Random cluster model on regular graphs

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Joint work with Ferenc Bencs and Márton Borbényi.

Graph limits, Nonparametric Models, and Estimation
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Partition function of the random cluster model

For a graph $G = (V, E)$ the partition function of the random cluster model is defined by

$$Z_G(q, w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|},$$

where $k(A)$ denotes the number of connected components of the graph (V, A).

Tutte polynomial

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-v(G)},$$

where $k(A)$ denotes the number of connected components of the graph (V, A), and $v(G)$ denotes the number of vertices of the graph G.

$$T_G(x,y) = (x-1)^{-k(E)} (y-1)^{-v(G)} Z_G((x-1)(y-1), y-1).$$

Tutte polynomial and random cluster model

$T_G(x, y)$

$Z_G(q, w)$

Combinatorics

- $T_G(1, 1)$ spanning trees
- $T_G(2, 1)$ spanning forests
- $T_G(1, 2)$ connected subgraphs
- $T_G(2, 2) = 2^{e(G)}$
- $T_G(2, 0)$ acyclic orientations
- $T_G(0, 2)$ strong orientations
- chromatic polynomial
- flow polynomial

Statistical physics

- $q = 2$ Ising model
- $q \in \mathbb{Z}_{>0}$ Potts model
- $q > 0$ and $w \geq -1$ random cluster model

Main problem

Let $(G_n)_n$ be an essentially large girth sequence of d-regular graphs. Let $v(G)$ denote the number of vertices of G. Does the limit

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}(q,w)$$

exist?

Essentially large girth: for every fixed g,

$$\frac{\text{number of cycles of length } g \text{ in } G_n}{\text{number of vertices of } G_n} \to 0$$

Earlier results

Dembo and Montanari: Ising model $(q = 2)$

Dembo, Montanari, Sun: Potts model (positive integer q), except an interval (w_0, w_1)

Dembo, Montanari, Sly and Sun: even d and positive integer q

Helmuth, Jenssen and Perkins: proof of convergence for large q assuming some expansion property of $(G_n)_n$

Bandyopadhyay and Gamarnik: graph coloring, integer $q \geq d + 1$ and $w = -1$.

Bencs and Csikvári: Tutte-polynomial with $x \geq 1$ and $0 \leq y \leq 1$.

Main theorem

Theorem (Bencs, Borbényi and Cs.)

For a graph $G = (V, E)$ *let* $Z_G(q, w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|}$. *If* $(G_n)_n$ *is an essentially large girth sequence of* d*-regular graphs, then the limit*

$$\lim_{n \to \infty} \frac{1}{v(G_n)} \ln Z_{G_n}(q, w) = \ln \Phi_{d,q,w}$$

exists for $q \geq 2$ *and* $w \geq 0$. *The quantity* $\Phi_{d,q,w}$ *can be computed as follows. Let*

$$\left(\sqrt{1 + \frac{w}{q}} \cos(t) + \sqrt{\frac{(q-1)w}{q}} \sin(t) \right)^d + (q-1) \left(\sqrt{1 + \frac{w}{q}} \cos(t) - \sqrt{\frac{w}{q(q-1)}} \sin(t) \right)^d ,$$

then

$$\Phi_{d,q,w} := \max_{t \in [-\pi, \pi]} \Phi_{d,q,w}(t).$$

The same conclusion holds true with probability 1 *for a sequence of random* d*-regular graphs.*

Phase transition

Theorem (BBC)

Let $q \geq 2$ *and*

$$w_c := \frac{q-2}{(q-1)^{1-2/d}-1} - 1.$$

If $0 \leq w \leq w_c$, *then* $\Phi_{d,q,w} = q\left(1+\frac{w}{q}\right)^{d/2}$. *If* $w > w_c$, *then* $\Phi_{d,q,w} > q\left(1+\frac{w}{q}\right)^{d/2}$.

