Common graphs with large chromatic number

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Czech Technical University in Prague

Joint work with D. Král' and F. Wei.

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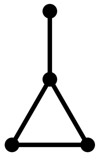

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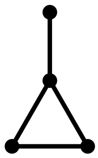

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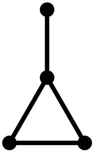

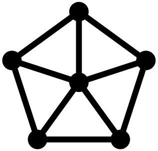

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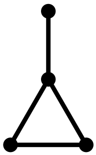

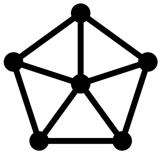

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Fox-Wei ('17): H locally Sidorenko $\iff$ the girth of H is even

Common graphs H with $\chi(H) > 3$

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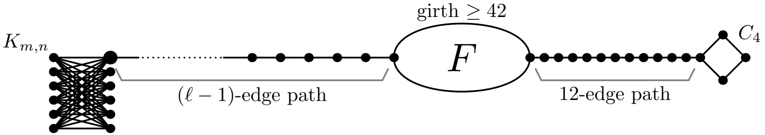

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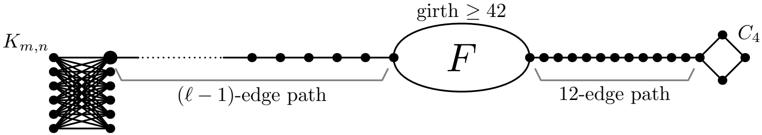

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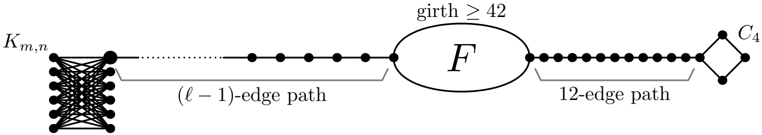

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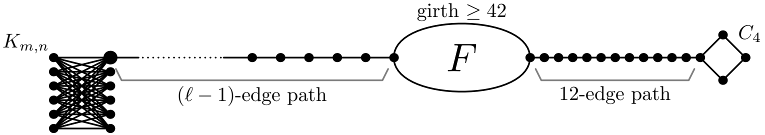

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or $\forall v \in [0, 1]$ is in $\delta_0 > 0$ of H's & use boost of $\# K_{m,n}$ by non-random

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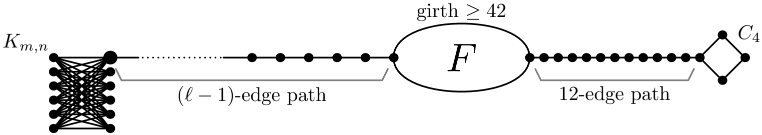

$\forall F$: girth(F) ≥ 42 $\exists N_0$ s.t. $\forall m \geq n \geq N_0$ and $\ell \approx 2n \rightarrow$ common

---

Proof idea: if $(W, 1 - W)$ is FAR from the constant $1/2$, then

either find sparse spot S in (say) red $\longrightarrow$ induct on S in blue

or $\forall v \in [0, 1]$ is in $\delta_0 > 0$ of H's & use boost of $\#K_{m,n}$ by non-random

---

When W (and also $1 - W$) is CLOSE to the constant $1/2$, then

Common graphs H with $\chi(H) > 3$

Jagger-Štovíček-Thomason ('96): Is there any? What about the 5-wheel?

Hatami, Hladký, Kráľ, Norin, Razborov ('12): YES, 5-wheel is common

Hatami, Hladký, Kráľ, Norin, Razborov ('12), Conlon, Fox, Sudakov ('15): Do there exist common graphs of all chromatic numbers?

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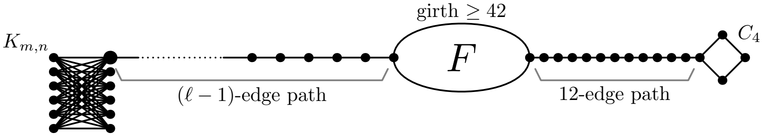

$\forall F : \text{girth(F)} \geq 42\ \exists N_0$ s.t. $\forall m \geq n \geq N_0$ and $\ell \approx 2n \to$ common

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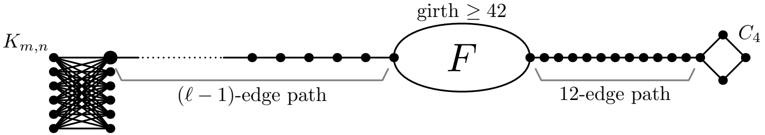

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When W (and also $1-W$) is CLOSE to the constant $1/2$, then

- Girth of H is four so Fox-Wei local-Sidorenko does apply...
- ...however, we need control on $m, n, \ell \rightarrow$ restrict only to H_k

Conclusion (1/2)

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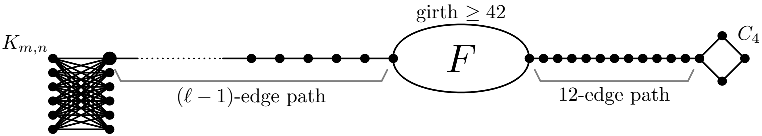

