Embedded Graphon Mean Field Games

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Program

- **Large Population Systems and MFG Equilibria**
- **Networks and Graphons**
- **Graphon MFG Systems and GMFG Equilibria**
- **Critical Nodes in LQG GMFG Systems +Examples**
- **Overview and Conclusion**

Large Population Systems and MFG Equilibria

Fundamentals of Mean Field Game Theory

Problem Formulation:

- **Notation: Integer valued subscript for finite population minor agents** $\{\mathcal{A}_i : 1 \leq i \leq N\}$
- $\mathbb{R}^n$ **valued states of** $\mathcal{A}_i$ **denoted** $x_i^N(t)$

Agent Dynamics:

$$dx_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f(t, x_i^N(t), u_i^N(t), x_j^N(t))dt + \frac{1}{N}\sum_{j=1}^{N} \sigma(t, x_i^N(t), x_j^N(t))dw_i(t), \quad x_i^N(0) = x_i^0 \quad 1 \leq i \leq N.$$

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t^N\}_{t\geq 0}, \mathbb{P})$: a complete filtered probability space
- $\mathcal{F}_t^N := \sigma\{x_j^0, w_j(s) : 1 \leq j \leq N, 0 \leq s \leq t\}$.
  $\{x_j^0\}_1^N$ $i.i.d.$ $L^2 \perp\!\!\!\perp$ $i.i.d$ Brownian motions $\{w_j\}_1^N$

Cost - or Performance - Functions for a Generic Agent:

$$J_i^N(u_i^N; u_{-i}^N) \quad := \quad E\int_0^T \Big(\frac{1}{N}\sum_{j=1}^{N} l[t, x_i^N(t), u_i^N(t), x_j^N(t)]\Big)dt$$

$$l[.,.,.,.] \;\geq\; 0$$

Large Population Systems and MFG Equilibria

Infinite Populations: Controlled McKean-Vlasov Equations:

- **McKean-Vlasov Equation describes the infinite population limit dynamics for uniform agents using a uniform control law (when a soln. exists):**

$$dx_t = f[x_t, u_t, \mu_t]dt + \sigma dw_t$$

$$f[x, u, \mu_t] \triangleq \int_{\mathbb{R}} f(x, u, y)\mu_t(dy) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} f(x, u, y_j)$$

$\mu_t(\cdot)$ = **measure of the popn. state distribution.**

McKean-Vlasov Systems are Markovian in the joint (x, μ) state.

- **Similar representation of infinite population limit cost:**

$$J(u, \mu) \triangleq \mathbb{E}\int_0^T l[x_t, u_t, \mu_t]dt$$

Information Patterns and Nash Equilibria

Information Patterns:

Information Pattern of MFG Systems: Decentralized and Individual to each Agent i:

$\mathcal{F}_i^N(t) \triangleq \sigma(x_i(\tau); \tau \le t), \;\; 1 \le i \le N$

$\mathcal{U}_{loc,i}$ **:=** $\mathcal{F}_i^N$ **adapted controls (+ system parameters)**

Information Pattern of MF Control Systems: Global/Centralized with respect to the Population :

$\mathcal{F}^N(t) \triangleq \sigma(x_j(\tau); \tau \le t, 1 \le j \le N)$

$\mathcal{U}$**:=** $\mathcal{F}^N$ **adapted controls (+ system parameters)**

Information Patterns and Nash Equilibria

Fundamental Notion of Non-cooperative Game Equilibrium:

The controls $\mathcal{U}^0 = \{u_i^0; u_i^0$ adapted to $\mathcal{U}_{loc,i}, 1 \leq i \leq N\}$ generate an ε*-Nash Equilibrium* w.r.t. $\{J_i; 1 \leq i \leq N\}$ if, for all i, a unilateral control law u_i utilizing the global information pattern $\mathcal{U}$ satisfies

$$J_i(u_i^0, u_{-i}^0) - \varepsilon \leq \inf_{u_i \in \mathcal{U}} J_i(u_i, u_{-i}^0) \leq J_i(u_i^0, u_{-i}^0)$$

So, by definition, a unilateral move against a population of agents all of whom are utilizing a Nash strategy cannot yield a benefit of more than $\varepsilon > 0$ for the unilateral agent.

