Graphon limit and large independent sets in uniform random cographs

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joint work with F. Bassino, M. Bouvel,
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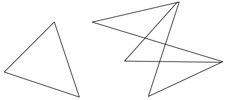
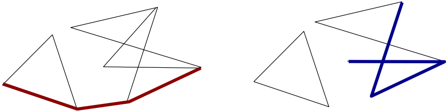
Definition

A cograph is a P_4-free graph, i.e. not containing P_4 as an induced subgraph.



A (disconnected) cograph



These are not cographs

Questions: asymptotic behaviour of a uniform random cograph

- What is its graphon limit?
- Existence of independent sets/cliques of linear size?

Motivations:

- When studying H-free graphs, the case $H = P_4$ is special with an interesting limit object;
- Probabilistic work around Erdős-Hajnal conjecture.

First part

Asymptotic enumeration

of H-free graphs

Background: enumerating H-free graphs

Fix a graph H, and consider its chromatic number $k = \chi(H)$.
Observation: if $\chi(G) \leq k-1$, then G is H free.

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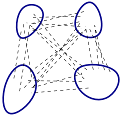
Easy: There are $\geq 2^{\left(1-\frac{1}{k-1}+o(1)\right)\binom{n}{2}}$ graphs G with n vertices and $\chi(G) \leq k-1$



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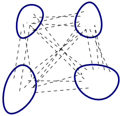

Conclusion:

$$\left| \left\{ \begin{array}{c} H\text{-free graphs} \\ \text{with } n \text{ vertices} \end{array} \right\} \right| \geq 2^{\left(1-\frac{1}{k-1}+o(1)\right)\binom{n}{2}}.$$

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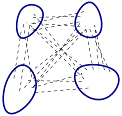

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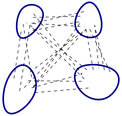

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→ works also if $\chi(\overline{H}) = k$ (putting cliques in the blue clusters), or if V_H cannot be partitioned in s cliques and t independent sets for some given $s + t = k-1$.

Background: enumerating H-free graphs

Theorem (Prömel–Steger, '92)

$$\left|\left\{ \begin{array}{c} H\text{-free graphs} \\ \text{with } n \text{ vertices} \end{array} \right\}\right| = 2^{\left(1-\frac{1}{r}+o(1)\right)\binom{n}{2}},$$

where r is maximal such that V_H cannot be partitioned into s cliques and t independent sets, for some (s,t) with $s+t=r$.

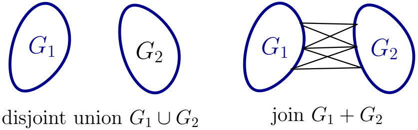
$r+1$: coloring number of H.

$H = P_4$ is one of the few cases with $r=1$. Hence

$$c_n := \left|\left\{ \begin{array}{c} \text{cographs} \\ \text{with } n \text{ vertices} \end{array} \right\}\right| = 2^{o(n^2)}.$$

Background: enumerating cographs

Operations on graphs:



Proposition (Corneil–Lerchs–Stewart Burlingham '81)

The class of cographs is the smallest set of graphs containing the one-vertex graph, and stable by disjoint unions and joins (cographs are sometimes called complement-reducible graphs).

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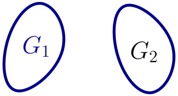
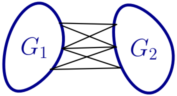


disjoint union $G_1 \cup G_2$



join $G_1 + G_2$

Proposition (Corneil–Lerchs–Stewart Burlingham '81)

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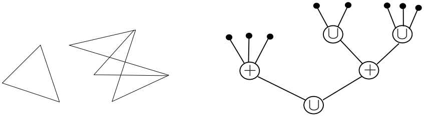
Consequence: cographs can be encoded by decorated trees



A cograph and the associated decorated tree

Easy to enumerate: $c_n \sim C\, n!\, \kappa^n\, n^{-3/2}$, with $\kappa = (2\log(2) - 1)^{-1}$ and some explicit constant C (labeled case).

Second part

Graphon limit

of uniform random cographs

A general result for H-free graphs

Fix H and let $\boldsymbol{G_n}$ be a uniform H-free graph on n vertices.

Question: what is the graphon limit of $\boldsymbol{G_n}$ (if it exists)?

