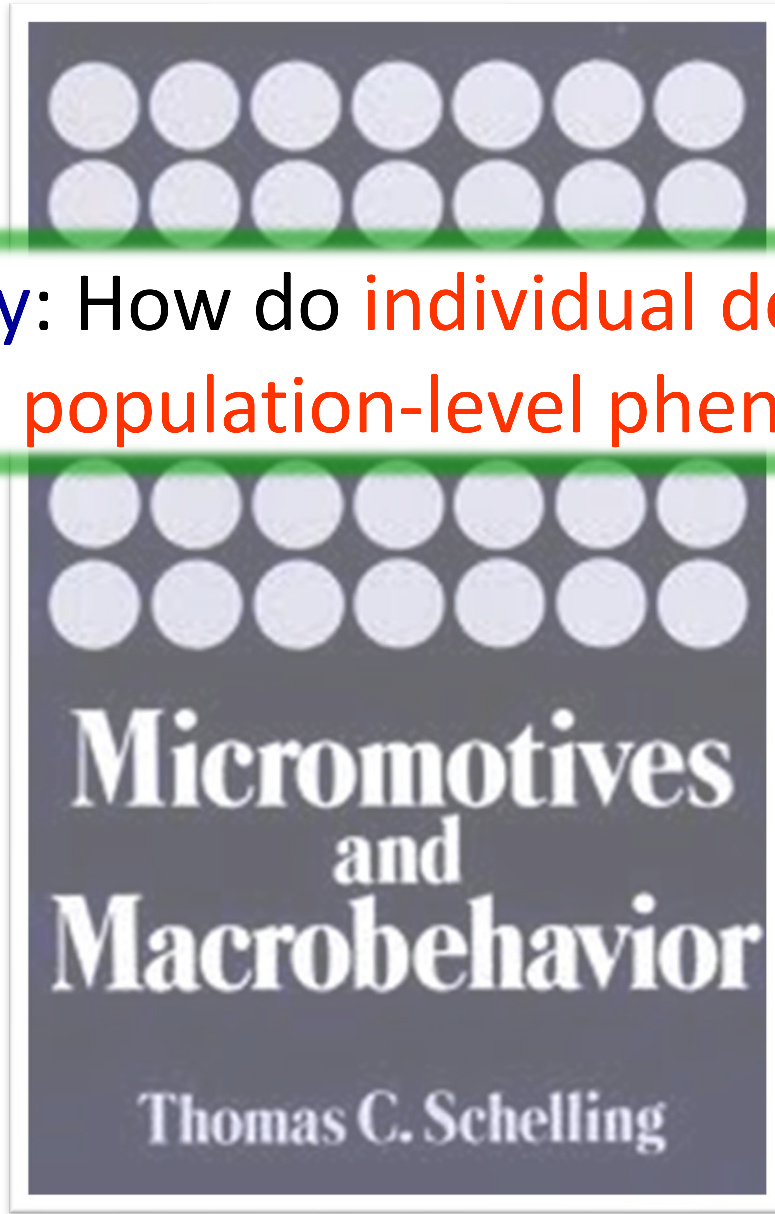
Schelling Segregation

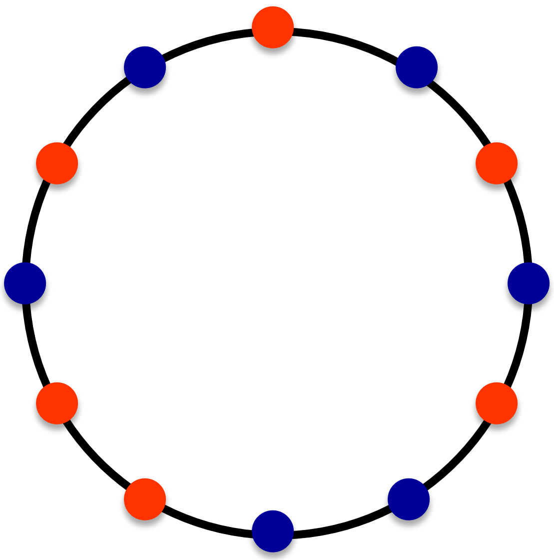
NICOLE IMMORLICA, MICROSOFT

BASED ON JOINT WORKS WITH CHRISTINA BRANDT, GAUTAM KAMATH, ROBERT D. KLEINBERG, BRENDAN LUCIER, AND M. ZADOMIGHADDAM