Mean Field Game MV HJB-FPK Theory

- **Mean Field Game Equations**

  Formally, if an infinite agent population system with uniform agent dynamics and uniform performance functions has a Nash Equilibrium with generic agent Nash value V, generic agent state measure (i.e. mean field) μ and best response φ, it would satsify the MV-HJB MV-SDE (or FPK) equations:

[MF-HJB]
$$-\frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x,u,\mu_t] \frac{\partial V}{\partial x} + l[x,u,\mu_t] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2}$$
$$V(T,x) = 0, \quad p(0,x) = p_0, \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

[MF-FPK (if μ a.c.Lb.)]
$$\frac{\partial p(t,x)}{\partial t} = -\frac{\partial \{ f[x,u,\mu] p(t,x) \}}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p(t,x)}{\partial x^2}$$

[MF-MKV SDE]
$$dx_t = f[x_t, \varphi(t,x|\mu_\cdot), \mu_t] dt + \sigma dw_t$$

[MF-BR]
$$u_t = \arg \inf_{u \in U} H(x,u,\mu_t) =: \varphi(t,x|\mu_t), \quad (t,x) \in [0,T] \times \mathbb{R}^{\ltimes}$$

Mean Field Game MV HJB-FPK Theory

Theorem

Subject to technical conditions (U compact, Lipschitz and boundedness conditions on all functions on $\mathbb{R} \times U \times \mathbb{R}$, and existence of a unique continuous minimizer of the Hamiltonian):

(i) (HMC 2006, LL 2006) The MKV MFG Equations have a unique solution with the Nash equilibrium generated by the best response control:

$$u_i^0 = \varphi(t, x|\mu_t), \quad 0 \le t \le T, \ \ 1 \le i \le N.$$

(ii) (HMC 2006) Furthermore, $\forall \epsilon > 0 \ \ \exists N(\epsilon)$ s.t. $\forall N \ge N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \le \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \le J_i^N(u_i^0, u_{-i}^0),$$

$u_i(t) \in \mathcal{U}$ adapted to $\mathcal{F}^N(t) := \{\sigma(x_j(\tau); 0 \le \tau \le t, 1 \le j \le N)\}$.

Significance (ii): Finite population use of infinite pop. MFG BRs.

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The Classical MFG Model:

Key variables are simply averaged when, as a mass, they play a role in the behaviour of a large population system.

Equivalent to a Uniformity Assumption:

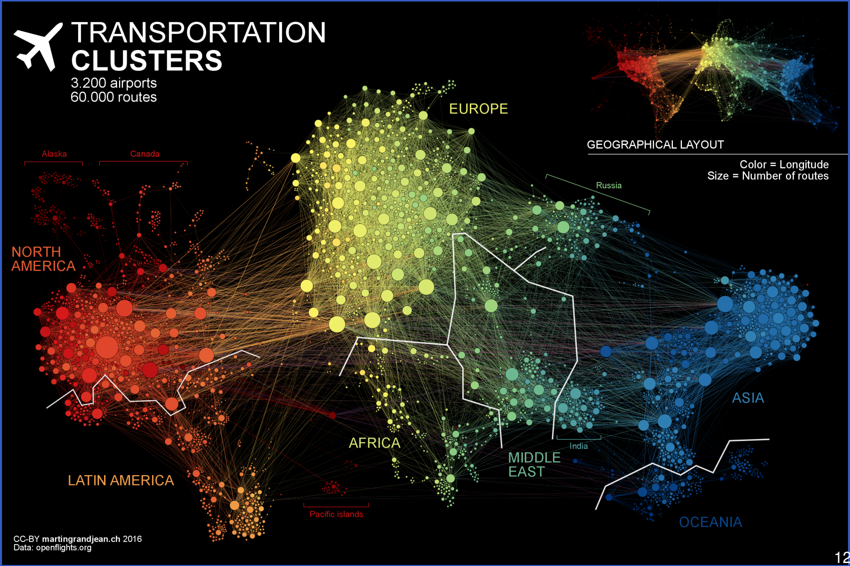
Equivalent to individual agents being distributed over the nodes of a large scale network which is completely connected and where all edges have equal weight.

