Sparse random graphs:
Interplay of local and global structrure

Mihyun Kang

TU Graz

Workshop on Graph Limits, Non-Parametric Models, and Estimation

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Outline of the talk

I. From local to global structure

II. From global to local structure

Part I.

From local to global structure

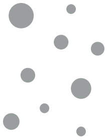
Emergence of giant component

- $|L| = \#$ vertices in the largest component in $G(n,p)$
- $d = p\,(n-1) \in (0,\infty)$

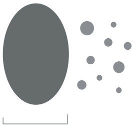
Theorem [ERDŐS–RÉNYI 60]

If $d < 1$, whp $|L| = O(\log n)$

If $d > 1$, whp $|L| = \Theta(n)$



O(log n)



Θ(n)

$*$ whp = with high probability = with prob tending to one as $n \to \infty$

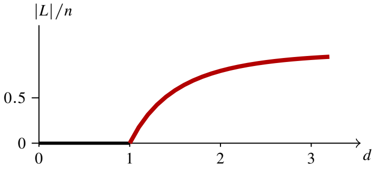
Largest component in ER random graph

- $|L| \;=\; \#$ vertices in the largest component in $G(n, p)$
- $d \;=\; p\,(n-1) \;\in\; (1, \infty)$
- $\rho \;=\;$ survival prob of $\mathrm{Po}(d)$ Galton-Watson branching process

  $=\;$ unique positive solution of $1 - \rho \;=\; \exp(-d\,\rho)$

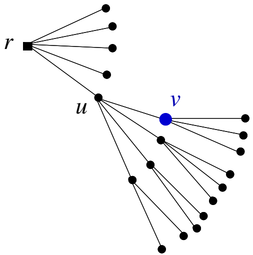
Theorem

whp $$|L| \;=\; (1 + o(1))\,\rho\,n$$



Local structure of ER random graph

- $d \;=\; p\,(n-1) \;\in\; (0,\infty)$
- $r \;=\;$ vertex chosen uniformly at random from $V(G(n,p))$



$d^{+}(r) \;\sim\; \mathrm{Po}(d)$

$d^{+}(u) \;\sim\; \mathrm{Po}(d)$

$d^{+}(v) \;\sim\; \mathrm{Po}(d)$

Local weak convergence

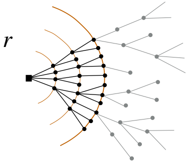
[BENJAMINI–SCHRAMM 2001], [ALDOUS–STEELE 2004]

- A rooted graph (H, r)

  $=$ a connected locally finite graph H with a vertex $r \in V(H)$ as the root

- Given a rooted graph (H, r) and $\ell \in \mathbb{N} := \{1, 2, \dots\}$, let

$$B_\ell(H, r) := H\left[\{v \in V(H) : d_H(v, r) \leq \ell\}\right]$$



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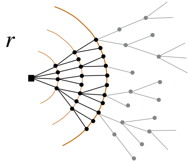


- Two rooted graphs (H, r) and (H', r') are isomorphic,

$$(H, r) \quad \cong \quad (H', r')$$

  if $\exists$ isomorphism ϕ from H onto H with $\phi(r) = r'$

Local weak convergence — cont'd

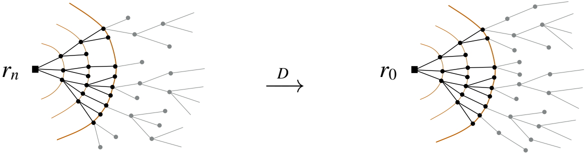
Given a sequence $\big((G_n, r_n)\big)_n$ of random rooted graphs,

a random rooted graph (G_0, r_0) is the local weak limit of (G_n, r_n)

$$(G_n, r_n) \xrightarrow{D} (G_0, r_0)$$

if for each fixed rooted graph (H, r_H) and $\ell \in \mathbb{N}$,

$$\mathbb{P}\Big[B_\ell(G_n, r_n) \cong (H, r_H)\Big] \xrightarrow{n\to\infty} \mathbb{P}\Big[B_\ell(G_0, r_0) \cong (H, r_H)\Big]$$



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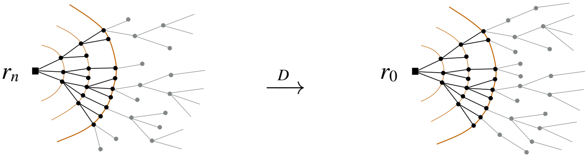


For not necessarily connected (G_n, r_n), its local weak limit?