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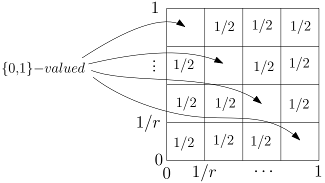
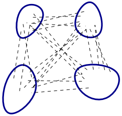
Let H be a graph and take r as before. Then any subsequential limit of $\boldsymbol{G_n}$ is supported on the set of graphons of the following form:



Reminiscent of the picture 

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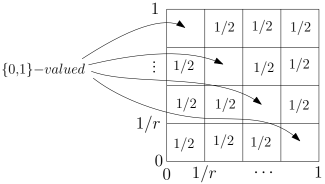

It does not say much for $H = P_4$ (where $r = 1$). In fact, all P_4-free graphons are $\{0,1\}$-valued.

Limit of uniform random cograph

Theorem (Bouvel–Bassino–F.–Gerin–Maazoun–Pierrot '22, Stufler '22)

Let $\boldsymbol{G}_n$ be a uniform random (either labeled or unlabeled) cograph with n vertices. Then $W_{\boldsymbol{G}_n}$ converges in distribution to a random graphon $\boldsymbol{W}^{Br}$, which we call Brownian cographon.

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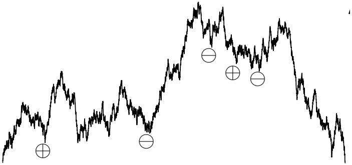
Construction of $\boldsymbol{W}^{Br}$: start from a Brownian excursion $\mathfrak{e}$ with i.i.d. balanced signs $(S(m))$ on local minima m of $\mathfrak{e}$ and set

$$\boldsymbol{W}^{Br}(x,y) = \begin{cases} 1 & \text{if } S\left(\operatorname{argmin}_{[x,y]} \mathfrak{e}\right) = \oplus; \\ 0 & \text{if } S\left(\operatorname{argmin}_{[x,y]} \mathfrak{e}\right) = \ominus. \end{cases}$$



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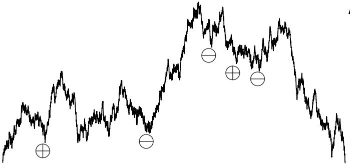
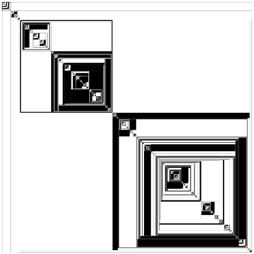
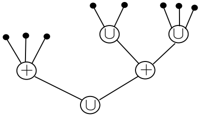


Adjacency matrix of a large uniform cograph (with a well-chosen order of vertices)



Heuristic for the theorem

- A cograph G is encoded by a decorated tree T;



- Vertices in G correspond to leaves in T;
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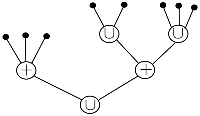

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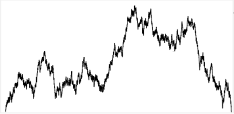

- Leaves of T_∞ form a measure 1 subset of $[0,1]$;
- Youngest common ancestor between x and y correspond to $\text{argmin}_{[x,y]}\,\mathfrak{e}$. Thus x, y are linked in $\boldsymbol{W}^{Br}$ if $S(\text{argmin}_{[x,y]}\,\mathfrak{e}) = \oplus$.

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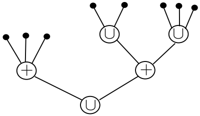

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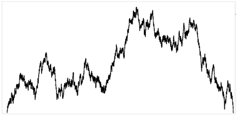

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Note: in the discrete, decorations alternate; in the continuous, they are independent.

Sampling from $\boldsymbol{W}^{Br}$

$G(n, \boldsymbol{W}^{Br})$ has vertex set $[n]$ and $i \sim j$ if and only if $\boldsymbol{W}^{Br}(U_i, U_j) = 1$ ($U_1, \dots, U_n$ i.i.d. unif. in $[0,1]$).

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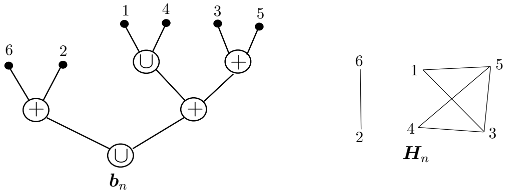
Proposition

Let $\boldsymbol{T}_n$ be a uniform random binary tree with n leaves and choose independent $\{\cup, +\}$ decoration for its internal node. Let $\boldsymbol{H}_n$ be the associated cograph. Then

$$\boldsymbol{G}(n, \boldsymbol{W}^{Br}) \stackrel{d}{=} \boldsymbol{H}_n.$$



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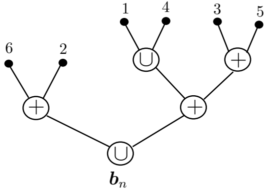

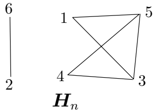

Expected degree distribution of $\boldsymbol{W}^{Br}$

Consider $\boldsymbol{G}(n, \boldsymbol{W}^{Br})$ the random graph with n vertices sampled from $\boldsymbol{W}^{Br}$.