Uniformity Assumption Often Does Not Hold.

The network examples depicted below do not satisfy this assumption globally, but locally some do (approximately).

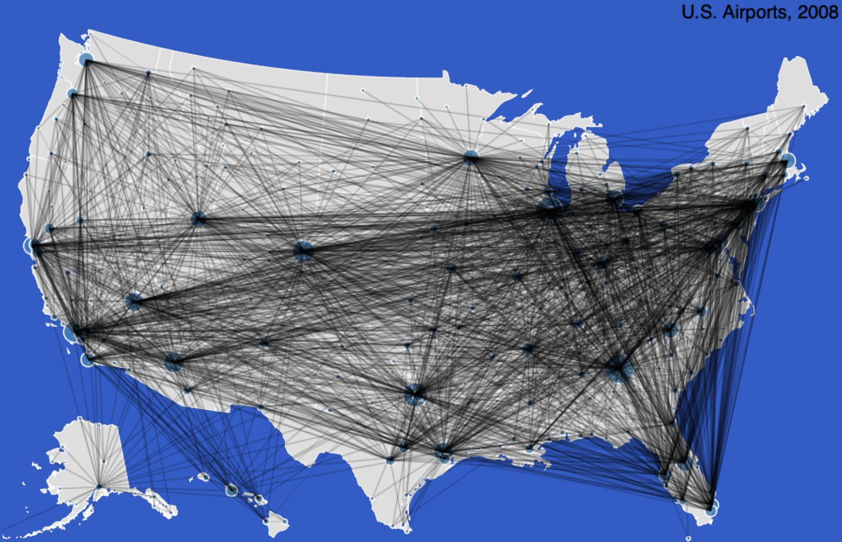
Graphon Mean Field Games: Motivation

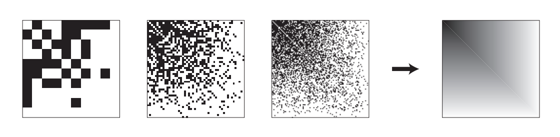
Global non-uniform Connections - Dense Network of Clusters



Graphon Mean Field Games: Motivation

National Non-uniform Connections - Dense Network of Clusters
https://vega.github.io/vega/tutorials/airports





Figures: Convergence of a Uniform Attachment Graph Sequence to a Limit Graphon. (Each Cycle: $N-1$ node graph; new node: attached with prob. $1/N$ to each old $N-1$ node, and old unattached pairs attached with prob. $1/N$.)

Definition: Graphon (Lovasz, AMS 2012) : A bounded symmetric Lebesgue measurable function $\mathbf{W} : [0,1]^2 \to [0,1]$. May be interpreted as weighted undirected edge graphs on vertex set $[0,1]$.

Principal Graphon Spaces

$$\mathcal{W} := \{\mathbf{W} : [0,1]^2 \to [0,1]\}$$

$$\mathcal{W}_I := \{\mathbf{W} : [0,1]^2 \to I\}$$

Cut norm

$$\|\mathbf{W}\|_\square := \sup_{M,T\subset[0,1]} \left| \int_{M\times T} \mathbf{W}(x,y)dxdy \right| \quad (1)$$

Cut distance

$$d_\square(\mathbf{W},\mathbf{V}) := \|\mathbf{W}^\phi - \mathbf{V}\|_\square \quad (2)$$

Cut metric obtained by infimizing over all measure preserving bijections on $[0,1]$:

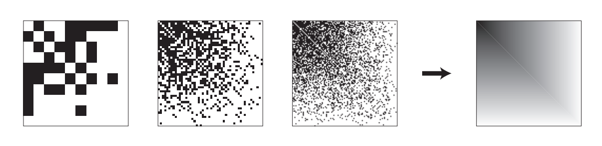
$$\delta_\square(\mathbf{W},\mathbf{V}) := \inf_\phi \|\mathbf{W}^\phi - \mathbf{V}\|_\square \quad (3)$$

δ_{L^2} metric

$$\delta_{L^2}(\mathbf{W},\mathbf{V}) := \inf_\phi \|\mathbf{W}^\phi - \mathbf{V}\|_2 \quad (4)$$

where $\mathbf{W}^\phi(x,y) = \mathbf{W}(\phi(x),\phi(y))$.