Picture

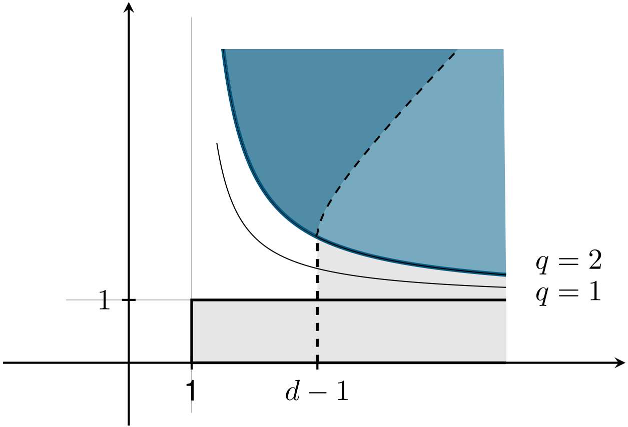

Figure: The investigated parameters are in blue. The dashed lines are $x = d - 1$ and the phase transition parametrized in x, y. We have $q = (x - 1)(y - 1)$ and $w = y - 1$.

Plan of the proof

The proof consists of two parts:

- Approximations of the partition function $Z_G(q, w)$
- Study of ferromagnetic 2-spin models

Approximations of the partition function $Z_G(q, w)$

Spin models

Given a graph $G = (V, E)$, a symmetric matrix $N \in \mathbb{R}^{r \times r}$ and $\underline{\mu} \in \mathbb{R}^r$ let

$$Z_G(N, \underline{\mu}) := \sum_{\sigma: V \to [r]} \prod_{v \in V} \mu_{\sigma(v)} \prod_{(u,v) \in E(G)} N_{\sigma(u),\sigma(v)}.$$

Potts model

Potts model with q spins: When q is a positive integer and M is the $q \times q$ matrix with diagonal elements $1 + w$ and off-diagonal elements 1, and $\underline{\mu} \equiv 1$, then

$$Z_G(M, \underline{1}) = Z_G(q, w).$$

Rank 1 approximation

Motivation: assume that q is a positive integer.

$$M = \begin{pmatrix} 1+w & 1 & \dots & 1 \\ 1 & 1+w & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1+w \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} 1+\frac{w}{q} & 1+\frac{w}{q} & \dots & 1+\frac{w}{q} \\ 1+\frac{w}{q} & 1+\frac{w}{q} & \dots & 1+\frac{w}{q} \\ \vdots & \vdots & \ddots & \vdots \\ 1+\frac{w}{q} & 1+\frac{w}{q} & \dots & 1+\frac{w}{q} \end{pmatrix}.$$

Idea: approximate $Z_G(M)$ with $Z_G(M_1)$. Let

$$Z_G^{(1)}(q,w) := Z_G(M_1) = q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)},$$

the rank 1 approximation of $Z_G(q,w)$. Make sense for any $q > 0$.

Lemma

If $q \geq 1$, *then*

$$Z_G(q,w) \geq Z_G^{(1)}(q,w).$$

If $0 < q \leq 1$, *then*

$$Z_G(q,w) \leq Z_G^{(1)}(q,w).$$

Proof.

Using the fact that $k(A) \geq v(G) - |A|$ for an $A \subseteq E(G)$ we get that for $q \geq 1$ we have

$$Z_G(q,w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|} \geq \sum_{A \subseteq E(G)} q^{v(G)-|A|} w^{|A|} = q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)}.$$

For $q \leq 1$ we have the opposite inequality in the above computation. □

Rank 2 approximation

What is better than a rank 1 approximation? Of course, a rank 2...

Motivation: again assume that q is a positive integer.

$$M = \begin{pmatrix} 1+w & 1 & \dots & 1 \\ 1 & 1+w & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1+w \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1+w & 1 & \dots & 1 \\ 1 & 1+\frac{w}{q-1} & \dots & 1+\frac{w}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1+\frac{w}{q-1} & \dots & 1+\frac{w}{q-1} \end{pmatrix}.$$

Then

$$Z_G^{(2)}(q,w) := Z_G(M_2) = \sum_{S\subseteq V} (1+w)^{e(S)}(q-1)^{v(G)-|S|}\left(1+\frac{w}{q-1}\right)^{e(G-S)}.$$

Makes sense if $q>1$.