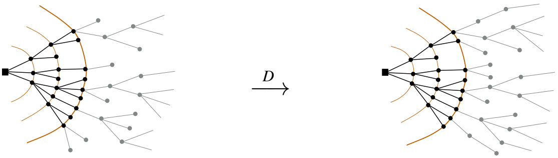
$\Longrightarrow$ define it as the local weak limit of the component of G_n containing r_n

ER random graph vs Galton–Watson tree

- $G = G(n,p)$ and $d = p(n-1) \in (0,\infty)$
- $r \in_R V(G)$ = vertex chosen uniformly at random from $V(G)$
- $\mathrm{GWT}(d)$ = Galton–Watson tree with offspring distribution $\mathrm{Po}(d)$

Theorem [Dembo–Montanari 2010], [van der Hofstad 2022+]

$$(G,r) \xrightarrow{D} \mathrm{GWT}(d)$$



i.e., $\mathbb{P}\Big[B_\ell(G,r) \cong (H,r_H)\Big] \xrightarrow{n\to\infty} \mathbb{P}\Big[B_\ell(\mathrm{GWT}(d)) \cong (H,r_H)\Big]$

Why local structure?

- Percolation threshold
  - Universality principle in percolation theory
- Giant component
  - Coupling component exploration processs via BFS with Galton-Watson branching process
  - High-dimensional analogues
  - Percolated hypercubes
  - • • •
- Message passing algorithms
  - Belief Propagation on random k-SAT
  - Warning Propagation for the k-core and rank of parity matrix
  - • • •

Part II.

From global to local structure

Planarity of ER random graph

- $d = p(n-1) \in (0, \infty)$

Theorem [Erdős-Rényi 1959–60]

- If $d < 1$, whp
  - each component is either a tree or unicyclic component
  - $G(n,p)$ is planar
- If $d > 1$, whp
  - largest component contains $\geq$ two cycles
  - $G(n,p)$ is not planar

Random graphs with topological constraints

How does a topological constraint such as

– being planar

– being embeddable on the orientable surface with given genus

affect the global and local structure of a random graph, e.g.,

– component structures

– local weak limits?

Random graphs on surfaces

- $g \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$
- $\mathbb{S}_g \;=\;$ the orientable surface of genus g (i.e., with g handles)

Random graphs on surfaces

- $g \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$
- $\mathbb{S}_g$ = the orientable surface of genus g (i.e., with g handles)
- $\mathcal{S}_g(n, m)$ = set of all vertex-labelled simple graphs on $[n]$ with $m = m(n)$ edges that are embeddable on $\mathbb{S}_g$

$S_g(n, m)$ = a graph chosen uniformly at random from $\mathcal{S}_g(n, m)$

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* $\mathcal{S}_0(n, m) \subset \ldots \subset \mathcal{S}_g(n, m) \subset \mathcal{S}_{g+1}(n, m) \subset \ldots \subset \mathcal{G}(n, m)$

Random graphs on surfaces – cont'd

Note

* If $1 \leq m < \frac{n}{2}$, then $\frac{|\mathcal{S}_0(n,m)|}{|\mathcal{G}(n,m)|} \xrightarrow{n\to\infty} 1$

* If $m > 3n - 6 + 6g$, then $\mathcal{S}_g(n,m) = \emptyset$

- Assume $2m/n \to d \in (1, 6]$

Random graphs on surfaces – cont'd

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- Assume $2m/n \to d \in (1,6]$