$\|\mathbf{W}\|_\square \leq \|\mathbf{W}\|_{L^2}$, so convgc. in δ_{L^2} implies convgc. in $\delta_\square$.



Convergence to a Limit Graphon. Uniform Attachment Graph Sequence (Each Cycle: $N-1$ node graph; new node: attached with prob. $1/N$ to each old $N-1$ node, and for all old unattached pairs, attach them with prob. $1/N$.)

Theorem [Lovasz and Szegedy, 2006; LL AMS2012]
Under the cut metric the graphon spaces $(\mathcal{W}_I, \delta_\square)$, are compact, where I any closed interval in $\mathbb{R}$.

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Finite Network Finite Population Mean Field Games

Consider a finite population distributed over a finite graph G_k.

Let $\mathbf{x}_{G_k} = \bigoplus_{l=1}^{M_k} \{x_i | i \in \mathcal{C}_l\}$ denote the states of all agents in the total set of clusters of the population.

This gives a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ individual states.

For $\mathcal{A}_i$ in the cluster $\mathcal{C}(i)$, the two coupling terms in the dynamics take the form

$$f_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} f_0(x_i, u_i, x_j), \tag{5}$$

$$f_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in C_l} f(x_i, u_i, x_j). \tag{6}$$

They model intra- and inter-cluster couplings, respectively. The defn. of f_{G_k} uses the sectional (i.e. vertex) information $g^k_{\mathcal{C}(i)\bullet}$.

Finite Network Finite Population Games

The state process of $\mathcal{A}_i$ is then given by the SDE

$$
\begin{aligned}
dx_i(t) &= \frac{1}{|C(i)|}\sum_{j\in\mathcal{C}(i)} f_0(x_i,u_i,x_j)dt \\
&\quad + \frac{1}{M_k}\sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l}\frac{1}{|\mathcal{C}_l|}\sum_{j\in C_l} f(x_i,u_i,x_j)dt + \sigma dw_i \qquad (7)
\end{aligned}
$$

$$
= \quad f_0(x_i,u_i,\mathcal{C}(i))dt + f_{G_k}(x_i,u_i,g^k_{\mathcal{C}(i)})dt + \sigma dw_i \qquad (8)
$$

$$1 \le i \le N$$

Finite Network Finite Population Costs

Analogously, in the GMFG case, we define the running cost coupling terms for agent $\mathcal{A}_i$ to be

$$l_0(x_i, u_i, \mathcal{C}(i)) = \frac{1}{|\mathcal{C}(i)|} \sum_{j \in \mathcal{C}(i)} l_0(x_i, u_i, x_j),$$

$$l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = \frac{1}{M_k} \sum_{l=1}^{M_k} g^k_{\mathcal{C}(i)\mathcal{C}_l} \frac{1}{|\mathcal{C}_l|} \sum_{j \in \mathcal{C}_l} l(x_i, u_i, x_j).$$

Define the complete running cost as
$\tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) = l_0(x_i, u_i, \mathcal{C}(i)) + l_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)})$.