2-spin model

Note that $Z_G^{(2)}(q,w) = Z_G(M_2', \underline{\nu}_2)$, where

$$M_2' = \begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix} \quad \text{and} \quad \underline{\nu}_2 = \begin{pmatrix} 1 \\ q-1 \end{pmatrix}$$

even if q is not an integer. Also observe that

$$Z_G^{(2)}(q,w) = \sum_{S \subseteq V(G)} (1+w)^{e(S)} Z_{G-S}^{(1)}(q-1,w).$$

Rank 2 approximation

Lemma

We have

$$Z_G(q,w) = \sum_{S \subseteq V} (1+w)^{e(S)} Z_{G-S}(q-1,w).$$

Lemma

For $q \geq 2$ we have

$$Z_G(q,w) \geq Z_G^{(2)}(q,w).$$

For $1 < q \leq 2$ we have

$$Z_G(q,w) \leq Z_G^{(2)}(q,w).$$

Rank 2 approximation

Lemma

We have

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For $1 < q \leq 2$ we have

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Large girth graphs

Theorem (BBC)

Let G be a graph with $L = L(G, g)$ cycles of length at most $g - 1$. Let $q \geq 2$. Then

$$Z_G^{(2)}(q, w) \leq Z_G(q, w) \leq q^{n/g+L} Z_G^{(2)}(q, w).$$

Theorem (BBC)

Let $q \geq 2$ and $w \geq 0$. Let $(G_n)_n$ be an essentially large girth sequence of d-regular graphs. If the limit $\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}^{(2)}(q, w)$ exists, then the limit $\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}(q, w)$ exists too, and they have the same value.

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Picture

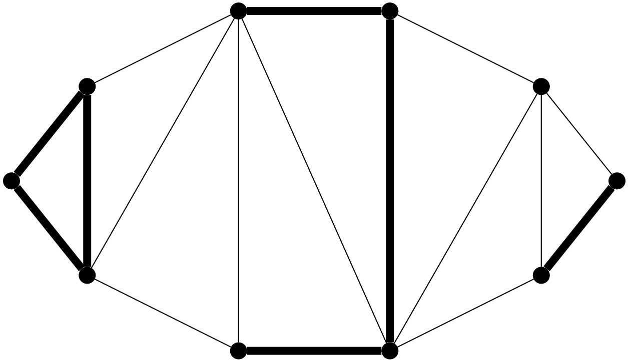

Figure: Large component: a component containing cycle. Set $\mathcal{L}_A$.
Small component: component not containing cycle. Set $\mathcal{S}_A$.
Compatible vertex set R: union of some small components (it may be empty), notation $R \sim A$.

Computation

- $k(A) = |\mathcal{L}_A| + |\mathcal{S}_A|$
- $|\mathcal{L}_A| \leq \frac{n}{g} + L(G, g)$
- $q^{|\mathcal{S}_A|} = ((q-1)+1)^{|\mathcal{S}_A|} = \sum_{R \sim A} (q-1)^{k(R,A)}$.

$$\begin{aligned}
Z_G(q, w) &= \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|} = \sum_{A \subseteq E(G)} q^{|\mathcal{L}_A| + |\mathcal{S}_A|} w^{|A|} \\
&\leq q^{n/g+L} \sum_{A \subseteq E(G)} q^{|\mathcal{S}_A|} w^{|A|} \\
&= q^{n/g+L} \sum_{A \subseteq E(G)} \sum_{R : R \sim A} (q-1)^{k(R,A)} w^{|A|} \\
&= q^{n/g+L} \sum_{R \subseteq V(G)} \sum_{A : A \sim R} (q-1)^{k(R,A)} w^{|A[R]| + |A[V \setminus R]|} \\
&= q^{n/g+L} \sum_{R \subseteq V(G)} (1+w)^{e(V \setminus R)} \sum_{D} (q-1)^{k(D)} w^{|D|},
\end{aligned}$$

Proof continued...

