Long range dependence in evolving networks

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Outline

- Brief motivation and structure
- Seed detection in dynamic networks
- Change point detection
- Co-evolving networks

In collaboration with Sayan Banerjee, Iain Carmichael and Zoe Huang

Structure of talk

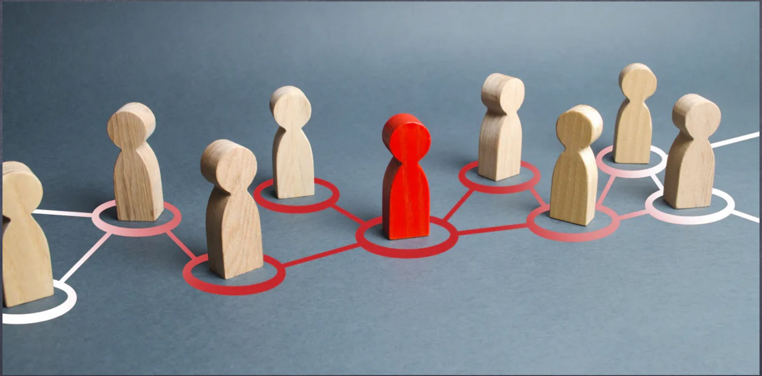
- For each problem area I will describe the motivation of the area in words

  ↪ Important

- I will describe our specific contributions

  ↪ Potentially Irrelevant

Seed detection in evolving networks

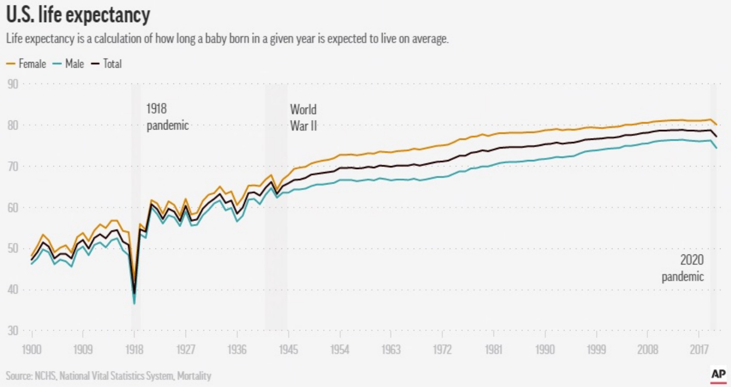


https://theconversation.com/patient-zero-why-its-such-a-toxic-term-134721

Our motivation in words

- Dynamic network started with a single node ("patient zero") or seed graph at time zero.
- observe network when it is of large size e.g. $n = 10^6$. with no temporal information only network topology (adjacency matrix)
- Have a fixed budget say $K = 30$.
- GOAL: Output 30 vertices such that with high prob. seed is in the output.

Change Point Detection



Source: Associated Press

Our motivation in words

- Suppose you have temporal network data.
  - Ex: Adjacency matrix at all or sub-sample of time points
  - Ex: Time series observations at each node etc
- Suppose network experiences a shock at some point.
- Can we detect this change point from observations?
- Changes in structural properties of the system?

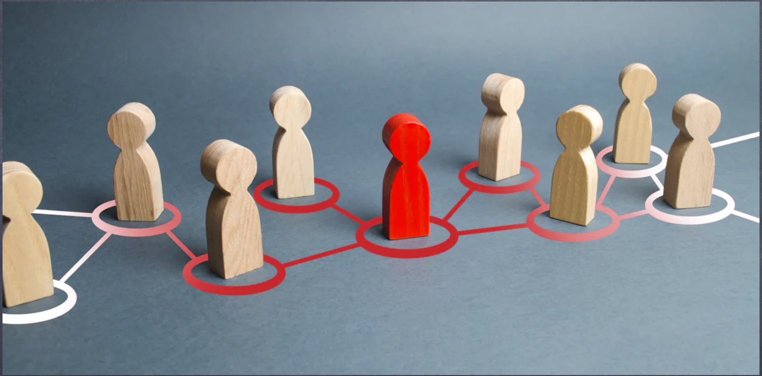
Network co-evolution: our motivation

- Most real world networks support some particular purpose (e.g. diffusion of information on Twitter)
- Co-evolution: Network influences individuals
  Individuals influence networks

- Till date majority of models deal either with
    - Dynamics on a fixed network (e.g. random walk or epidemics on a fixed network).
    - Dynamics "of" a network: Network itself changing in some fashion.
- However kept these two disciplines largely "separate". Most network practitioners believe co-evolving networks is the next frontier.