The performance function for agent $\mathcal{A}_i$ in a finite population on a finite graph G_k is then

$$J_i = E \int_0^T \tilde{l}_{G_k}(x_i, u_i, g^k_{\mathcal{C}(i)}) dt. \tag{9}$$

Infinite Network Infinite Population Games

Assume :
(i) The number of nodes of the graph G_k tends to infinity with assumed unique graphon limit $g(\alpha, \beta)$.
(ii) The subpopulation at each node tends to infinity. giving the local mean field μ_α, the global set of mean fields $\mu_G = \{\mu_\beta; 0 \leq \beta \leq 1\}$, and the graphon $g(\alpha, \beta)$:

$$f_0[x_\alpha, u_\alpha, \mu_\alpha] := \int_{\mathbb{R}^n} f_0(x_\alpha, u_\alpha, z)\mu_\alpha(dz), \tag{10}$$

$$f[x_\alpha, u_\alpha, \mu_G; g_\alpha] \ := \ \int_0^1 \int_{\mathbb{R}^n} f(x_\alpha, u_\alpha, z)g(\alpha, \beta)\mu_\beta(dz)d\beta, \tag{11}$$

This yields the complete local limit graphon drift dynamics:

$$\widetilde{f}[x_\alpha, u_\alpha, \mu_G; g_\alpha] \ := \ f_0[x_\alpha, u_\alpha, \mu_\alpha] + f[x_\alpha, u_\alpha, \mu_G; g_\alpha]. \tag{12}$$

Infinite Network Infinite Population Games

Parallel to the standard MFG case, in the infinite population graphon case the generic agent state SDE is:

$$\textbf{[MV-SDE]}(\alpha) \quad dx_\alpha(t) = \widetilde{f}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha]dt + \sigma dw_t^\alpha, \\ 0 \le t \le T, \quad \alpha \in [0,1], \tag{13}$$

with $\widetilde{l}[.,.]$ defined similarly to $\widetilde{f}[.,.]$, the generic agent α has the cost, or performance, function

$$J_\alpha(u_\alpha; \mu_G(\cdot)) = E\int_0^T \widetilde{l}[x_\alpha(t), u_\alpha(t), \mu_G(t); g_\alpha]dt. \tag{14}$$

Graphon Mean Field Game (GMFG) Equations

$$
\begin{aligned}
\text{[HJB]}(\alpha) \quad -\frac{\partial V^\alpha(t,x)}{\partial t} &= \inf_{u\in U}\Big\{\widetilde{f}[x,u,\mu_G;g_\alpha]\frac{\partial V^\alpha(t,x)}{\partial x} \\
&\quad + \widetilde{l}[x,u,\mu_G;g_\alpha]\Big\} + \frac{\sigma^2}{2}\frac{\partial^2 V^\alpha(t,x)}{\partial x^2}, \\
V^\alpha(T,x) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^n, &\quad \alpha\in[0,1]^m,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\text{[FPK]}(\alpha) \quad \frac{\partial p_\alpha(t,x)}{\partial t} &= -\frac{\partial\{\widetilde{f}[x,u^0,\mu_G;g_\alpha]p_\alpha(t,x)\}}{\partial x} \\
&\quad + \frac{\sigma^2}{2}\frac{\partial^2 p_\alpha(t,x)}{\partial x^2}, \quad p_\alpha(0)=p_0 \\
\text{[BR]}(\alpha) \quad u^0 &= \varphi(t,x|\mu_G;g_\alpha).
\end{aligned} \tag{16}
$$

Graphon Mean Field Game (GMFG) Equations

Theorem: GMFG Existence and Uniqueness (GMFG E+U)
[PEC-Minyi Huang CDC2018,CDC 2019, SICON 2021]

For U compact, subject to boundedness and Lipschitz conditions on all functions on $\mathbb{R} \times U \times \mathbb{R}$, together with the existence of a unique continuous minimizer of the Hamiltonian, there exists a unique Nash equilibrium solution $(V^\alpha, \mu_\alpha(\cdot))_{\alpha\in[0,1]}$ to the GMFG equations (15) and (16).

The feedback control best response *(BR)* strategy $\varphi(t, x_\alpha|\mu_G(\cdot); g_\alpha)$ for each agent depends only upon the agent's state and the graphon mean fields: (x_α, μ_G).

Graphon Mean Field Games

Graph Convergence Assumption (GCA)

The sequence $\{G_k; 1 \leq k < \infty\}$ and the graphon limit satisfy

$$\lim_{k\to\infty} \max_i \sum_{j=1}^{M_k} \left| \frac{1}{M_k} g^k_{\mathcal{C}_i,\mathcal{C}_j} - \int_{\beta\in I_j} g_{I_i^*,\beta} d\beta \right| = 0,$$

where I_i^* is the midpoint of the subinterval $I_i \in \{I_1 \cdots I_{M_k}\}$ of length $1/M_k$.