Proposition (Bouvel–Bassino–F.–Gerin–Maazoun–Pierrot '22)

The degree of vertex 1 *in* $\boldsymbol{G}(n, \boldsymbol{W}^{Br})$ *is a random variable following the uniform distribution in* $\{0, 1, \dots, n-1\}$.

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Corollary 1

The expected degree distribution of $\boldsymbol{W}^{Br}$

$$\mathrm{Law}\left(\mathbb{E}\left[\int_0^1 \boldsymbol{W}^{Br}(U,y)dy \middle| U\right]\right)$$

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Corollary 2

The normalized degree $\frac{d_{\boldsymbol{v}}}{n}$ of a uniform random vertex $\boldsymbol{v}$ in a uniform random cograph on n vertices is asymptotically uniform in $[0, 1]$.

Third part

Independent sets

in uniform random cographs

Erdős-Hajnal conjecture and the probabilistic version

Erdős-Hajnal conjecture ('89)

Fix a graph H. There exists $\varepsilon = \varepsilon(H)$ such that every H-free graph contains a homogeneous set of size n^{ε}.

homogeneous set = clique or independent set

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Theorem (Loebl–Reed–Scott–Thomason–Thomassé, '14)

Fix a graph H. There exists $\varepsilon = \varepsilon(H)$ such that a uniform random H-free graph contains a homogeneous set of size n^ε (with high probability).

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For most H, the exists $b = b(H)$ such that a uniform random H-free graph contains a homogeneous set of size $b\,n$ (with high probability).

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Question (KMRS, '14)

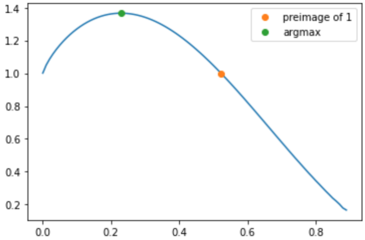
Does this hold for $H = P_4$?

Large independent sets in uniform random cographs

Theorem (Bouvel–Bassino–Drmota–F.–Gerin–Maazoun–Pierrot '22)

Let $\boldsymbol{G}_n$ be a uniform random cograph of size n.

1. *There exists $\beta_0 > 0$ s.t. for any $\beta < \beta_0$, the expected number $\mathbb{E}(X_{n,\lfloor \beta n \rfloor})$ of independent sets of size $\lfloor \beta n \rfloor$ in $\boldsymbol{G}_n$ grows exponentially fast.*



Exponential growth rate of $\mathbb{E}(X_{n,\lfloor \beta n \rfloor})$ as a function of β

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From 2, for any $\beta > 0$, we have $X_{n,\lfloor \beta n \rfloor} = 0$ with high probability (the expectation is misleading).

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- 2 answers KMRS's question from previous slide by the negative;
- Proof of 1 uses analytic combinatorics (and the tree encoding);
- Proof of 2 uses the graphon limit (next few slides).

Independence number of a graphon

Definition (Hladkỳ and Rocha, '20)

An independent set I of a graphon W is a subset $I \subseteq [0,1]$ such that $W(x,y) = 0$ for almost all (x,y) in $I \times I$.
The independence number of W, denoted $\alpha(W)$, is the maximum measure of an independent set of W.

Clearly, if G is a graph with n vertices, then $n\,\alpha(W_G)$ is the maximum size of an independent set of G.

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Proposition (Hladkỳ and Rocha, '20)

α is a lower semi-continuous function, i.e. if W_n converges to W, then $\limsup \alpha(W_n) \leq \alpha(W)$.

Independence number of the Brownian cographon

We only need to prove:

Proposition

$\alpha(\boldsymbol{W}^{Br}) = 0$ *a.s.*

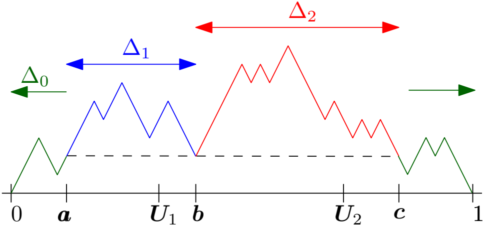
Indeed, let $\boldsymbol{G_n}$ be a uniform random cograph on n vertices. Since $W_{\boldsymbol{G}_n}$ tends to $\boldsymbol{W}^{Br}$ and α is lower semi-continuous, it would imply $W_{\boldsymbol{G}_n} \to 0$.