⟶ Goal: Understand conjectured phase transitions in one tractable model

Seed detection in evolving networks



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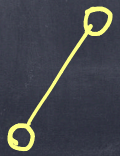
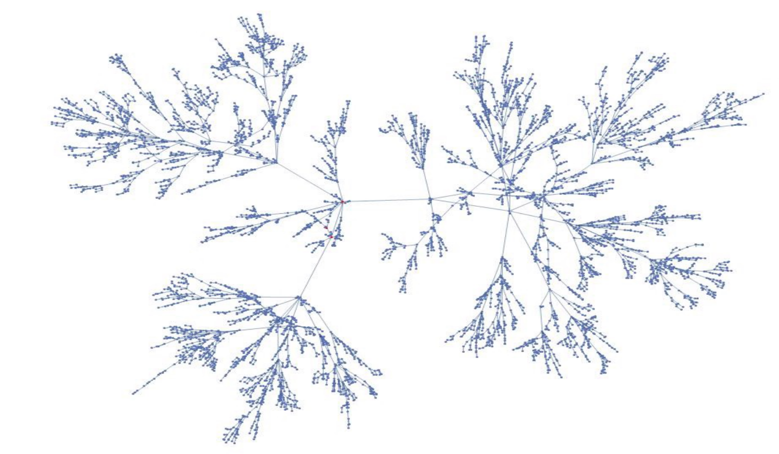
Probabilistic foundations

- Network model: Fix attachment function f. Start with single seed.
- At each stage new vertex v enters system. Connects to one pre-existing vertex
- Probability connecting to a vertex u in the system proportional to $f(\text{degree}(u))$.
- $\mathcal{T}_n$ = network of size n

Example: $f = 1$ (Random recursive tree)

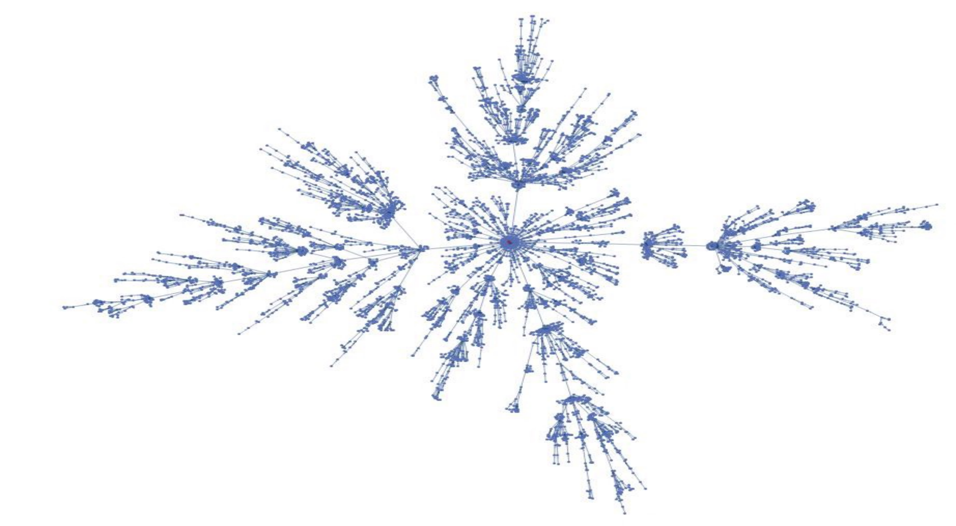


SIMULATION (n = 3000 ?)



Example $f(k) = k$ Preferential attachment

Simulation $(n = 5000)$



Setup:

- **G**: space of equivalence classes (upto isomorphisms) of finite unlabelled graphs.
- For finite labelled graph $\mathcal{G}$: $\mathcal{G}^\circ$ for the isomorphism class of $\mathcal{G}$ in **G**.
- Root finding algorithm: Fix $K \geq 1$ and a mapping H_K on **G** that takes an input finite unlabelled graph $\mathbf{g} \in \mathbf{G}$ and outputs a subset of K vertices from **g**.

Root finding algorithms

Let $\{\mathcal{G}_n : n \geq 0\}$ be a sequence of growing random networks. Fix $0 < \varepsilon < 1$ and $K \geq 1$. A mapping H_K is called a budget K root finding algorithm with error tolerance ε for the sequence of networks if,

$$\liminf_{n\to\infty} \mathbb{P}(1 \in H_K(\mathcal{G}_n^\circ)) \geq 1 - \varepsilon.$$

Question: can we choose K independent of n? Dependence on ε?

Class of seed detection algorithms

- Centrality based measures
- For each vertex obtain some measure of centrality so collection of numbers $\{\phi(u) : u = \text{vertex in } \mathcal{T}_n\}$
- Example:
  - Degree centrality: $\phi(u) =$ degree of u
  - Eigen-vector centrality
  - Centroid or Jordan centrality

ALGORITHM

- Suppose budget $= K$
- Output the "top" K vertices(Could be smallest or largest depending on the measure)
- Say that above has error tolerance ε if

$$\lim_{n \to \infty} \mathbb{P}(\text{seed} \in \text{outputted set of } \mathcal{T}_n) \geq 1 - \varepsilon$$

Fundamental questions

- For given error tolerance ε (e.g. $\varepsilon = .01$)
  Can we select K independent of n = size of network?