For the ϵ-Nash equilibrium analysis, we consider a sequence of games each defined on a finite graph G_k. Recall that there is a total of $N = \sum_{l=1}^{M_k} |\mathcal{C}_l|$ agents.

Suppose the cluster $\mathcal{C}(i)$ of agent $\mathcal{A}_i$ corresponds to the subinterval $I(i) \in \{I_1, \cdots, I_{M_k}\}$. Then the agent $\mathcal{A}_i$ uses the Midpoint BR Control, namely it takes the midpoint $I^*(i)$ of the subinterval $I(i)$ and uses the GMFG equations solution to determine its control law.

Graphon Mean Field Games

Theorem: GMFG epsilon Nash Property ($\epsilon-$ NP)
[PEC-M.Huang CDC2018,CDC 2019, SICON 2021]

In addition to the conditions of the GMFG E+U Theorem, assume (GCA) holds.
Then when the Midpoint BR Controls are applied to a sequence of finite graph systems $\{G_k; 1 \leq k < \infty\}$ with limit G the ϵ-Nash equilibrium property holds:

$$\forall \epsilon > 0 \;\; \exists N(\epsilon) \quad s.t. \; \forall N \geq N(\epsilon)$$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}} J_i^N(u_i, u_{-i}^0) \leq J_i^N(u_i^0, u_{-i}^0)$$

where $\epsilon \to 0$ as $k \to \infty$, and where the unilateral agent $\mathcal{A}_i$ uses a centralized Lipschitz feedback strategy $u_i \in \mathcal{U}$ adapted to $\mathcal{F}^N := \{\sigma(x_j(\tau); \tau \leq t, 1 \leq j \leq N)\}$.

Significance: Finite network & population use of GMFG BRs.

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Critical Nodes in Graphon Mean Field Games

Definition Critical Nodes for GMFG Systems

Assume a GMFG system has Nash values which are differentiable with respect to node values in $[0,1]$, then α is a critical node for the GMFG system if the local Nash value stationarity condition holds at α:

$$\frac{\partial}{\partial \lambda} V_t^{\lambda,g}\bigg|_{\lambda=\alpha} = 0, \qquad \forall t \in [0,T]. \tag{17}$$

LQG GMFG Critical Nodes [Foguen-Tcheundom, CDC 2021, (extended in Tcheundom-Gao, CDC 2022)]

Assume a sequence of graphs G_k converges to a unique graphon limit in the cut metric where the metric is defined without infimization over measurable bijections i.e. with fixed indexing:

$$g : [0,1] \times [0,1] \longrightarrow [0,1], \quad (\alpha, \beta) \mapsto g(\alpha, \beta).$$

and a representative (aka generic) agent at a graphon node $\alpha \in [0,1]$ has the linear controlled dynamics:

$$\begin{aligned} dx_t^\alpha &= \left(a x_t^\alpha + b u_t^\alpha\right) dt + \sigma dw_t^\alpha, \\ x_0^\alpha &= \xi^\alpha \sim \mathcal{N}(m, v^2) \quad \forall t \in [0,T], \quad \forall \alpha \in [0,1], \end{aligned} \tag{18}$$

where $(\xi^\alpha)_{\alpha \in [0,1]}$ are pairwise independent.

For the specific LQG-GMFG problem under consideration, take the generic agent's performance function to be

$$J_\alpha(u^\alpha, \mu) := \mathbb{E} \int_0^T \left[\frac{r}{2} |u_t^\alpha|^2 + \frac{q}{2} \big(x_t^\alpha - z_t^{\alpha,g} \big)^2 \right] dt, \tag{19}$$

where at $\alpha \in [0, 1]$, the α component of the global (mean) mean field term, denoted $z_t^{\alpha,g}, t \in [0, T]$, is defined as

$$z_t^{\alpha,g} := \int_0^1 g(\alpha, \beta) \int_{\mathbb{R}} y d\mu(\beta, t)(y) d\beta, \ t \in [0, T], \tag{20}$$

where for all $\alpha \in [0, 1], t \in [0, T]$, $\mu(\alpha, t)$ is assumed to lie in the set of probability measures with finite second moment, $\mathcal{P}_2\big(\mathbb{R}\big)$.