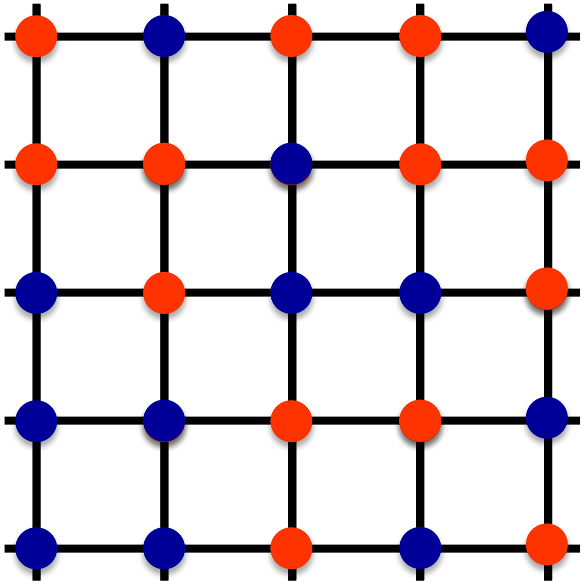
Theory: How do individual decisions impact population-level phenomena?



MODEL:

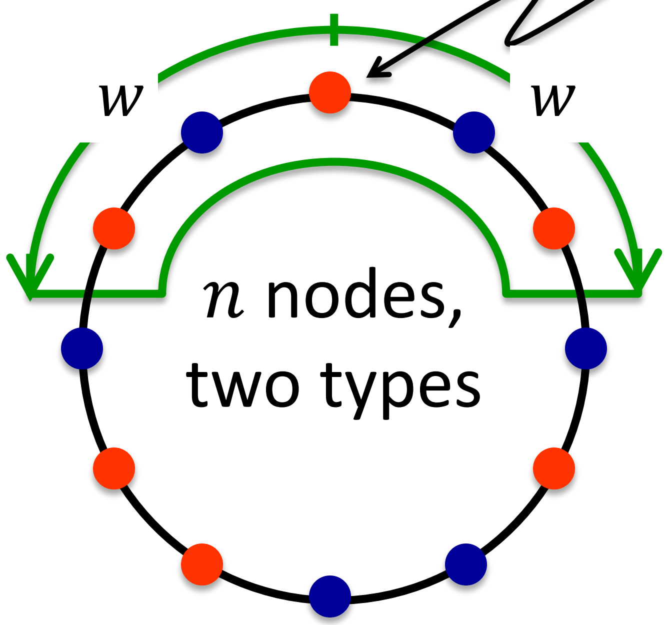


one-dimensional



two-dimensional

INDIVIDUAL PREFERENCES:



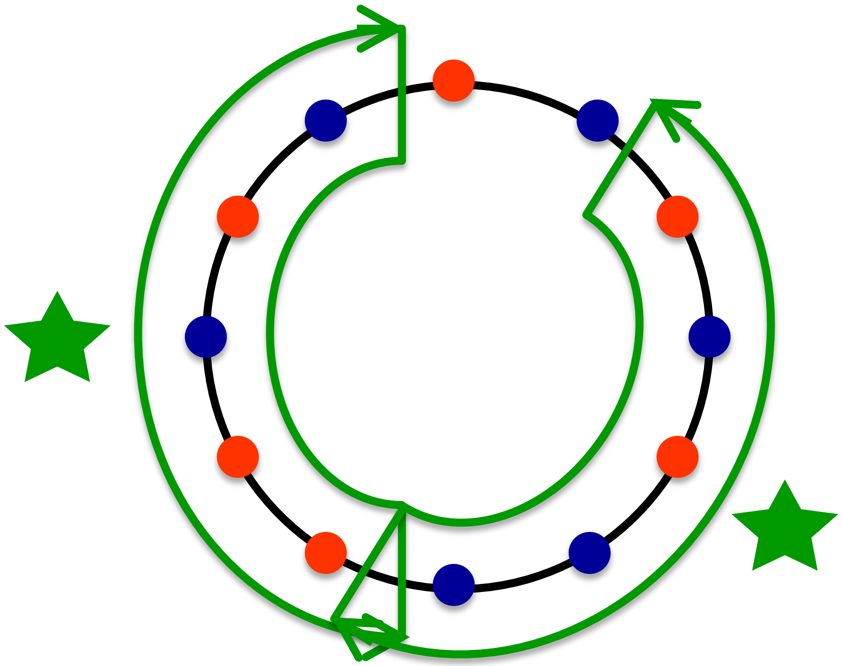
neighborhood =
$2w + 1$ nearest neighbors,
happy if at least $\tau = 0.5$
neighbors of like-type

model parameters:

- tolerance $\tau = 0.5$
- window size w
- society size n

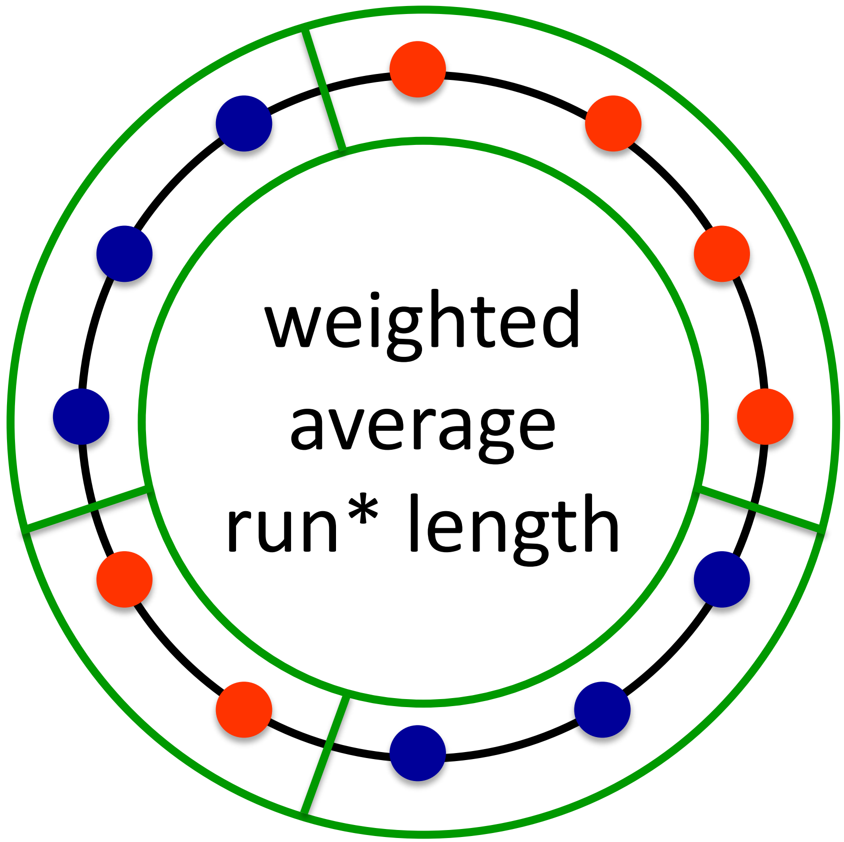
Pr[● = ●] = Pr[● = ●] = $^1/_2$

INDIVIDUAL BEHAVIOR:



