Graphon Dynamics

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Key Question

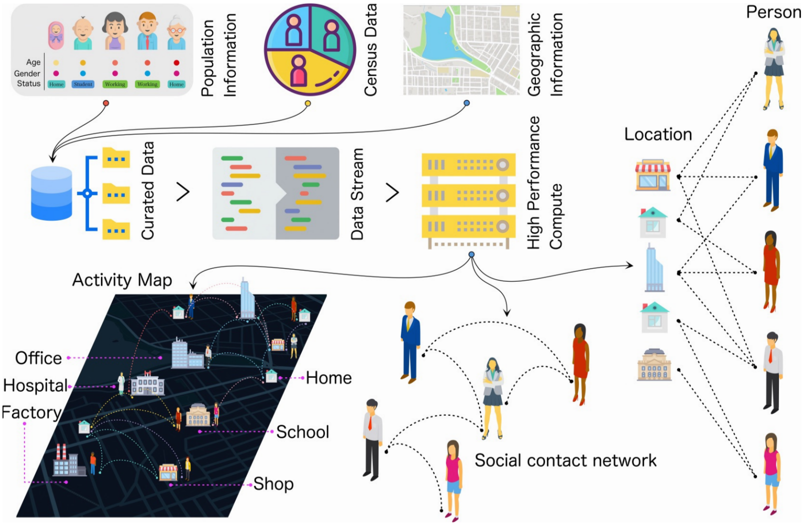
- Graphs and Graphons viewed in: $(\widetilde{\mathcal{W}}, d_{\text{sub}})$ is a compact metric space.
- Introduce any stochastic dynamics for $n \geq 1$, to get $\{G_n(t)\}_{n\geq 1, t\geq 0}$

Question: In the $n \to \infty$ limit, does this provide a diffusion on $\{h_t\}_{t\geq 0}$ on $(\widetilde{\mathcal{W}}, d_{\text{sub}})$?

Motivation: - modelling of real-world network dynamics

- Applications:
  - Model the evolution of social networks.
  - Spread of pathogens in (dynamic) social contact networks.
  - Propagation of information in a dynamic network.
  - Analysis: specific models, ODE, Agent-based models and Simulation.
- Mathematical Treatment:
  - Dynamic Percolation
  - Dynamic Very Sparse Graph: mixing times of Random walks
  - Dense Networks it is in early stages.

Ferguson et. al - Epidemic Spread - IISc. Bangalore city Simulation



Alter mixing times by changing the network

Mathematics > Probability

[Submitted on 24 Jun 2016 (this version), latest version 15 Jan 2018 (v3)]

Mixing times of random walks on dynamic configuration models

Luca Avena, Hakan Guldas, Remco van der Hofstad, Frank den Hollander

The mixing time of a simple random walk on a random graph generated according to the configuration model is known to be of order $\log n$ when n is the number of vertices. In this paper we investigate what happens when the random graph becomes dynamic, namely, at each unit of time a fraction α_n of the edges is randomly relocated. For degree distributions that converge and have a second moment that is bounded in n, we show that the mixing time is of order $1/\sqrt{\alpha_n}$, provided $\lim_{n\to\infty} \alpha_n (\log n)^2 = \infty$. We identify the sharp asymptotics of the mixing time when we additionally require that $\lim_{n\to\infty} \alpha_n = 0$, and relate the relevant proportionality constant to the average probability of escape from the root by a simple random walk on an augmented Galton-Watson tree which is obtained by taking a Galton-Watson tree whose offspring distribution is the size-biased version of the limiting degree distribution and attaching to its root another Galton-Watson tree with the same offspring distribution. Our proofs are based on a randomised stopping time argument in combination with coupling estimates.