- How does $K = K(\varepsilon)$ depend on ε?

  $\frac{1}{\varepsilon}$? $\frac{1}{\varepsilon^{100}}$? $\frac{1}{\varepsilon^{10000}}$?

Related notion: Robustness (Morters-Dietrich; Jog-Loh)

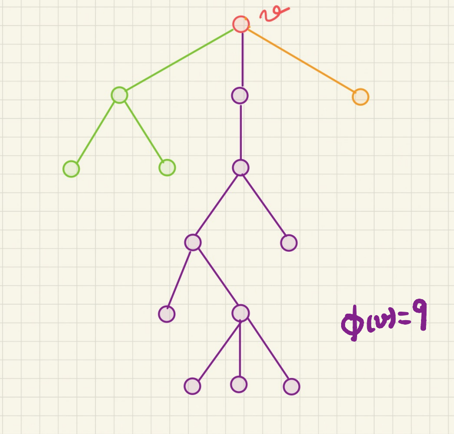
Persistence

Fix $K \geq 1$ and a network centrality measure Ψ. For a family of network models $\{\mathcal{G}_n : n \geq 1\}$ say that this sequence is (Ψ, K) **persistent** if $\exists\, n^* < \infty$ a.s. such that for all $n \geq n^*$ the optimal K vertices $(v_{1,\Psi}(\mathcal{G}_n^\circ), v_{2,\Psi}(\mathcal{G}_2^\circ), \ldots, v_{K,\Psi}(\mathcal{G}_n^\circ))$ remain the same and further the relative ordering amongst these K optimal vertices remains the same.

Example: If degree centrality was persistent this implies, the *identity* of the maximal degree vertex becomes fixed within finite time and no other vertex can overtake the degree of this vertex after this time.

Such phenomenon once again a hallmark of long range dependence.

Jordan or centroid Centrality*



$\phi(v)$ = size of the largest subtree of a child of v

* Only works for trees. First analyzed by Bubeck-Devroye-Lugosi.

Centroid centrality sufficiency bounds

technical

Banerjee and B(2020)

Under ~~above~~ assumptions:

1. Suppose for some $\overline{C}_f > 0$, $\beta \geq 0$, f satisfies $f_* \leq f(i) \leq \overline{C}_f \cdot i + \beta$ for all $i \geq 1$. Then $\exists$ positive constants C_1, C_2 such that for any error tolerance $0 < \varepsilon < 1$, the budget requirement satisfies,

$$K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon^{(2\overline{C}_f+\beta)/f_*}} \exp(\sqrt{C_2 \log 1/\varepsilon}).$$

2. If further the attachment function f is in fact bounded with $f(i) \leq f^*$ for all $i \geq 1$ then one has for any error tolerance $0 < \varepsilon < 1$,

$$K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon^{f^*/f_*}} \exp(\sqrt{C_2 \log 1/\varepsilon}).$$

Centroid centrality necessary bounds

- If $\exists\, \underline{C}_f > 0$ and $\beta \geq 0$ such that $f(i) \geq \underline{C}_f \cdot i + \beta$ for all $i \geq 1$ then $\exists$ a positive constant C_1' such that for any error tolerance $0 < \varepsilon < 1$,

$$K_\Psi(\varepsilon) \geq \frac{C_1'}{\varepsilon^{(2\underline{C}_f + \beta)/f(1)}}.$$

- For general f one has for any error tolerance $0 < \varepsilon < 1$,

$$K_\Psi(\varepsilon) \geq \frac{C_1'}{\varepsilon^{f_*/f(1)}}.$$

Special cases

- **Uniform attachment:** $f(k) = 1$

$$\frac{C_1'}{\varepsilon} \leq K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon}\exp(\sqrt{C_2 \log\frac{1}{\varepsilon}})$$

- **Pure Preferential attachment:** $f(k) = k$

$$\frac{C_1'}{\varepsilon^2} \leq K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon^2}\exp(\sqrt{C_2 \log\frac{1}{\varepsilon}}).$$

- **Affine preferential attachment:** $f(k) = k + \beta$

$$\frac{C_1'}{\varepsilon^{\frac{2+\beta}{1+\beta}}} \leq K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon^{\frac{2+\beta}{1+\beta}}}\exp(\sqrt{C_2 \log\frac{1}{\varepsilon}}).$$

- **Sublinear preferential attachment:**

$$\frac{C_1'}{\varepsilon} \leq K_\Psi(\varepsilon) \leq \frac{C_1}{\varepsilon^2}\exp(\sqrt{C_2 \log\frac{1}{\varepsilon}}).$$

Disadvantages of Centroid centrality

- Essentially need quite precise information of entire network
- *Natural question:* How do more local measures like degree centrality perform? Does there exist a *persistent hub* (i.e. maximal degree vertex fixates within finite time)?
- *Fake popularity:* Suppose i-th vertex enters the system with m_i edges that it attaches to the current existing system (again with popularity of vertices measured via some function f). How quickly does $m_i \uparrow \infty$ to break persistence phenomenon?