- $P = P(n,m) \in_R \mathcal{P}(n,m)$

  $\mathcal{P}(n,m)$ = set of all vertex-labelled simple graphs with vertex set $[n]$ and $m = m(n)$ edges that are embeddable on the sphere $\mathbb{S}_0$

  $P(n,m)$ = a graph chosen uniformly at random from $\mathcal{P}(n,m)$

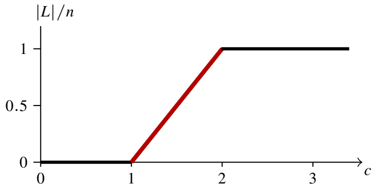
Phase transition in a random planar graph

- $P \;=\; P(n,m) \;\in_R\; \mathcal{P}(n,m)$
- $|L| \;=\; \#$ vertices in the largest component of P

Theorem [K.–Łuczak 2012], [K.–Mosshammer–Sprüssel 2020]

If $d \in (1,2]$, whp $\qquad |L| \;=\; (1+o(1))\,(d-1)\,n$

If If $d \in [2,6]$, whp $\qquad |L| \;=\; (1+o(1))\,n$



Local weak limit of a random planar graph

Theorem [K.–MISSETHAN 2022+]

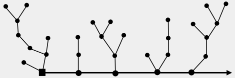
Assume $2m/n \xrightarrow{n\to\infty} d \in (1,2)$ and $r \in_R V(P)$. Then

$$(P,r) \xrightarrow{D} (2-d)\,\mathrm{GWT}(1) + (d-1)\,T_\infty$$

i.e., for each rooted graph (H, r_H) and $\ell \in \mathbb{N}$, we have

$$\mathbb{P}\Big[B_\ell(P,r) \cong (H,r_H)\Big] \xrightarrow{n\to\infty}$$
$$(2-d)\,\mathbb{P}\Big[B_\ell(\mathrm{GWT}(1)) \cong (H,r_H)\Big] + (d-1)\,\mathbb{P}\Big[B_\ell(T_\infty) \cong (H,r_H)\Big]$$

Skeleton tree T_∞

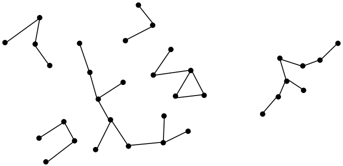


$=$ an infinite path whose vertices are replaced by independent $\mathrm{GWT}(1)$

From global to local structure

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \xrightarrow{n\to\infty} d \in (1,2)$
- L largest component of P
- $S = P \setminus L$ 'small' part of P

- S 'behaves similarly' like a critical ER random graph $G(\bar{n},\bar{m})$ with $\bar{n} = (2-d)\,n$ and $2\bar{m}/\bar{n} \to 1$



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- $r_S \in_R V(S)$

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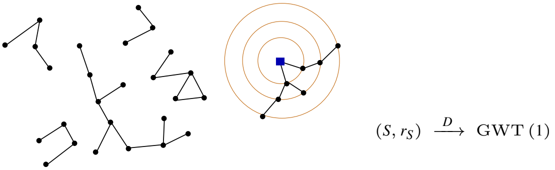
- $(S, r_S) \xrightarrow{D} \mathrm{GWT}\,(1)$



Giant component and its 2-core

- $P = P(n,m) \in_R \mathcal{P}(n,m)$ and $2m/n \xrightarrow{n\to\infty} d \in (1,2)$
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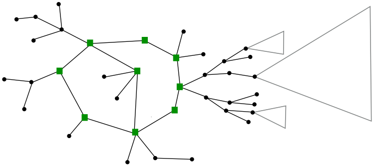
Theorem [K.–MOSSHAMMER–SPRÜSSEL 2020]

- $|L| \sim (d-1)\,n$
- $|C| \sim o(n)$
- $L = C$ + each vertex in $V(C)$ replaced by a rooted tree