In the last sum, D is a subset of the edges induced by R such that none of the induced connected components contains a cycle. Then

$$\sum_D (q-1)^{k(D)} w^{|D|} = \sum_D (q-1)^{|R|-|D|} w^{|D|} \leq (q-1)^{|R|} \left(1 + \frac{w}{q-1}\right)^{e(R)}.$$

Hence

$$Z_G(q,w) \leq q^{n/g+L} \sum_{R \subseteq V(G)} (1+w)^{e(V \setminus R)} Z^{(1)}_{G[R]}(q-1,w),$$

that is

$$Z_G(q,w) \leq q^{n/g+L} Z^{(2)}_G(q,w).$$

Analysis of ferromagnetic 2-spin models

Ferromagnetic 2-spin models

Recall that $Z_G^{(2)}(q,w) = Z_G(M_2', \underline{\nu}_2)$, where

$$M_2' = \begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix} \quad \text{and} \quad \underline{\nu}_2 = \begin{pmatrix} 1 \\ q-1 \end{pmatrix}$$

So we need to show that the limit $\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}^{(2)}(q,w)$ exists for an essentially large girth sequence of d-regular graphs $(G_n)_n$. **This is already done!**

The work of Dembo, Montanari, Sly and Sun

- Amir Dembo and Andrea Montanari. Ising models on locally tree-like graphs.
- Allan Sly and Nike Sun. Counting in two-spin models on d-regular graphs.
- Amir Dembo, Andrea Montanari, and Nike Sun. Factor models on locally tree-like graphs.
- Amir Dembo, Andrea Montanari, Allan Sly, and Nike Sun. The replica symmetric solution for Potts models on d-regular graphs.

Limit theorem

Theorem (Sly and Sun building on Dembo and Montanari)

Let N be a 2×2 positive definite matrix with positive entries and let $\underline{\mu} \in \mathbb{R}^2_{>0}$. Then there exists a $\Phi_d(N, \underline{\mu})$ such that if $(G_n)_n$ is an essentially large girth sequence of d-regular graphs, then

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}(N, \underline{\mu}) = \ln \Phi_d(N, \underline{\mu}).$$

The same statement holds true for a sequence of random d-regular graphs with probability 1.

Some improvements

Theorem (BBC)

Let N be a 2×2 positive definite matrix with positive entries and let $\underline{\mu} \in \mathbb{R}^2_{>0}$. Let $(G_n)_n$ be a Benjamini–Schramm convergent sequence of d-regular graphs. Then

$$\lim_{n \to \infty} \frac{1}{v(G_n)} \ln Z_{G_n}(N, \underline{\mu}).$$

exists.

Theorem (BBC)

Let N be a 2×2 positive definite matrix with positive entries and let $\underline{\mu} \in \mathbb{R}^2_{>0}$. For any d-regular graph G we have $Z_G(N, \underline{\mu}) \geq \Phi_d(N, \underline{\mu})^{v(G)}$. Furthermore, if G contains $\varepsilon v(G)$ cycles of length g, then there exists a $\delta = \delta(d, N, \underline{\mu}, \varepsilon, g) > 0$ such that $Z_G(N, \underline{\mu}) \geq ((1 + \delta)\Phi_d(N, \underline{\mu}))^{v(G)}$.

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Subgraph counting polynomial

Subgraph counting polynomial of a d-regular graph:

$$F_G(x_0, \ldots, x_d) = \sum_{A \subseteq E} \left(\prod_{v \in V} x_{d_A(v)} \right),$$

and a bit more generally,

$$F_G(x_0, \ldots, x_d | z) = \sum_{A \subseteq E} \left(\prod_{v \in V} x_{d_A(v)} \right) z^{2|A|} = F_G(x_0, x_1 z, x_2 z, \ldots, x_d z^d)$$