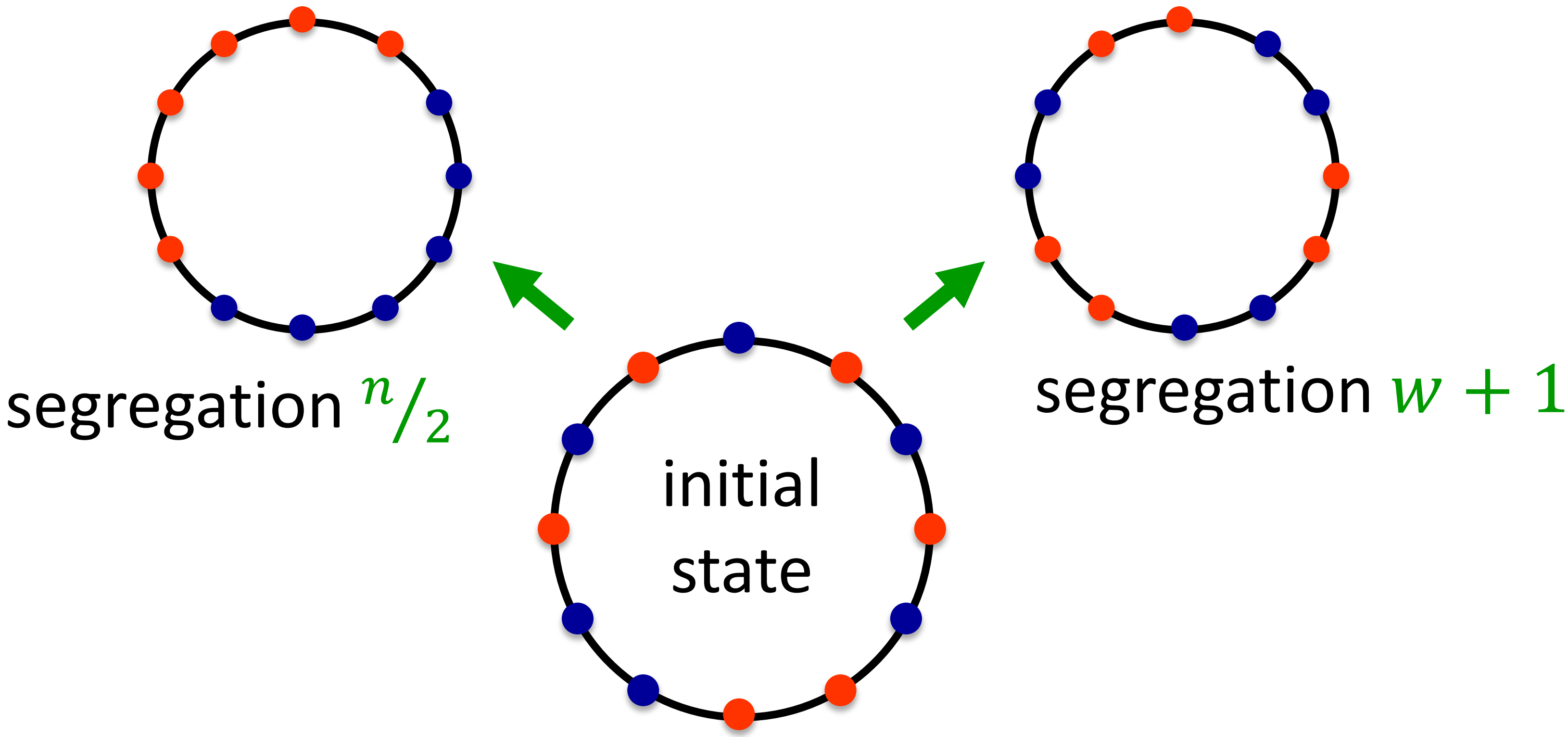
Each day, two randomly selected individuals swap nodes if both unhappy oppositely-colored.