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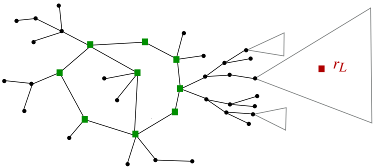

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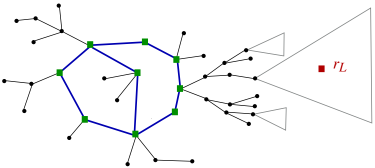

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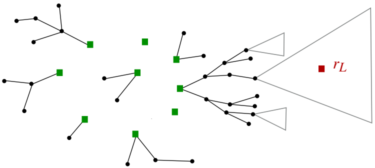
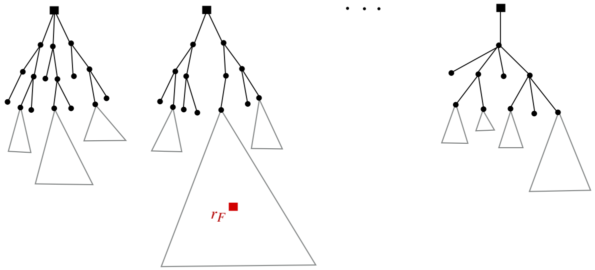


Local weak limit of a random forest

- $F \; = \; F(n,t) \; \in_R \; \mathcal{F}(n,t)$ a forest on $[n]$ with t tree components
- $r_F \; \in_R \; V(F)$ a vertex chosen uniformly at random from $V(F)$



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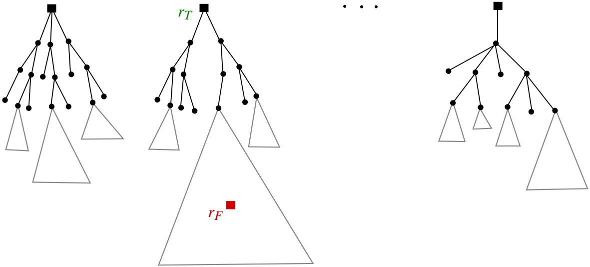
- $r_F \in_R V(F)$ a vertex chosen uniformly at random from $V(F)$
- r_T the root of tree component T that contains r_F



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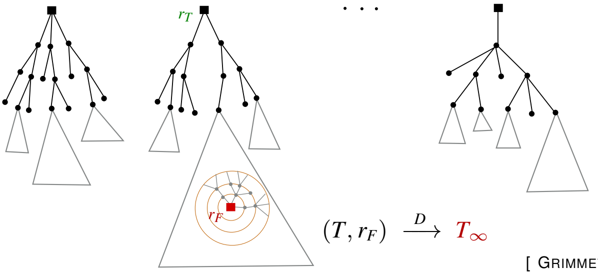
- $F \;=\; F(n,t) \;\in_R\; \mathcal{F}(n,t)$ — a forest on $[n]$ with t tree components
- $r_F \;\in_R\; V(F)$ — a vertex chosen uniformly at random from $V(F)$
- r_T — the root of tree component T that contains r_F

Lemma [K.–Missethan 2022+]

If $t = t(n) = o(n)$, then whp $d(r_F, r_T) = \omega(1)$ and

$$(F, r_F) \xrightarrow{D} T_\infty$$



$(T, r_F) \xrightarrow{D} T_\infty$

[Grimmett 1980/1981]

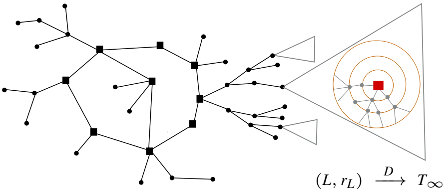
Finer view of local weak limits

- $P \; = \; P(n,m) \; \in_R \; \mathcal{P}(n,m)$, $\qquad 2m/n \xrightarrow{n\to\infty} d \in (1,2)$
- L largest component of P,