Critical Nodes in LQG Graphon Mean Field Games

1. (GMFG Control Problem) Find a family of $\{\mathcal{F}^t; 0 \leq s \leq t; 0 \leq t \leq T\}$ adapted square integrable optimal controls, denoted $u^{\alpha,o} := (u_t^{\alpha,o})_{t\in[0,T]} \in \mathbb{A}$, such that

$$J(u^{\alpha,o}, \mu) = \min_{u^\alpha \in \mathbb{A}} J(u^\alpha, \mu) \tag{21}$$
$$= \min_{u^\alpha \in \mathbb{A}} \mathbb{E}\Big[\int_0^T \Big(\frac{r}{2}(u_t^\alpha)^2 + \frac{q}{2}(x_t^\alpha - z_t^{\alpha,g})^2\Big)dt\Big]$$

$$dx_t^\alpha = \big(ax_t^\alpha + bu_t^\alpha\big)dt + \sigma dw_t^\alpha, \quad x_0^\alpha = \xi^\alpha, \tag{22}$$

$$z_t^{\alpha,g} = \int_0^1 g(\alpha,\beta)[\int_{\mathbb{R}} v\mu(\beta,t)(dv)]d\beta. \tag{23}$$

2. (MFG Consistency Conditions) And such that the solution family of optimal state trajectories $(x_t^{\alpha,\mu,o})_{t\in[0,T]}, \forall\alpha \in [0,1]$, solving (21) satisfies the MFG-consistency conditions:

$$\mu(\alpha,t) = \mathcal{L}\big(x_t^{\alpha,\mu,o}\big), \quad \forall(\alpha,t) \in [0,1]\times[0,T]. \tag{24}$$

Assume the LQG-GMFG above admits a unique solution.

Optimal tracking (BR) control for any agent in cluster C_α:

$$u_\alpha(t) = -r^{-1}b[\Pi_t x_\alpha(t) + s_\alpha(t)]$$
$$-\dot{\Pi}_t = a\Pi_t + \Pi_t a - \Pi_t b r^{-1} b \Pi_t + q, \quad \Pi_T = q_T$$
$$-\dot{s}_\alpha(t) = \left(a - br^{-1}b\Pi_t\right)s_\alpha(t) - qz_t^{\alpha,g}, \quad s_\alpha(T) = q_T \nu_\alpha(T)$$

Graphon local mean field (mean) and tracked process (cost coupling)

$$z_t^{\alpha,g} = \int_0^1 g(\alpha,\beta)[\int_{\mathbb{R}} v\mu(\beta,t)(dv)]d\beta, \quad \alpha \in [0,1],$$
$$\bar{x}_\beta \triangleq \lim_{|C_\beta|\to\infty} \frac{1}{|C_\beta|}\sum_{j\in C_\beta} x_j = \int_{R^n} x_\beta \mu_\beta(dx_\beta)$$

The GMFG scheme closes with the local mean state process of x_α

$$\dot{\bar{x}}_\alpha = \left(a - br^{-1}b\Pi_t\right)\bar{x}_\alpha - br^{-1}bs_\alpha, \quad \alpha \in [0,1].$$

Critical Nodes in LQG Graphon Mean Field Games

Definition [Critical Mean Field Nodes for LQG-GMFG]

$\lambda \in [0, 1]$ is a critical mean field node for an LQG-GMFG system if the local mean field stationarity condition holds:

$$\frac{\partial}{\partial \alpha} z_t^{\alpha,g} \Big|_{\alpha=\lambda} = 0, \quad \forall t \in [0, T]. \tag{25}$$

Critical Nodes in Graphon Mean Field Games

Two examples of graphons for which one can readily identify critical mean field nodes for the specified LQG-GMFG problem.