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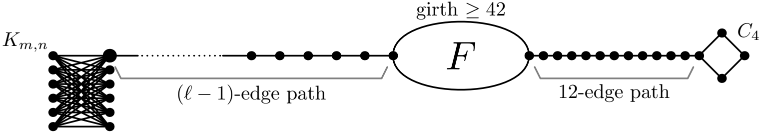

H is ℓ-common: $\forall$ $(W_1, \ldots, W_\ell)$ s.t. $\sum W_i = 1$: $\sum t(H, W_i) \geq \ell^{1-e(H)}$

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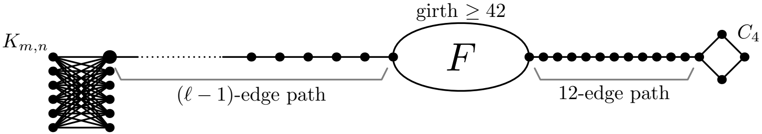

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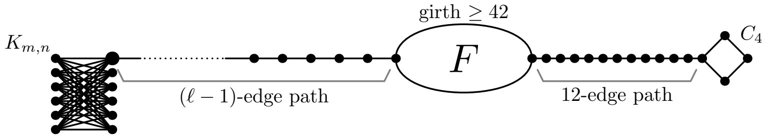

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- H is Sidorenko $\Longrightarrow$ H is ℓ-common for every ℓ

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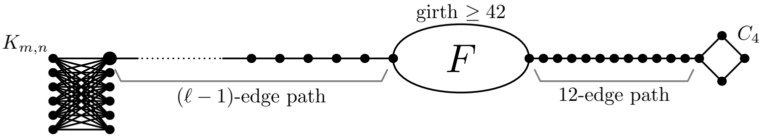

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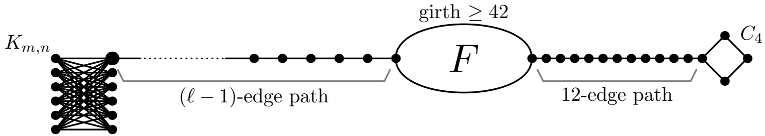

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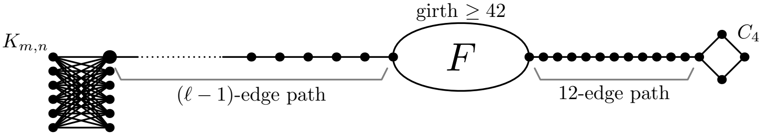

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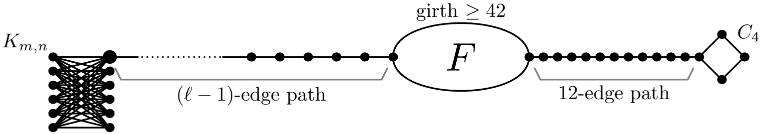

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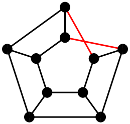
Is the graph $K_{5,5} - C_{10}$ common?



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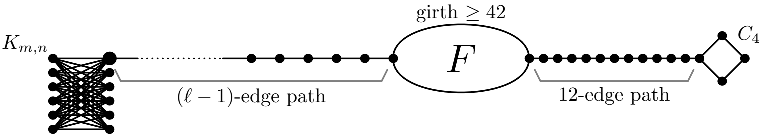

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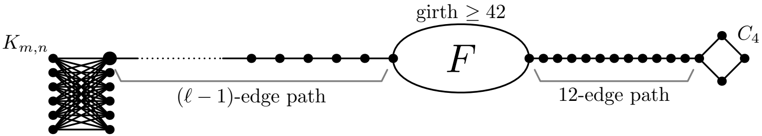

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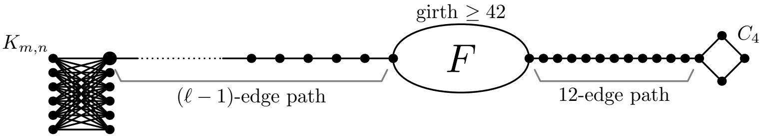

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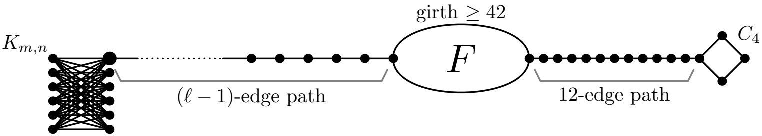

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Conclusion (2/2) Thank you for your attention!

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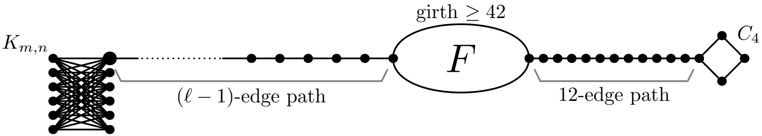

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