Example: $F_{K_5}(x_0, x_1, x_2, x_3, x_4)$

$$x_0^5 + 10x_0^3x_1^2 + 15x_0x_1^4 + 30x_0^2x_1^2x_2 + 30x_1^4x_2 + 60x_0x_1^2x_2^2 + 10x_0^2x_2^3 + 70x_1^2x_2^3 + 15x_0x_2^4$$
$$+ 12x_2^5 + 20x_0x_1^3x_3 + 60x_1^3x_2x_3 + 60x_0x_1x_2^2x_3 + 120x_1x_2^3x_3 + 60x_1^2x_2x_3^2 + 30x_0x_2^2x_3^2 + 70x_2^3x_3^2$$
$$+ 60x_1x_2x_3^3 + 5x_0x_3^4 + 30x_2x_3^4 + 5x_1^4x_4 + 30x_1^2x_2^2x_4 + 15x_2^4x_4 + 60x_1x_2^2x_3x_4 + 60x_2^2x_3^2x_4$$
$$+ 20x_1x_3^3x_4 + 15x_3^4x_4 + 10x_2^3x_4^2 + 30x_2x_3^2x_4^2 + 10x_3^2x_4^3 + x_4^5.$$

Ferromagnetic 2-spin models and subgraph counting polynomial

Suppose that we can write an $r \times r$ matrix N into the form $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$ and let $\underline{\mu} \in \mathbb{R}^r$. Then

$$\begin{aligned}
Z_G(N, \underline{\mu}) &= \sum_{\varphi: V \to [r]} \prod_{v \in V} \mu_{\varphi(v)} \prod_{(u,v) \in E} N_{\varphi(u)\varphi(v)} \\
&= \sum_{\varphi: V \to [r]} \prod_{v \in V} \mu_{\varphi(v)} \prod_{(u,v) \in E} (\underline{a}\underline{a}^T + \underline{b}\underline{b}^T)_{\varphi(u)\varphi(v)} \\
&= \sum_{S \subseteq E} \sum_{\varphi: V \to [r]} \prod_{v \in V} \mu_{\varphi(v)} \prod_{(u,v) \in E \setminus S} (\underline{a}\underline{a}^T)_{\varphi(u)\varphi(v)} \prod_{(u,v) \in S} (\underline{b}\underline{b}^T)_{\varphi(u)\varphi(v)} \\
&= \sum_{S \subseteq E} \sum_{\varphi: V \to [r]} \prod_{v \in V} \mu_{\varphi(v)} \prod_{(u,v) \in E \setminus S} (\underline{a}_{\varphi(u)} \underline{a}_{\varphi(v)}) \prod_{(u,v) \in S} (\underline{b}_{\varphi(u)} \underline{b}_{\varphi(v)}) \\
&= \sum_{S \subseteq E} \prod_{v \in V} \left(\sum_{k=1}^{r} \mu_k a_k^{d - d_S(v)} b_k^{d_S(v)} \right) \\
&= F_G(r_0, \dots, r_d),
\end{aligned}$$

where $r_j = \sum_{k=1}^{r} \mu_k a_k^{d-j} b_k^j$.

More than one way

$\underline{a}$ and $\underline{b}$ are not the only vectors satisfying $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$. Indeed, let us define the vectors $\underline{a}(t)$ and $\underline{b}(t)$ as follows:

$$\underline{a}(t)_j = a_j \cos(t) + b_j \sin(t),$$

and

$$\underline{b}(t)_j = -a_j \sin(t) + b_j \cos(t).$$

Then $N = \underline{a}(t)\underline{a}(t)^T + \underline{b}(t)\underline{b}(t)^T$. So each pairs $\underline{a}(t), \underline{b}(t)$ gives rise to a vector $\underline{v}(t) = (r_0(t), \ldots, r_d(t))$ such that

$$F_G(\underline{v}(t)) = Z_G(N, \underline{\mu}).$$

Specialization

We can apply our argument to $N = M_2'$, $\underline{\mu} = \underline{\nu}_2$ with the following vectors.