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Under stand neural network via gradient flows on graphons

Mathematics > Probability

[Submitted on 18 Nov 2021 (v1), last revised 9 May 2022 (this version, v2)]

Gradient flows on graphons: existence, convergence, continuity equations

Sewoong Oh, Soumik Pal, Raghav Somani, Raghav Tripathi

Wasserstein gradient flows on probability measures have found a host of applications in various optimization problems. They typically arise as the continuum limit of exchangeable particle systems evolving by some mean-field interaction involving a gradient-type potential. However, in many problems, such as in multi-layer neural networks, the so-called particles are edge weights on large graphs whose nodes are exchangeable. Such large graphs are known to converge to continuum limits called graphons as their size grow to infinity. We show that the Euclidean gradient flow of a suitable function of the edge-weights converges to a novel continuum limit given by a curve on the space of graphons that can be appropriately described as a gradient flow or, more technically, a curve of maximal slope. Several natural functions on graphons, such as homomorphism functions and the scalar entropy, are covered by our set-up, and the examples have been worked out in detail.

Comments: 41 pages, 3 figures
Subjects: **Probability (math.PR)**; Machine Learning (cs.LG); Machine Learning (stat.ML)
MSC classes: 05C60, 05C80, 68R10, 60K35
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Dynamics of Dense Networks to Graphon Processes

- Easy to visualise and construct random dynamics of graphs on n vertices for each fixed n,
- It is a challenge to get interesting $n \to \infty$ limits.
- Crane 2016- first to attempt:
  - established limiting dynamics of time-varying dense networks using the Aldous-Hoover Theory.
  - *Their Projections* onto the space of Graphons induce processes of locally bounded variation.
  - Exchangeability of Vertices ensures only jump processes or deterministic flows.
- Černý and Klimovsky(2018) have elaborated/explained Crane's work.

Aldous-Hoover Theory of Infinite Exchangeable Arrays

$(X_{ij})_{i,j\geq 1}$ is called *exchangeable array* if its distribution remains the same under any finite permutation of its index labels.

Special case: infinite exchangeable random graph (ierg) is case where $X_{ij} \in \{0,1\}$ and $X_{ij} = X_{ji}$ a.s.

Theorem (Aldous/Hoover, 70s/80s)

Let H be an ierg. Then $\exists f : [0,1]^3 \to [0,1]$ such that $f(u,x,y) = f(u,y,x)$ and such that

$$X_{ij} = \mathrm{I}[U_{ij} \leq f(U, U_i, U_j)] \quad \forall i, j,$$

where the U, U_i, U_{ij} are all i.i.d. uniform on $[0,1]$.

Theorem (Diaconis and Janson (2007))

The class of iergs and the class of distributions on $(\widetilde{W}, dsub)$ is one-to-one.

If H is ierg and h is random element of $\widetilde{W}$, then

$$Et_F(h) = P[F \subset H|_{\{1,\dots,k\}}] \quad \forall F$$

uniquely determines $h(\equiv h^H)$ from H and vice versa.

Crane's theory (2016, AoP)

- Theory of Markov processes on exchangeable arrays, aiming to understand graphon-valued processes.

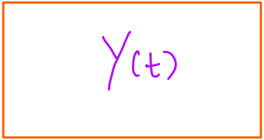
Theorem (Crane (2016))

If $H(t)$ is ierg-valued Markov process, then $h^H(t)$ has bounded variation.

- Hence, this way, we cannot obtain graphon-valued diffusions, e.g. "graphon-valued Brownian motion"

However...

Pick your preferred diffusion $Y(t)$ on $[0, 1]$ and define $h(t) \equiv Y(t)$, that is, constant graphon, the height of which changes in time.



Connect

i~j w.p Y(t)

- This is a perfectly fine Markov process with unbounded variation.
- It seems the assumption that the ierg $H(t)$
  $(h(t) \equiv h^H(t))$
  be Markov is very strong.

Dynamics of Dense Networks to Graphon Processes

- Diffusions are not captured through the lens of the Aldous-Hoover theory.
- Are there No diffusions on Graphons as $n \to \infty$ dynamics ?
- Technical difficulties:
  - $n \to \infty$ sub-graph counts have lot of averaging takes place.
  - Typical intuitive stochastic discrete dynamics will lead to a deterministic flow on Graphon space.