MEASURE OF SEGREGATION:



*A run is a maximal sequence of like-colored individuals.

EMERGENT STRUCTURE:



Behavior may converge to a variety of states.

HISTORY:

Simulations: For $n = 70$ and $w = 4$, the average segregation was 12.

- societal impact: shifted discourse about segregation which until then had been attributed to discrimination.
- theoretical impact: became archetypical example of global emergent structure from simple local rules.

[Schelling'69]

HISTORY:

App... matio... dic... on

...on

...ity ϵ,

[Zhang'04]

But approximations have
exponential mixing time;
predictions contradict simulations.

- spin systems ... onochr... atic with high probability a... emperatu... e approaches 0.

[Bhakta, Miracle, and Randall'14]

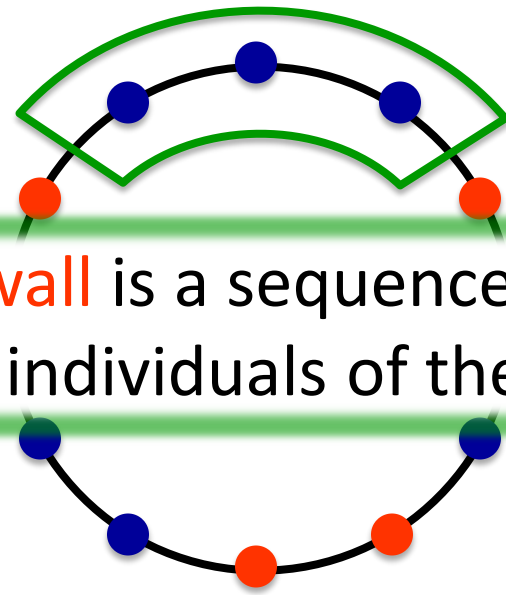
TODAY:

The dynamics converge after poly(n) steps with $O(w)$ segregation.

The distribution of run lengths is such that for all $\lambda > 0$, the probability a randomly selected node is in a run of length $> \lambda w$ is bounded above by c^{λ} for some constant $c < 1$.

[Brandt, Immorlica, Kamath, Kleinberg, 2012]

KEY STRUCTURE:

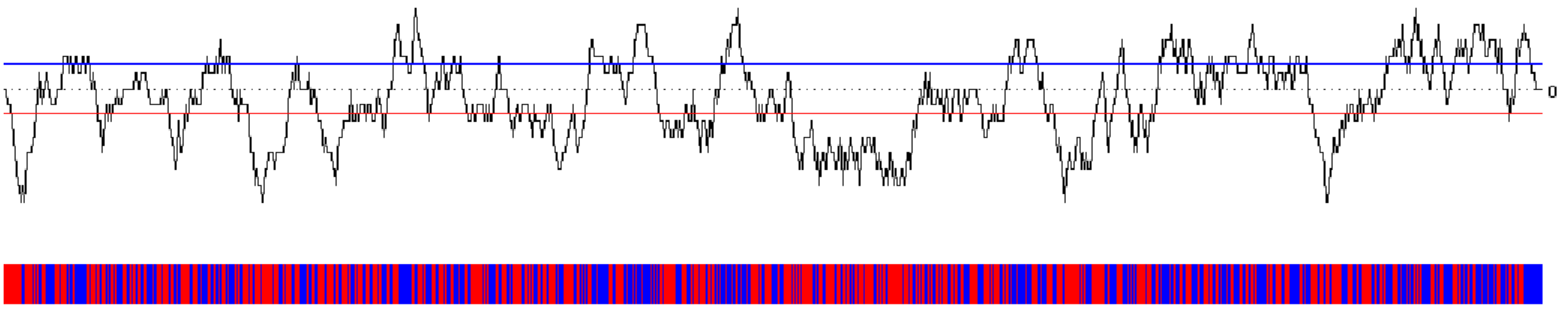


Defn. A firewall is a sequence of $w + 1$ consecutive individuals of the same type.