- $f_* := \inf_{i \geq 0} f(i) > 0$; further at most linear growth $f(i) \leq C_f(i)$.
- $\sum_{i=0}^{\infty} \frac{1}{f(i)} = \infty$.
- $\Phi_k(x) = \int_0^x \frac{1}{f^k(z)} dz$.
- $\mathcal{K}(t) = \Phi_2 \circ \Phi_1^{-1}(t), t \geq 0$.
- $d_{max}(n) := \max_{0 \leq k \leq n} d_k(n)$.
- *Index of the maximal degree:*

$$\mathcal{I}_n^* := \inf\{0 \leq i \leq n : d_i(n) \geq d_j(n) \text{ for all } j \leq n\}.$$

Banerjee + B(2020)

Under a few technical assumptions on f and *f is increasing*:

- Suppose $\Phi_2(\infty) < \infty$ (e.g. $f(k) = k^\alpha$ for $\alpha \in (1/2, 1]$) and that $\limsup_{n\to\infty} \frac{\Phi_1(m_n)}{\log s_n} \leq \frac{1}{8C_f}$. Then a persistent hub emerges almost surely in the random graph sequence

Do not need increasing assumption for trees.

$s_n = \sum_{i=1}^{n} m_i$

Lack of persistence

Banerjee + B(2020)

- Assume $\Phi_2(\infty) = \infty$ (e.g. $f(k) = k^\alpha$ for $\alpha \in (0, 1/2)$) and (we are working in the tree case) and $f(k) \to \infty$ as $k \to \infty$. Then index of maximal degree satisfies:

$$\frac{\log \mathcal{I}_n^*}{\mathcal{K}\left(\frac{1}{\lambda^*}\log n\right)} \xrightarrow{P} \frac{\lambda^{*2}}{2}, \quad \text{as } n \to \infty.$$

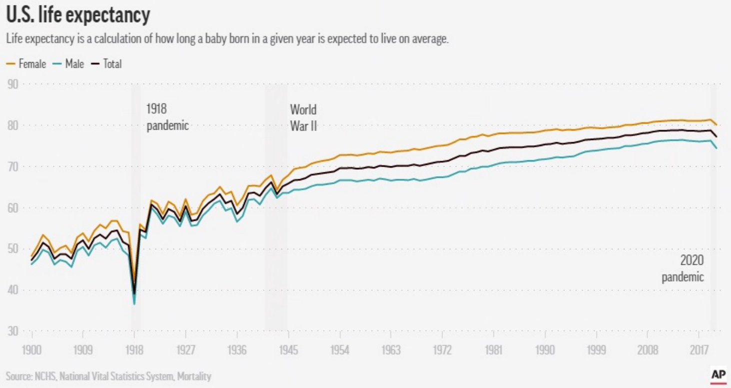
  where λ^* is the Malthusian rate of growth of the continuous time embedding.

- For $f(k) = k^\alpha$ for $\alpha \in (0, 1/2)$,

$$\frac{\log \mathcal{I}_n^*}{(\log n)^{\frac{1-2\alpha}{1-\alpha}}} \xrightarrow{P} \frac{(\lambda^*)^{\frac{1}{1-\alpha}}}{2}, \quad \text{as } n \to \infty.$$

  Inspired by Morters and Dietrich who proved similar results for a different evolving network model.

Change Point Detection



Source: Associated Press

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Recall: Probabilistic foundations

- Network model: Fix attachment function f. Start with single seed.
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Example $f(k) = k + \alpha$ Preferential attachment

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Known results for $f(k) = k + \alpha$

- $N_k(n)$ = # of vertices of degree k in $\mathcal{T}_n$

$$\frac{N_k(n)}{n} \longmapsto p_k$$

- $p_k \sim \dfrac{C}{k^{\alpha+3}}$ Degree exponent $= \alpha + 3$

- max-degree $= M_n \sim n^{\frac{1}{\alpha+2}}$

Example of Standard change point model

- Fix $\gamma \in (0,1)$.
- For $t \in [1, n\gamma]$, network uses attachment function

$$f(k) = k + \alpha$$

- For $t \in [n\gamma+1, n]$, network uses

$$g(k) = k + \beta$$

Any guesses on the degree exponent?

Guesses?

Recall under no change

$f(k) = k + \alpha$

degree exponent $= \alpha + 3$

$g(k) = k + \beta$

degree exponent $= \beta + 3$

Punchline of the Theorems

Irrespective of how small γ is (e.g $\gamma = .01$ or $\gamma = .00000001$), the initializer function Always wins !