1. Case 1: Consider the first graphon to be the limit of a sequence of finite Erdös-Rényi graphs. :

$$\text{For some } p \in (0,1),\ g(\alpha,\beta) := p\ \forall(\alpha,\beta) \in [0,1]^2.$$

Then the solution of the LQG-GMFG equations gives:

$$z_t^{\alpha,g} = p\int_0^1 \mathbb{E}\left[x_t^{\beta,o}\right] d\beta,\ \forall(\alpha,t) \in [0,1]\times[0,T].$$

From which it follows that, for all $\alpha \in [0,1]$:

$$\frac{\partial}{\partial\lambda} z_t^{\lambda,g}\bigg|_{\lambda=\alpha} = 0, \quad \forall t \in [0,T].$$

Hence for Erdös-Renyi graphons the associated LQG-GMFG problem is such that all nodes $\lambda \in [0,1]$ are critical mean field nodes.

Critical Nodes in LQG Graphon Mean Field Games

1. Case 2: The uniform attachment graphon (UAG) :

$$g(\alpha, \beta) = 1 - \max\{\alpha, \beta\}, \ \forall(\alpha, \beta) \in [0, 1]^2.$$

$$z_t^{\alpha,g} = \int_0^1 (1 - \max\{\alpha, \beta\})\mathbb{E}\left[x_t^{\beta,o}\right] d\beta \quad \forall(\alpha, t) \in [0, 1] \times [0, T]$$

$$= (1 - \alpha) \int_0^\alpha \mathbb{E}\left[x_t^{\beta,o}\right] d\beta + \int_\alpha^1 (1 - \beta)\mathbb{E}\left[x_t^{\beta,o}\right] d\beta. \tag{26}$$

Differentiation with respect to α yields:

$$\frac{\partial}{\partial\alpha} z_t^{\alpha,g} = -\int_0^\alpha \mathbb{E}\left[x_t^{\beta,o}\right] d\beta, \ \forall t \in [0, 1]. \tag{27}$$

Hence at $\alpha = 0 \in [0, 1]$:

$$\left.\frac{\partial}{\partial\alpha} z_t^{\alpha,g}\right|_{\alpha=0} = 0, \quad \forall t \in [0, T].$$

Consequently for the UAG the root node is a critical mean field node.

Critical Nodes in LQG Graphon Mean Field Games

Denote the LQG-GMFG equilibrium controls by $\{u_t^{\alpha,o},\ \forall(\alpha,t) \in [0,1] \times [0,T]\}$

Proposition

Assume that the LQG-GMFG problem (18),(19) *admits critical mean field nodes denoted* $\lambda \in [0,1]$. *Then the LQG-GMFG equilibrium controls are stationary at* $\lambda \in [0,1]$, *that is,* $\forall t \in [0,T]$

$$\frac{\partial u_t^{\alpha,o}}{\partial \alpha}|_{\alpha=\lambda} = -\frac{b}{r}\left[\Pi_t \frac{\partial x_t^{\alpha,o}}{\partial \alpha} + \frac{\partial s_t^{\alpha}}{\partial \alpha}\right]|_{\alpha=\lambda} = 0, \qquad (28)$$

and, further, they are critical nodes for the LQG-GMFG system (18),(19) *since the value function is stationary there:*

$$\frac{\partial V(\alpha,t,x)}{\partial \alpha}|_{\alpha=\lambda} = 0,\ \forall(t,x) \in [0,T] \times \mathbb{R}. \qquad (29)$$

Overview and Conclusion

- **Large Population Systems and MFG Equilibria**
- **Networks and Graphons**
- **Graphon MFG Systems and GMFG Equilibria**
- **Critical Nodes in LQG GMFG Systems + Examples**
- **Conclusion**

  The extension of the analysis above to parameterizations and thence differentiation in $R^m, m > 1,$ is enabled by the theory of vertexons and embedded graphons.

  The initial development of that work together with examples is to be presented at the IEEE Control Systems Society Conference on Decision and Control, Cancun, Mexico, 2022.