Topology: $G_n(\cdot)$ as $n \to \infty$ to $h(\cdot)$?

$$t(F, G_n(t)) = \frac{\text{copies of } F \text{ in } G_n}{\text{copies of } F \text{ in complete graph } K_n}$$

$$\xrightarrow{n \to \infty} \int_{[0,1]^k} \mathrm{d}x_1 \cdots \mathrm{d}x_k \prod_{(i,j) \in E(F)} h(t)(x_i, x_j)$$

$$=: t(F, h_t)$$

Proof:

- Averaging effect.
- concentration to the mean.
- Working with a generalised U-statistic.

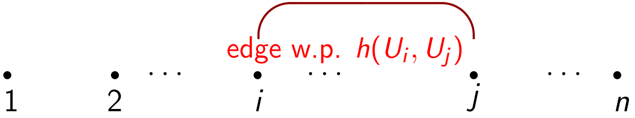
Graphon and Graph: Stochastic Block model

- Graphon for Stochastic Block model $h : [0,1]^2 \to [0,1]$

δ	β
α	δ

γ

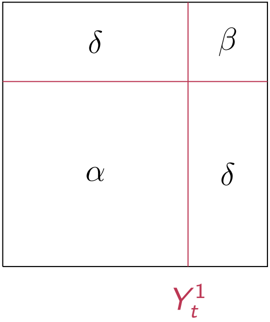
- Dense Graph $G(n, h)$ on $\{1, 2, \ldots, n\}$:



where $U_1, U_2, \ldots U_n$ be i.i.d. Uniform $[0,1]$.

Dynamic Stochastic Block Model

Can we construct a sequence of evolving dense graphs to achieve a scaling limit to :



where Y_t^1 is a diffusion ?

Dynamics of Networks

- Natural idea of discrete Dynamics:
  - Each edge evolves independently.
  - Random Time it remains active (i.e., present),
  - Random Time it remains inactive (i.e., absent).
- Will results in Deterministic limiting Dynamics for Y_t^1
- Something more needs to be done.

Moran Model to Wright Fisher

- Moran Model:
  - Consider n individuals, each carrying Type 0 or Type 1.
  - At rate 1, randomly draws an individual from the population (possibly itself) and adopts its type.
  - Let $X^n(s)$ be the number of individuals of type 0 at time s.

    $Y^n(s) = \frac{1}{n} X^n(ns)$ converges weakly to $Y(s)$
- Wright-Fisher:

$$Y(s) = Y(0) + \int_0^s \sqrt{Y(u)(1 - Y(u))}\, \mathrm{d}W(u)$$

with W being standard Brownian motion.

$G_n(s)$: discrete dynamics based on types

- At $s \geq 0$, i and j are connected by an edge with probability 1 if they are of the same type and remain disconnected if their types are different.
- For any connected graph F on k vertices, the sub-graph density of F in $G^n(s)$ is

$$t(F, G_n(s)) := \frac{\#\text{ of copies of } F \text{ in } G_n(s)}{\#\text{ of copies of } F \text{ in the complete graph}}$$

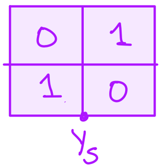
$$\simeq \frac{X^n(s)^k + (n - X^n(s))^k}{n^k}$$