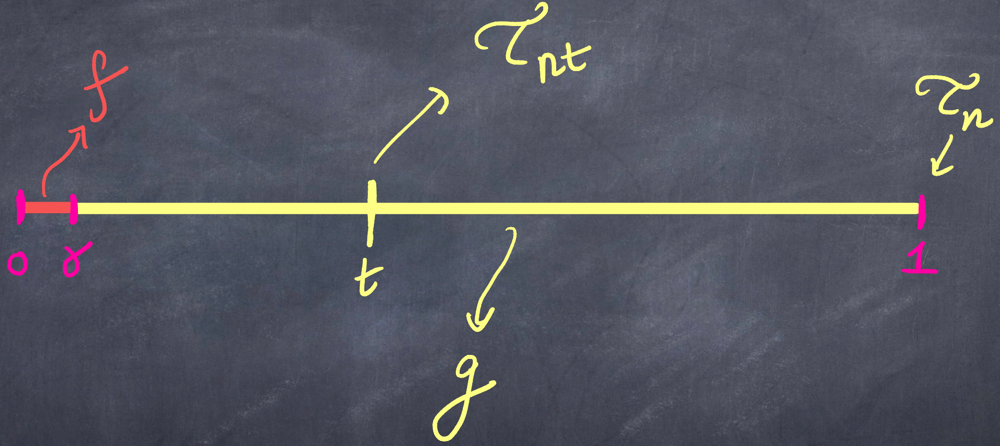
Standard change point model

- Fix $\gamma \in (0,1)$.
- For $t \in [1, n\gamma]$, network uses attachment function

$$f(k) = \text{general function}$$

- For $t \in [n\gamma+1, n]$, network uses

$$g(k) = \text{general function}$$



τ_{nt}
τ_n
f
0
γ
t
1
g

Fix $t \in [0,1]$. Let $N_k(nt)$ = # of vertices of degree k in $\mathcal{T}_{nt}$

Theorem [Banerjee, B, Carmichael]

Under conditions on f and g $\exists$ explicit probability mass functions $\{(p_k(t))_{k \geq 1} : t \in [0,1]\}$ such that

$$\sup_{t \in [0,1]} \left| \frac{N_k(nt)}{nt} - p_k(t) \right| \longrightarrow 0$$

__Theorem__ [Banerjee, B, Carmichael]

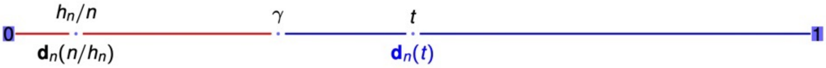
Under above technical conditions on f & g, irrespective of how small γ is f always wins!

- So if degree exponent with f and no change point is δ so is the model __with__ change point.

Change point estimator : For each $t \in (0,1)$ Compare degree distn $\left(\frac{N_k(nt)}{nt}\right)_{k \geq 1}$ with the degree distribution when network is of size $\frac{n}{\ln n}$ (recall change point at $n\gamma$)

and become alarmed the first time there seems to be a big change in degree distn.

Change point estimator



Nonparametric change point estimator

Fix any two sequences $h_n \to \infty, b_n \to \infty$: $\frac{\log h_n}{\log n} \to 0$, $\frac{\log b_n}{\log n} \to 0$. Define

$$\hat{\mathcal{T}}_n = \inf \left\{ t \geq \frac{1}{h_n} : \sum_{k=0}^{\infty} 2^{-k} \left| \frac{D_n(k, \mathcal{T}^{\boldsymbol{\theta}}_{\lfloor nt \rfloor})}{nt} - \frac{D_n(k, \mathcal{T}^{\boldsymbol{\theta}}_{\lfloor n/h_n \rfloor})}{n/h_n} \right| > \frac{1}{b_n} \right\}.$$

Then $\hat{\mathcal{T}}_n \xrightarrow{\mathrm{P}} \gamma$.

Lots of open problems

Simulations

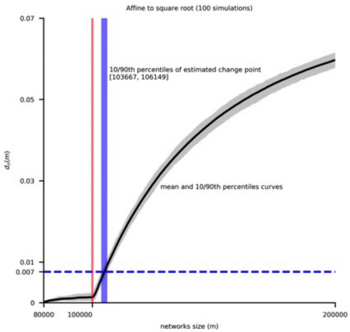


Figure: $n = 2 * 10^5$, $\gamma = 0.5$, $f_0(i) = i + 2$, $f_1(i) = \sqrt{i+2}$, $h_n = \log\log n$, $b_n = n^{1/\log\log n}$

$$d_n(m) := \sum_{k=0}^{\infty} 2^{-k} \left| \frac{D_n(k, \mathcal{T}_m^{\theta})}{m} - \frac{D_n(k, \mathcal{T}^{\theta}_{\lfloor n/h_n \rfloor})}{n/h_n} \right|, \qquad \frac{n}{\log\log n} < m \leq n.$$

The big bang model: What if the change happened very early in the system?