$r_L \; \in_R \; V(L)$

Theorem [K.–Missethan 2022+]

$$(L, r_L) \xrightarrow{D} T_\infty$$



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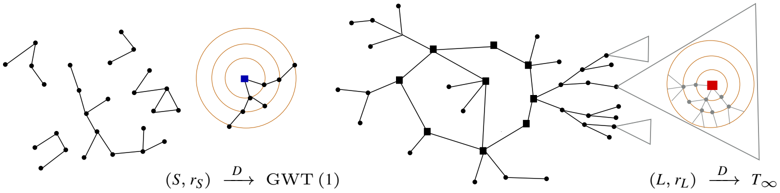
- L largest component of P,
- $S = P \setminus L \;\sim\;$ crtitical ER random graph,
- $r_S \;\in_R\; V(S), \quad r_L \;\in_R\; V(L)$

Theorem [K.–MISSETHAN 2022+]

$$(S, r_S) \xrightarrow{D} \mathrm{GWT}(1)$$

$$(L, r_L) \xrightarrow{D} T_\infty$$



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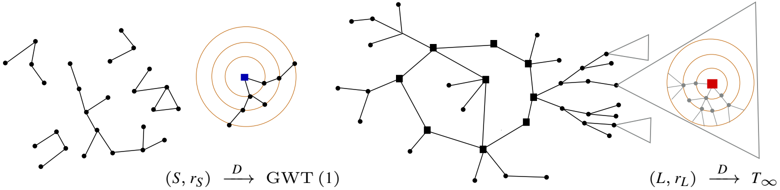
- $P \; = \; P(n,m) \;\; \in_R \;\; \mathcal{P}(n,m)$, $\qquad 2m/n \;\xrightarrow{n\to\infty}\; d \in (1,2)$
- L largest component of P, $\qquad |L| \;\sim\; (d-1)\,n$
- $S = P \setminus L \;\; \sim$ crtitical ER random graph, $\qquad |S| \;\sim\; (2-d)\,n$
- $r_S \;\in_R\; V(S)$, $\;\; r_L \;\in_R\; V(L)$ and $\;\; r_P \;\in_R\; V(P)$

Theorem [K.–MISSETHAN 2022+]

$$(S, r_S) \xrightarrow{D} \mathrm{GWT}(1)$$

$$(L, r_L) \xrightarrow{D} T_\infty$$

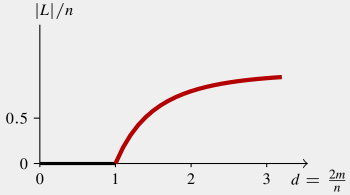
$$(P, r_P) \xrightarrow{D} (2-d)\,\mathrm{GWT}(1) \; + \; (d-1)\,T_\infty$$



Summary

(1) Phase transitions and critical phenomena

Uniform random graph $G(n,m)$



Random planar graph $P(n,m)$

$*$ $S = G(n,m) \setminus L$ 'behaves similarly' like a subcritical ER random graph

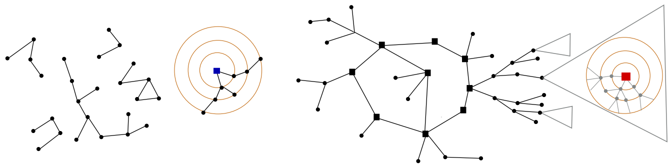
$*$ $S = P(n,m) \setminus L$ 'behaves similarly' like a critical ER random graph

Summary and an open question

(2) Local weak limit of a random planar graph

- $P \;=\; P(n,m) \;\in_R\; \mathcal{P}(n,m)$
- $r \;\in_R\; V(P)$
- $2m/n \xrightarrow{n\to\infty} d \in (1,2)$

$$(P,r) \quad \xrightarrow{D} \quad (2-d)\,\mathrm{GWT}\,(1) \;+\; (d-1)\,T_\infty$$



$Q.$ Local weak limit of (P,r) when $d \in (2,6)$?