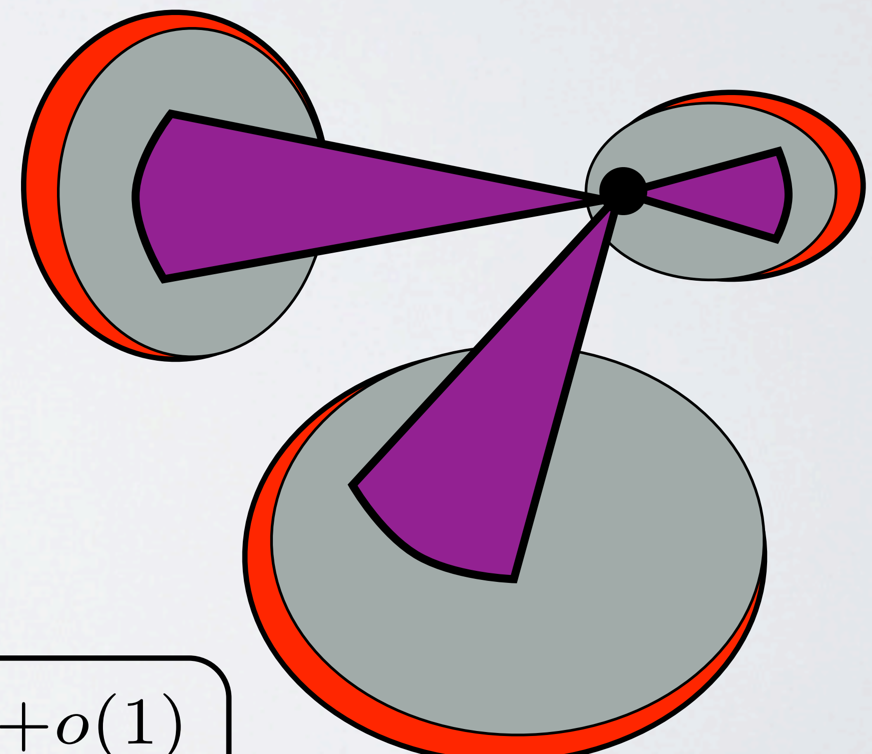
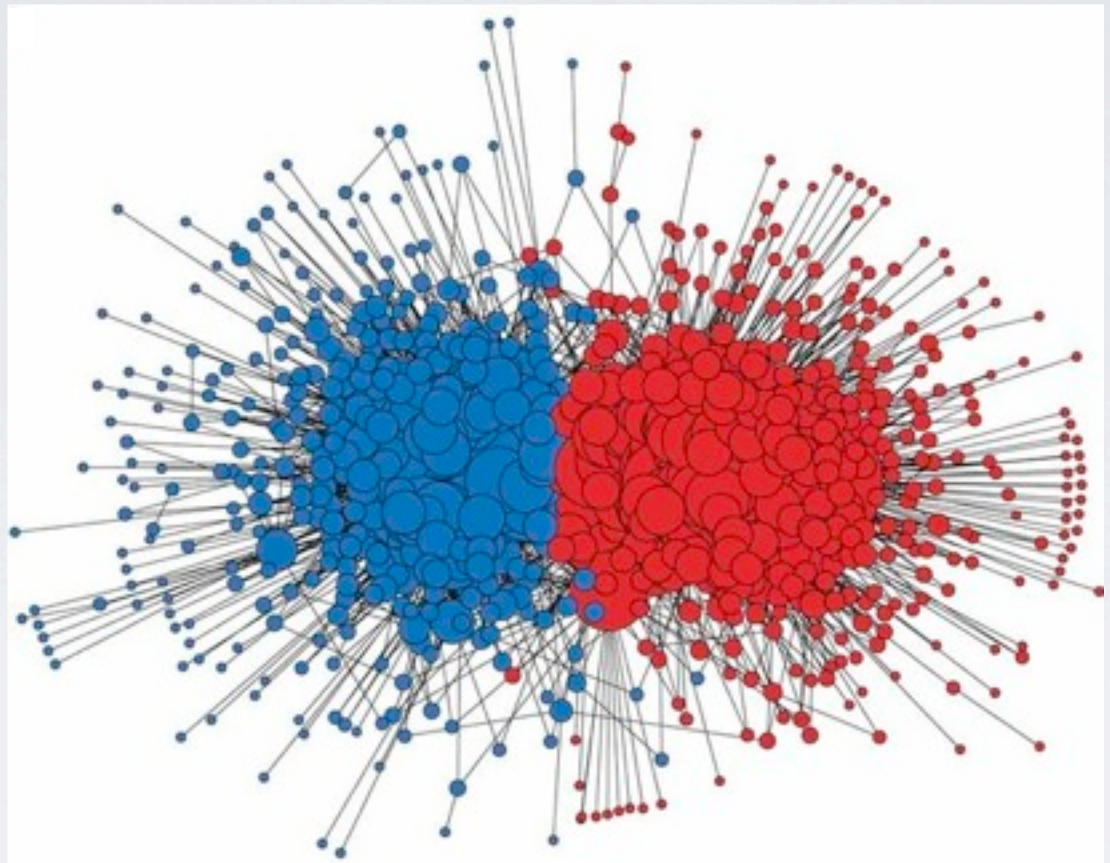


$$\boxed{P_e = n^{-D_+((PQ)_1,(PQ)_2)+o(1)}}$$

Some data: the blog network



1490 blogs
(left- and right-leaning)
[Adamic and Glance ’05]

$$Q_{11} \approx Q_{22} \approx 5.5 \log(n)/n$$
$$Q_{12} \approx 0.5 \log(n)/n$$

$$\sqrt{a} - \sqrt{b} \approx 1.6 > 1.41$$
$$95\%$$

Open problems

- exact distortion curve for partial recovery
- other models
- universal results
- detection with multiple symmetric clusters

Advertisement

- Tutorial on Information Theory and Machine Learning, ISIT 2015, Hong Kong