$G_n(s)$: discrete dynamics as $n \to \infty$

$$t(F, G_n(s)) := \frac{X^n(s)^k + (n - X^n(s))^k}{n^k}$$

$$= \left(\frac{X^n(s)}{n}\right)^k + \left(1 - \frac{X^n(s)}{n}\right)^k$$

Limit Graphon

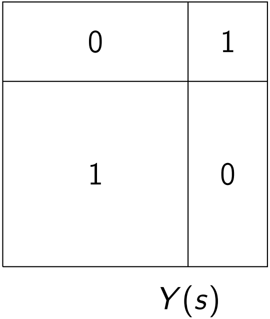


$= h(s)$

converges weakly as $n \to \infty$

$$= Y(s)^k + (1 - Y(s))^k \quad = t(F, h(s))$$

Graphon-Diffusion: $h(s) : [0,1]^2 \to [0,1]$

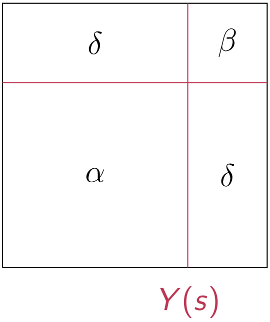


For any connected graph F on k vertices, the sub-graph density of F in $h(s)$ is

$$t(F, h(s)) = Y(s)^k + (1 - Y(s))^k$$

$G_n(s)$ converges weakly in Graphon space to $h(s)$

- $t_F(G_n(s))$ converge weakly to $t_F(h(s))$
- $t_F(h(s))$ is adapted to the filtration generated by $Y(s)$, and is a Markov process.



- Using the modulus of continuity of $t_F(h)$ and conclude in the sub-graph distance h is diffusive.

Result 1: [Set up] Convergence on Graphon Space

Weak Convergence: $G_n(\cdot) \Longrightarrow h(\cdot)$ or $h_n(\cdot) \Longrightarrow h(\cdot)$

- Convergence of $t(F, G_n(s))$ sub-graph density have hidden dependencies and are Generalised U-statistics.
- Establish tightness criteria in Graphon space based on $t(F, G_n(s))$ for finite graphs F.
- Obtain Process Convergence in sub-graph metric on Graphon space.

Proof Of Concept

Theorem

The following are equivalent:

- $\tilde{h}^n \Leftrightarrow \tilde{h}$ *as* $n \to \infty$ *in* $D([0,\infty),\widetilde{\mathcal{W}})$.
- *For all* $d \geq 1$ *and all graphs* $F_1, \ldots, F_d \in \mathcal{F}$,

$$(t_{F_1}(\tilde{h}^n), \ldots, t_{F_d}(\tilde{h}^n)) \Leftrightarrow (t_{F_1}(\tilde{h}), \ldots, t_{F_d}(\tilde{h})) \quad (n \to \infty),$$

  where weak convergence takes place in $D([0,\infty),[0,1]^d)$.
- *For every graph* $F \in \mathcal{F}$, *the sequence* $(t_F(\tilde{h}^n))_{n\geq 1}$ *is tight. Moreover, for all* $d \geq 1$, *all time points* $0 \leq s_1 < \cdots < s_d < \infty$ *where* $\tilde{h}$ *is continuous almost surely, and all graphs* $F_1, \ldots, F_d \in \mathcal{F}$,

$$\lim_{n\to\infty} E[t_{F_1}(\tilde{h}^n(s_1)) \times \cdots \times t_{F_d}(\tilde{h}^n(s_d))] = E[t_{F_1}(\tilde{h}(s_1)) \times \cdots \times t_{F_d}(\tilde{h}(s_d))].$$

Result 2: Multi-type Moran Model $m = 3$-types

- Multi-type Moran model to Multi-dimensional Wright-Fisher
- Connection Probability also changing dynamically.

$\delta(s)$	$\eta(s)$	$\beta(s)$
$\gamma(s)$	$\lambda(s)$	$\eta(s)$
$\alpha(s)$	$\gamma(s)$	$\xi(s)$

$Y^1(s)$ $Y^2(s)$

Result 3: Fleming-Viot Model ($m = \infty$)

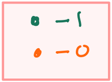
- M-Multi-dimensional Wright-Fisher converges to Fleming-Viot model on [0,1]-Continuum Type Space.

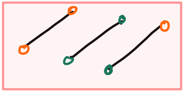
- h^M -Graphon induced by Wright-Fisher diffusion.

- Show h^M converge weakly to h
- h-Graphon which is measurable function of Fleming-viot.

A rich class of nontrivial diffusions in the space of graphons well beyond the stochastic block model framework.

Work in Progress : $G_n(t)$





Dynamics:

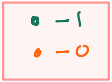
- [n]– Vertices: Individual is either Type 0 or Type 1.
- Edges: Turn on and off based on rates that depend on vertex type.
- Vertices: Voter model on existing Network.