Figure: Big Bang: Getty images

Fix functions $f_0, f_1 : \{0, 1, 2, \ldots\} \to \mathbb{R}_+$ and $\gamma \in (0, 1)$. Let $\boldsymbol{\theta} = (f_0, f_1, \gamma)$.

Model

- **Time** $1 \leq m \leq n^\gamma$ Vertices perform attachment with probability proportional to $f_0(out - deg)$.
- **Time** $n^\gamma < m \leq n$ Vertices perform attachment with probability probability proportional to $f_1(out - deg)$.

Result 1

- Here change point at n^{γ} (e.g. $\sqrt{n}$).
- Here
$$\frac{N_n(k)}{n} \xrightarrow{\mathrm{P}} p_k^1$$
namely the degree distribution of the model run purely with attachment function f_1

So what changes?

1. **Uniform** $\rightsquigarrow$ **Linear:** $f_0 \equiv 1$ whilst $f_1(k) = k + 1 + \alpha$ for fixed $\alpha > 0$. Then for $\omega_n \uparrow \infty$,
$$\frac{n^{\frac{1-\gamma}{2+\alpha}} \log n}{\omega_n} \ll M_n(1) \ll n^{\frac{1-\gamma}{2+\alpha}} (\log n)^2.$$
2. **Linear** $\rightsquigarrow$ **Uniform:** $f_0(k) = k + 1 + \alpha$ whilst $f_1(\cdot) \equiv 1$.
$$\frac{n^{\frac{\gamma}{2+\alpha}} \log n}{\omega_n} \ll M_n(1) \ll n^{\frac{\gamma}{2+\alpha}} (\log n)^2.$$
3. **Linear** $\rightsquigarrow$ **Linear:** $f_0(k) = k + 1 + \alpha$ whilst $f_1(k) = k + 1 + \beta$ where $\alpha \neq \beta$. Then $M_n(1)/n^{\eta(\alpha,\beta)}$ is tight where
$$\eta(\alpha, \beta) := \frac{\gamma(2+\beta) + (1-\gamma)(2+\alpha)}{(2+\alpha)(2+\beta)}. \qquad (5)$$

Motivation

- Most real world networks support some particular purpose (e.g. diffusion of information on Twitter)
- Co- evolution: Network influences individuals and vice- versa

Motivation 2: More sophisticated models for PA

Motivations Despite PA being heavily used, number of limitations

[I] Assumes global knowledge of network. Each new vertex needs complete knowledge of network

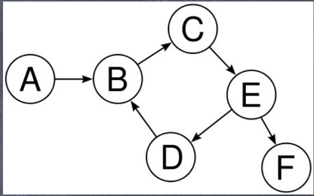
[II] In principle attractiveness should not depend ONLY on degree but potentially on "attenuated" neighborhood features.

Example: Page rank score attachment scheme.

Defn [Page rank Scores] Fix "damping factor" c.

For directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, page rank score $(\pi_v : v \in \mathcal{V})$ is the stationary distr^n of a random walk that at each step

- with prob c does usual random walk using outgoing edges
- with prob $1-c$ jumps to a randomly selected vertex uniformly at random



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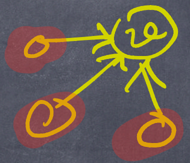
Thus (π_v) satisfies linear system of equations

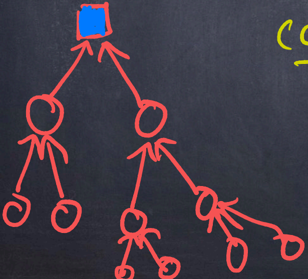
$$\pi_v = \frac{1-c}{n} + c \sum_{u \in N^-(v)} \pi_u / d^+(u)$$

out-degree of u



$N^-(v)$

Special case: Directed tree, directions to the root

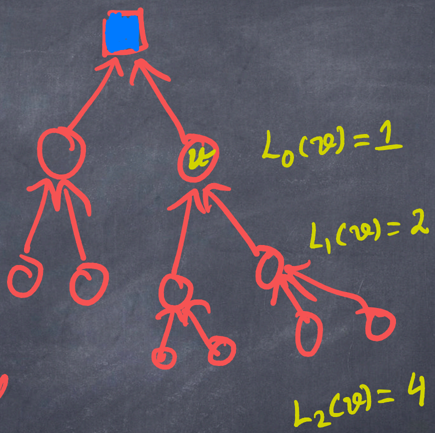


can check For $v \neq$ root

$$\pi_v = \frac{1-c}{n} \sum_{k=0}^{\infty} c^k L_k(v)$$

$L_k(v)$ = # of individuals at level k below v

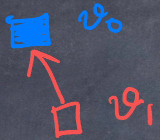
Thus if one does Preferential attachment using Page rank scores, then one does attachment using more global attractiveness function



III Motivation which might be contradictory to the previous motivations: Local exploration based attachment schemes

- Might want network evolution schemes where vertices decide to attach to a previous vertex after exploring the network "web-surfing" for some time.

