Faster (Spectral) Algorithms via Approximation Theory

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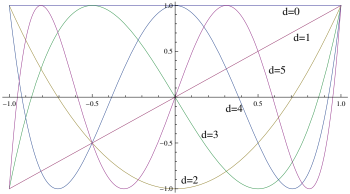
EPFL



Based on a recent monograph with Sushant Sachdeva (Yale)

Simons Institute, Dec. 3, 2014

The Goal

Many algorithms today rely on our ability to quickly compute good approximations to matrix-function-vector products: e.g.,

- $A^s v$, $A^{-1} v$, $\exp(-A) v$, ...
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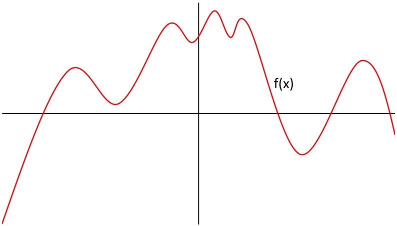
The classical area in analysis of **approximation theory** provides the right framework to study these questions.

Approximation Theory



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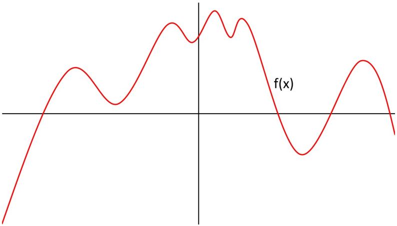

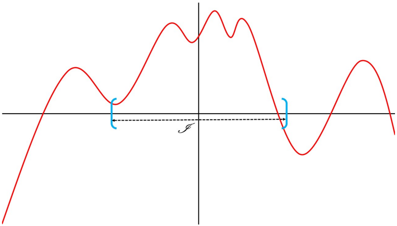

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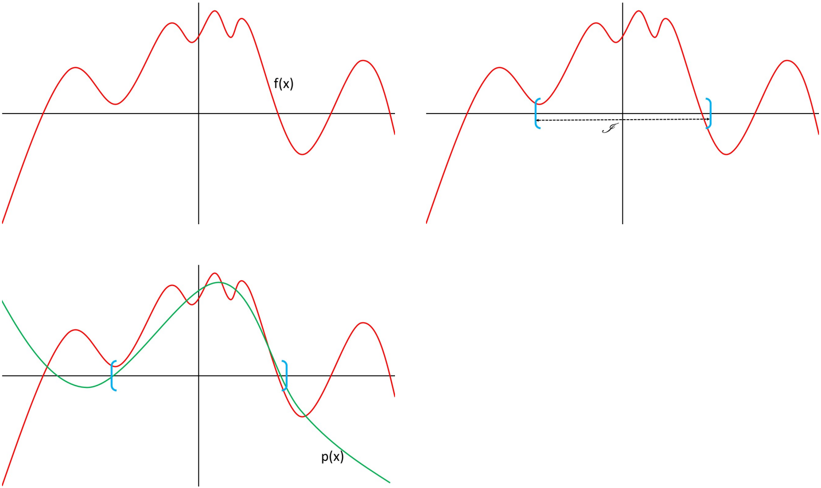

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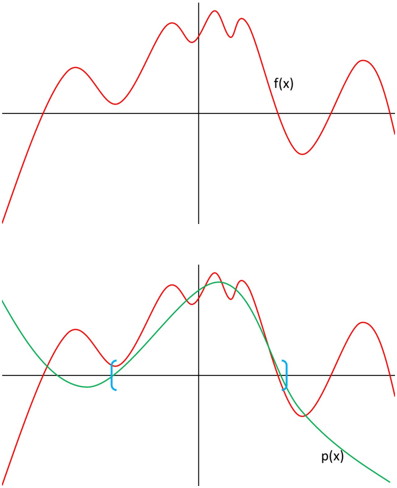

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- For our applications good enough approximations suffice.

Algorithms/Numerical Linear Alg.- $f(A)v$, Eigenvalues, ..

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- $\| \sum_{i=0}^{d} a_i A^i v - A^s v\| \leq \delta \|v\|$ since
  - all the eigenvalues of A lie in $[-1, 1]$, and
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How small can d be?

Example: Approximating the Monomial

For any s, for any $\delta > 0$, and $d \sim \sqrt{s \log(1/\delta)}$, there is a polynomial $p_{s,d}$ s.t. $\sup_{x \in [-1,1]} |p_{s,d}(x) - x^s| \leq \delta$.

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- **Quadratic speedup over the Power Method**: Given A, in time $\sim {}^{m}\!/_{\sqrt{\delta}}$ can compute a value $\mu \in [(1-\delta)\lambda_1(A), \lambda_1(A)]$.

Chebyshev Polynomials

The Chebyshev polynomial of deg. d is defined recursively to be:

$$T_d(x) \stackrel{\text{def}}{=} 2xT_{d-1}(x) - T_{d-2}(x)$$

for $d \geq 2$ with $T_0(x) \stackrel{\text{def}}{=} 1$, $T_1(x) \stackrel{\text{def}}{=} x$.

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Thus, $|T_d(x)| \leq 1$ for all $x \in [-1, 1]$.

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$D_s \stackrel{\text{def}}{=} \sum_{i=1}^{s} Y_i$ where $Y_1, \ldots, Y_s$ i.i.d. ± 1 w.p. $1/2$ ($D_0 \stackrel{\text{def}}{=} 0$).

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How far can polynomial approximations take us?

Lower Bounds for Polynomial Approximations

Bad News [see Sachdeva-V. 2014]

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Bypass this barrier via rational functions!

Example: Approximating the Exponential

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How do we compute $(S_d(A))^{-1} v$?

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Desire: A rational approximation with negative poles.

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For every d, there exists a degree-d polynomial p_d s.t.,

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Sachdeva-V. 2014

Moreover, the coefficients of p_d are bounded by $d^{O(d)}$, and can be approximated up to an error of $d^{-\Theta(d)}$ using $\mathrm{poly}(d)$ arithmetic operations, where all intermediate numbers $\mathrm{poly}(d)$ bits.

Computing the Matrix Exponential- Summary

Orecchia-Sachdeva-V. 2012, Sachdeva-V. 2014

Given an **SDD** $A \succeq 0$, a vector v with $\|v\| = 1$ and δ, we compute a vector u s.t. $\|\exp(-A)v - u\| \leq \delta$, in time $\tilde{O}\left(m \log\|A\| \log 1/\delta\right)$.

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Given $Lx = b$, L is SDD, and $\varepsilon > 0$, obtain a vector u s.t., $\|u - L^{-1}b\|_L \leq \varepsilon\|L^{-1}b\|_L$. Time required $\tilde{O}\left(m \log 1/\varepsilon\right)$

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Are Laplacian solvers necessary for the matrix exponential?

Matrix Inversion via Exponentiation

Belykin-Monzon 2010, Sachdeva-V. 2014

For $\varepsilon, \delta \in (0,1]$, there exist $\mathrm{poly}(\log(1/\varepsilon\delta))$ numbers $0 < w_j, t_j$ s.t. for all symm. $\varepsilon I \preceq A \preceq I$, $(1-\delta)A^{-1} \preceq \sum_j w_j e^{-t_j A} \preceq (1+\delta)A^{-1}$.

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- Discretizing this integral, we bound the error using the Euler-Maclaurin formula, Riemann zeta fn.; global error analysis!

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Theorem

For large enough p, $e^{(B_1+\cdots+B_k)} \approx \left(e^{\frac{B_1}{p}} \cdots e^{\frac{B_k}{p}}\right)^p$.

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Thanks for your attention!