Scaling :

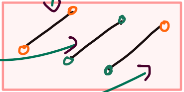
- Total number of edge flips are of same order as Total number of type changes (per unit time)
- Vertex changes same rate as its degree changes by 1.

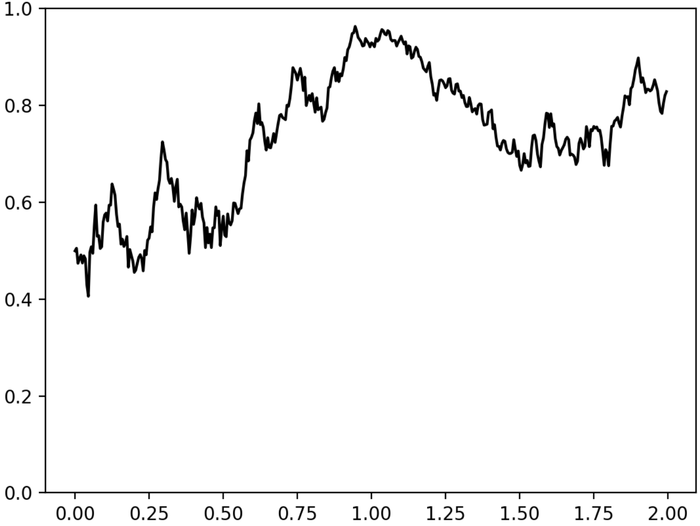
Work in Progress : Generator

vertex flips

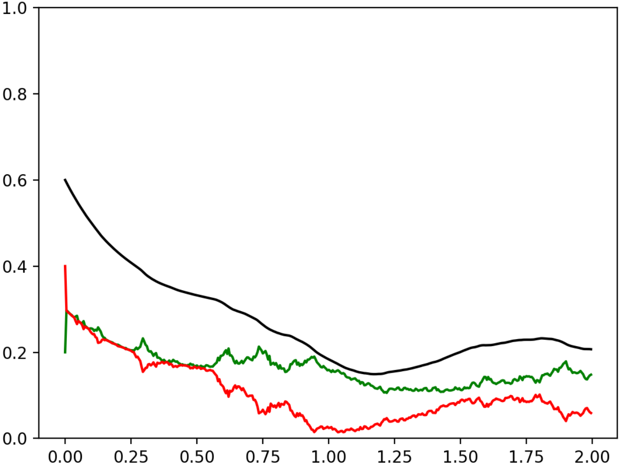


$$
\begin{aligned}
&\mathcal{A}_n f(G) \\
&= \eta \sum_{1\leq i\leq n} \mathrm{op}_i(G) f(\text{v-fl}_i(G)) - f(G) \\
&\quad + \rho \sum_{1\leq i<j\leq n} s_{d,0} + e_{i,j}(G)(s_{d,1} - s_{d,0}) + c_{i,j}(G)(s_{c,0} - s_{d,0}) \\
&\qquad e_{i,j}(G) c_{i,j}(G)(s_{c,1} - s_{c,0} + s_{d,0} - s_{d,1}) \times f(\text{e-fl}_{i,j}(G)) - f(G).
\end{aligned}
$$





Fraction of 0s in the population ($t_\bullet$)



edge density $(t_{\bullet\!-\!\bullet} + t_{\circ\!-\!\circ} + t_{\bullet\!-\!\circ})$,
concordant edge density $(t_{\bullet\!-\!\bullet} + t_{\circ\!-\!\circ})$,
discordant edge density $(t_{\bullet\!-\!\circ})$)

Proof Techniques among Friends

- Need coloured subgraph densities theory.
- Work out the n-th Level Generator of the coloured subgraph densities.
- $\mathcal{A}^n f(t_F(h)) = n(\qquad\qquad) + \ldots$ nice terms → characterize process
- Lyapunov function to make it go to 0 & process to rush into manifold
- Convergence to a graphon valued stochastic process in Meyer-Zheng Topology.

Grand Claims among Friends

- Complete description of all Subgraph densities of an interacting system.
- No moment closure and no mean field approximation.
- A rich class of models– very exciting.

Thank You !