$$\underline{a} = \begin{pmatrix} \sqrt{1 + \frac{w}{q}} \\ \sqrt{1 + \frac{w}{q}} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} \sqrt{\frac{(q-1)w}{q}} \\ -\sqrt{\frac{w}{q(q-1)}} \end{pmatrix}.$$

One can check that $M_2' = \underline{aa}^T + \underline{bb}^T$ indeed holds true. We can again introduce the vectors $\underline{a}(t), \underline{b}(t)$ giving rise to a vector $\underline{v}(t) = (r_0(t), \ldots, r_d(t))$ such that

$$F_G(\underline{v}(t)) = Z_G(M_2', \underline{\nu}_2) = Z_G^{(2)}(q, w).$$

Example

Let $d = 8$, $q = 5$ and $w = 1$. Then the vector

$$\underline{v}(0) = (10.368, 0, 1.728, 1.058, 0.936, 0.749, 0.615, 0.501, 0.409),$$

where we kept only the first three digits everywhere. Note that $10.368 = 5 \cdot \left(1 + \frac{1}{5}\right)^{8/2}$. So for every 8-regular graph G we have

$$Z_G^{(2)}(5,1) = F_G(10.368, 0, 1.728, 1.058, 0.936, 0.749, 0.615, 0.501, 0.409).$$

Using $t_0 = 0.6619549492373429$ we get the vector

$$\underline{v}(t_0) = (16.277, 0, 0.433, -0.496, 0.581, -0.679, 0.794, -0.929, 1.086)$$

and

$$Z_G^{(2)}(5,1) = F_G(\underline{v}(t_0)) \geq 16.277^{v(G)}$$

for every 8-regular graph G.

Bethe limit

Lemma (BBC)

Let N *be a* 2×2 *positive definite matrix and* $\underline{\mu} \in \mathbb{R}^2$*. Suppose that* $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$*. Let* t_0 *be the maximizer of*

$$r_0(t) = \mu_1(a_1 \cos(t) + b_1 \sin(t))^d + \mu_2(a_2 \cos(t) + b_2 \sin(t))^d.$$

Let

$$r_j(t) = \mu_1(a_1 \cos(t) + b_1 \sin(t))^{d-j}(-a_1 \sin(t) + b_1 \cos(t))^j + \mu_2(a_2 \cos(t) + b_2 \sin(t))^{d-j}(-a_2 \sin(t) + b_2 \cos(t))^j.$$

Then $r_1(t_0) = 0$ *and either*
(i) $r_j(t_0) \geq 0$ *for* $j = 0, \ldots, d$ *or*
(ii) $r_j(t_0) \geq 0$ *for even* j*, and* $r_j(t_0) \leq 0$ *for odd* j*.*

$$\Phi_d(N, \underline{\mu}) = \max_{t \in [-\pi, \pi]} r_0(t).$$

Lee-Yang theory

Theorem (BBC)

Let N be a 2×2 positive definite matrix with positive entries and let $\mu_1, \mu_2 > 0$. Then, there exists a $t_1 \in [0, 2\pi)$ such that for any d-regular graph G all the complex zeros of $F_G(\underline{v}(t_1)|z)$ are on a circle around 0 of radius $R_c(N, \underline{\mu})$.

Example

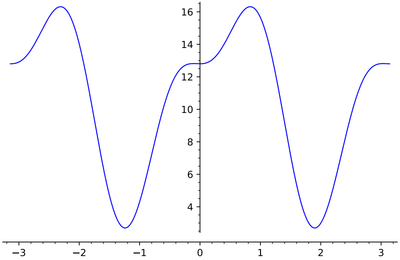

Figure: For $d = 4$, $q = 5$ and $w = 3$. The graph of the trigonometric polynomial $\Phi_{4,5,3}(t)$ is depicted in the figure.