Co-evolutionary network model ($\mathbf{P}$)

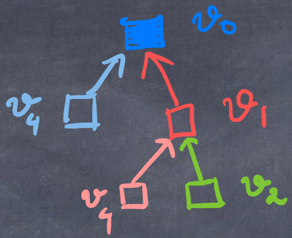
$T_1 =$ 

1. Having Constructed $\mathcal{T}_n$
2. At time $n+1$ a new vertex v_{n+1} enters system
3. Selects vertex V_n u.a.r in $\mathcal{T}_n$
4. Selects # of "exploration steps to root" variable

$$Z_{n+1} \sim \mathbf{P}$$

5. Goes up that many steps and attaches, stopping at root if need be.

$$\mathbf{P} = \text{pmf} = \{p_0, p_1, \ldots\} \qquad P(Z=i) = p_i \quad i \geq 0$$



Example

$C(2) = v_1, \quad Z_2 = 0$

$C(3) = v_2, \quad z_3 = 4$

$C(4) = v_2, \quad z_4 = 1$

$C(5) = v_0, \quad z_5 = 2$

$C(6) = v_2, \quad z_1 = 1$

$\vdots$

Special cases

① $p_0 \equiv 1$ $\rightarrow$ Random recursive tree
(Uniform Attachment)

② $p_0 = p$, $p_1 = 1-p$ $\rightarrow$ Preferential attachment
$$f(k) = k + \frac{(1-2p)}{p}$$

③ $p_0 = p$, $p_1 = p(1-p)$, $p_2 = p(1-p)^2$, ...

"Page rank model"

Theorem [Chebolu + Melsted 200x]

③ $\equiv$ Page rank attachment scheme with $1-c = p$

Theorems [Chebolut Melsted] Phase transition !

- If $P \leq \frac{1}{2}$

$$E(\text{degree of root}) = \tilde{\textcircled{H}}^{*}(n)$$

- If $P > \frac{1}{2}$

$$E(\text{degree of root}) = \tilde{\textcircled{H}}(n^{4pq})$$

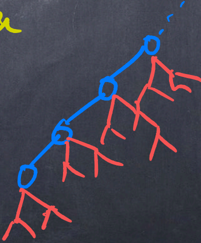
$* \; \tilde{\textcircled{H}} \Leftrightarrow O(n \log^{k}(n)) \qquad k \in \mathbb{Z}$

→ LOCAL WEAK CONVERGENCE

Theorem(s) [Banerjee, SB, Huang]

Let $Z \sim \mathbb{P}$

$\boxed{1}$ Assume $E(Z) < \infty$. Then the sequence of trees $\{\mathcal{T}_n\}_{n \geq 1}$ converge in the local weak convergene sense to a limiting infinite sin-tree



sin: tree with single infinite path to ∞

$\Rightarrow$ for example for every fixed $k \geq 0$

$$N_k(n) = \text{\# of vertices with } k \text{ children}$$

then $$\frac{N_k(n)}{n} \xmapsto{a.s} p_k \to \{p_k\}_{k \geq 0} = \text{PMF}$$

$\to$ If $E(Z) \leq 1$ $$\sum_{k=0}^{\infty} k p_k = 1$$

$\to$ If $E(Z) > 1$ $$\sum_{k=0}^{\infty} k p_k < 1$$

$\to$ Intuition for mass escaping to ∞. CONDENSAtion

$\boxed{2}$ Let $\{Z_i\}_{i\geq 1} \overset{IID}{=} \mathbb{P}$ $\quad f(s) = \text{pgf} = \sum_{k=0}^{\infty} p_k s^k$

Let $S_n = S_0 + \sum_{k=1}^{n} (Z_i - 1)$ = random walk started at S_0

Assumptions

① $p_0 \in (0,1)$, $p_0 + p_1 < 1$ [else if $p_0 + p_1 = 1 \Leftrightarrow$ preferential attachment regime]

② $f(s)$ is analytic at $s = 1$

Assumptions $\Rightarrow$ by work of [Daley 69] with a few more technical assumptions*

$\Rightarrow$ if we let $T_0 = \inf\{n \geq 1 : S_n = 0\}$ then

☐ If $E(Z) \neq 1$ then

$$P_1(n < T_0 < \infty) \approx e^{-n \log R}$$

i.e. $S_0 = 1$

for $R = R(P) > 1$

☐ If $E(Z) = 1$ then $R = 1$

* Aperiodicity + analyticity of pgf at $s = 1$.