Independence number of the Brownian cographon

Proposition

$\alpha(\boldsymbol{W}^{Br}) = 0$ *a.s.*

Sketch of proof: we use Aldous' self-similarity property of the Brownian excursion



Starting from $\mathfrak{e}$, U_1, U_2, we get three independent excursions $(\mathfrak{e}_0, \mathfrak{e}_1, \mathfrak{e}_2)$ scaled by random lengths $(\Delta_0, \Delta_1, \Delta_2)$.

Independence number of the Brownian cographon

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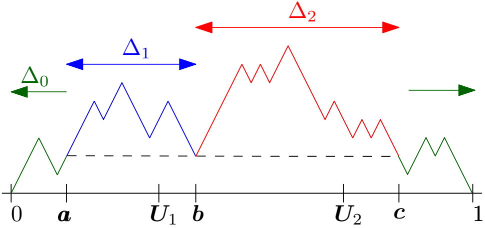

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Starting from W^{Br}, we get three independent copies W_0^{Br}, W_1^{Br} and W_2^{Br} of the Brownian cographon.

Independence number of the Brownian cographon

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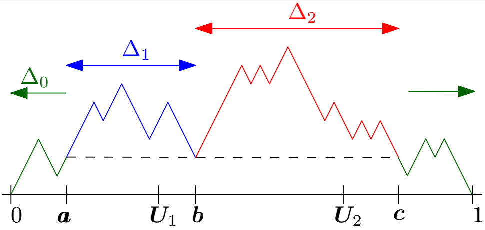

An independent set of W^{Br} consists of

- an independent set of W_0^{Br} (scaled by Δ_0);
- if $S(b) = \ominus$, an independent set of W_1^{Br} (scaled by Δ_1) and an independent set of W_2^{Br} (scaled by Δ_2);
- if $S(b) = \oplus$, an independent set of W_1^{Br} (scaled by Δ_1) or an independent set of W_2^{Br} (scaled by Δ_2);

Independence number of the Brownian cographon

Proposition

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Therefore, we have:

$$\alpha(W^{Br}) \leq \mathbf{\Delta}_0 \alpha(W_0^{Br}) + \begin{cases} \mathbf{\Delta}_1 \alpha(W_1^{Br}) + \mathbf{\Delta}_2 \alpha(W_2^{Br}) & \text{if } S(b) = \ominus; \\ \max\Big(\mathbf{\Delta}_1 \alpha(W_1^{Br}), \mathbf{\Delta}_2 \alpha(W_2^{Br})\Big) & \text{if } S(b) = \oplus. \end{cases}$$

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$\alpha(W_0^{Br})$, $\alpha(W_1^{Br})$ and $\alpha(W_2^{Br})$ are independent copies of $\alpha(W^{Br})$. This is an almost sure inequality, which implies an inequality satisfied by the law of $\alpha(W_0^{Br})$.

Independence number of the Brownian cographon

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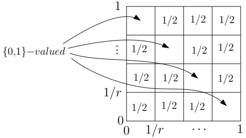
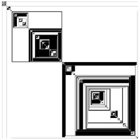
$\alpha(W_0^{Br})$, $\alpha(W_1^{Br})$ and $\alpha(W_2^{Br})$ are independent copies of $\alpha(W^{Br})$. This is an almost sure inequality, which implies an inequality satisfied by the law of $\alpha(W_0^{Br})$.

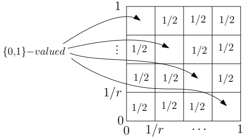
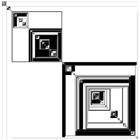
We show by a fixed point + monotonicity argument that δ_0 is the only distribution satisfying this inequality. Thus $\alpha(\boldsymbol{W}^{Br}) = 0$ a.s. □

☹ We do not control the speed of convergence of $\alpha(W_{\boldsymbol{G}_n})$ to 0.

Summary

<table>
<tr><th></th><th>enumeration</th><th>graphon limit</th><th>independent set</th></tr>
<tr><td>H-free ($r > 1$)</td><td>$e^{\Theta(n^2)}$</td><td></td><td>$\Theta_P(n)$
(for most H)</td></tr>
<tr><td>cographs</td><td>$e^{n\log(n)+\Theta(n)}$
(labeled)</td><td>W^{Br} </td><td>$o_P(n)$</td></tr>
</table>

Summary

	enumeration	graphon limit	independent set
H-free ($r > 1$)	$e^{\Theta(n^2)}$		$\Theta_P(n)$ (for most H)
cographs	$e^{n\log(n)+\Theta(n)}$ (labeled)	W^{Br} 	$o_P(n)$

Thank you for your attention