Let $d = 4$, $q = 5$ and $w = 3$. Then

$$\underline{v}(0) = (12.8, 0, 4.8, 4.409, 5.85).$$

Let $t_0 = 0.8316331320342567$ and $\Phi_{4,5,3} = 16.315621073058985$ while

$$\underline{v}(t_0) = (16.315, 0, 1.878, -3.867, 8.176).$$

Example continued

Let $t_1 = 1.06627054934707$ and the corresponding vector

$$\underline{v}(t_1) = (15.010, -2.835, 0.994, -2.454, 11.249).$$

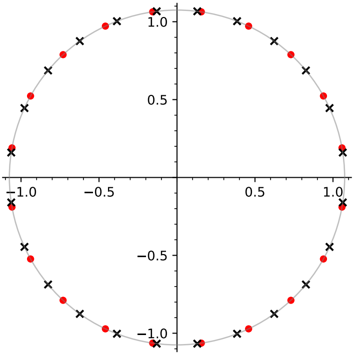

Figure: The zeros of $F_G(15.010, -2.835, 0.994, -2.454, 11.249 \mid z)$, where G is K_5 (red) and G is the octahedron (black x). The radius is approximately 1.0747696.

Convergence

Given a vector $\underline{a} \in \mathbb{R}^{d+1}$ and a graph G on n vertices let $\lambda_1(G), \ldots, \lambda_{nd}(G)$ be the zeros of the polynomial $F_G(\underline{a}|z)$. Let us define the probability measure $\rho_{G,\underline{a}}$ on $\mathbb{C}$ as follows:

$$\rho_{G,\underline{a}} := \frac{1}{nd} \sum_{k=1}^{nd} \delta_{\lambda_i(G)},$$

where δ_λ is the Dirac-measure on the number λ.

Convergence continued

Lemma

(a) For any integer $k \geq 0$, a vector $\underline{a} \in \mathbb{R}^{d+1}$ and a Benjamini–Schramm convergent sequence of d-regular graphs $(G_n)_n$ the sequence

$$\int z^k \, d\rho_{G_n,\underline{a}}(z)$$

is convergent.

(b) Let t_1 be such that the zeros of $F_G(\underline{v}(t_1)|z)$ lie on a circle of radius R_c for all graph G. If $(G_n)_n$ is a Benjamini–Schramm convergent sequence of d-regular graphs, then the sequence of measures $\rho_{G_n,\underline{v}(t_1)}$ converges weakly.

If $R_c(N,\underline{\mu}) \neq 1$, then $\ln|z-1|$ is a continuous function on an appropriate region.

Phase transition

Definition

We say that $(N, \underline{\mu})$ *exhibits a mixed state for a fixed positive integer* d *if* $R_c(N, \underline{\mu}) = 1$*.*

Note that $R_c(N, \underline{\mu}) = 1$ does not depend on which representation $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$ we choose. We also know that $R = R_c(N, \underline{\mu})$ is a solution of

$$(N_{11}N_{22}-N_{12}^2)R^4+(-N_{22}^2T+2N_{12}^2-N_{11}^2T^{-1})R^2+(N_{11}N_{22}-N_{12}^2)=0,$$

where $T = \left(\frac{\mu_2}{\mu_1}\right)^{2/d}$. This shows that $(N, \underline{\mu})$ exhibits a mixed state for d if

$$2(N_{11}N_{22} - N_{12}^2) - (N_{22}^2T - 2N_{12}^2 + N_{11}^2T^{-1}) = 0.$$

Specialization

$(M_2', \underline{\nu}_2)$ exhibits mixed state for some d if $q = 2$ or

$$w_c = \frac{q-2}{(q-1)^{1-2/d} - 1} - 1.$$

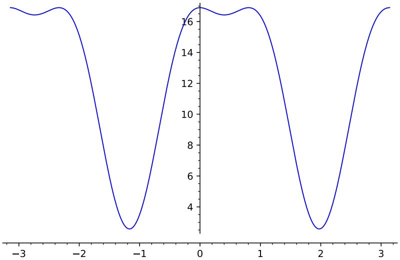

Figure: For $d = 4$ and $q = 10$ we have $w_c = 3$. The graph of the trigonometric polynomial $\Phi_{4,10,3}(t)$ is depicted in the figure.

Thank for your attention!