Results [non-root degree]

Fix $k > 0$ and consider $D_{v_k}(n)$ = degree of v_k at time n

then $\forall \ \delta > 0$

$$\frac{D_{v_k}(n)}{n^{\frac{1}{k}-\delta}} \to \infty \qquad \frac{D_{v_k}(n)}{n^{1/k}(\log n)^{1+\delta}} \to 0$$

Results

If $E(Z) \le 1$

Let D = random variable with limiting degree distr^n

$$\lim_{k \to \infty} \frac{\log P(D \ge k)}{\log k} = -R$$

Intuitively $P(D \ge k) \simeq \dfrac{C}{k^{R}}$

↳ for $E(Z) > 1$ get upper + lower bounds for degree exponent

CONDENSATION

- Assume $E(Z) > 1$
- Few more technical conditions

Then

$$\frac{D_{v_0}(n)}{n} \xrightarrow{a.s.} \text{limit random variable} > 0$$

root

E.g. Random surfer model $p < \frac{1}{2}$ above is true!

No phase transition in Height

$$\varkappa_0 = \inf_{s \in (0,1)} \frac{f(s)}{s \log(1/s)}$$

$$\frac{H_n}{\log n} \xrightarrow{P} \varkappa_0$$

Connection between Random Walks + Trees

- what is the first step in studying such models?
- [Chebolu + Melsted idea for Page rank driven model]
- Fix a vertex $u \neq$ root
- Fix a vertex $t+1$ at time $\geq u$
- What is $P($ $t+1$ attaches to u / info till time $t)$?

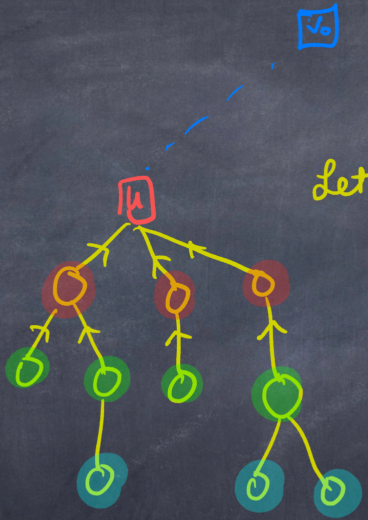
$L_0(t;u) = 1$

$L_1(t;u) = 3$

$L_2(t;u) = 4$

$L_3(t;u) = 3$

$\vdots$



Let $L_i(t;u)$ = # of vertices at distance i below vertex u at time t

$$P(t+1 \rightsquigarrow u \mid \text{information till time } t)$$

$$= \frac{L_0(t;u)}{t} p_0 + \frac{L_1(t;u)}{t} p_1 + \frac{L_2(t;u)}{t} p_2 + \cdots$$

<u>Check</u>: This leads to evolution equation for $\{L_k(t;u) : k \geq 0\}$ as

$L_k(t) = L_k(t;u)$

$$P\left(L_k(t+1) = L_k(t)+1 \mid \underset{\sim}{L}(t)\right)$$

$$= p_0 \frac{L_{k-1}(t)}{t} + p_1 \frac{L_k(t)}{t} + \cdots$$

$$= \frac{\left[A \cdot \underset{\sim}{L}(t)\right]_k}{t} \quad \text{where}$$

mass escaping above u.

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ p_0 & p_1 & p_2 & \cdots \\ 0 & p_0 & p_1 & \cdots \\ 0 & 0 & p_0 & \cdots \\ 0 & \cdots & & \end{pmatrix}.$$

→ Easier to do things in continuous time. Continuous time version of what is happening below a vertex $u \neq$ root.

Let $\mathbb{T}$ denote the space of rooted, directed, labelled trees. Let $\mathcal{T}^*(\cdot)$ be the continuous time process of growing trees started with $\mathcal{T}^*(0) = \{v_0\}$,where v_0 is the root of the tree. The vertices in $\mathcal{T}^*(\cdot)$ are labelled $v_0, v_1, v_2, \ldots$ in order of appearance. $\mathcal{T}^*(\cdot)$ is generated by the following procedure:

"vertex u"

Each vertex reproduces at rate 1. When vertex v reproduces, a random variable Z following the law F is sampled independently.

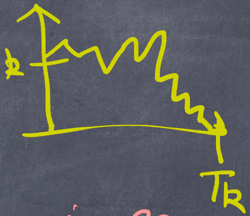
- If $Z \leq dist(v_0, v)$, then a new vertex $\tilde{v}$ is attached to the unique vertex u lying on the path between v and v_0 that satisfies $dist(v, u) = Z$ via a directed edge from $\tilde{v}$ to u.
- If $Z > dist(v_0, v)$, nothing occurs.

thus probability of new vertex being born to a current vertex = uniform distr^n

Two at first disjoint objects

Random walk: $S_n = S_0 + \sum_{i=1}^{n} (z_i - 1)$

$T_k = \inf\{n \geq 0 : S_n = 0 \mid S_0 = k\}$



Branching process: Let $L_k(t) = \#$ of vertices in generation k in the tree process described on previous page

Lemma:

$$E(L_k(t)) = \sum_{i=0}^{\infty} \frac{t^i}{i!} P(T_k = i)$$

Thank you for your attention!

ANY
QUESTIONS ?