Claim: Firewalls stable with respect to dynamics.

Corollary: Segregation at most distance bt firewalls.

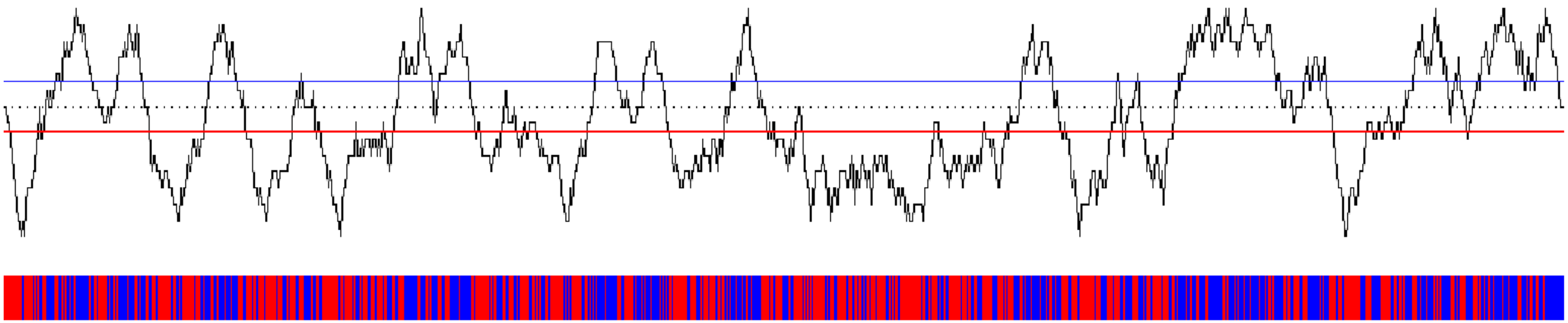
BIRTH OF FIREWALLS:



time $t \; = \; 0$

Simulation with $n \; = \; 10000, w \; = \; 10.$

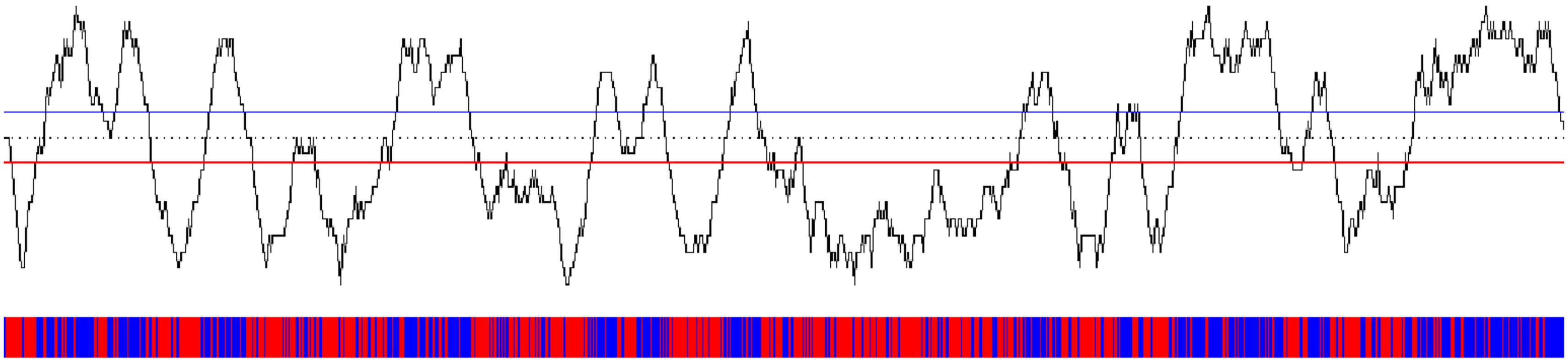
BIRTH OF FIREWALLS:



time $t = 40$

Simulation with $n = 10000, w = 10.$

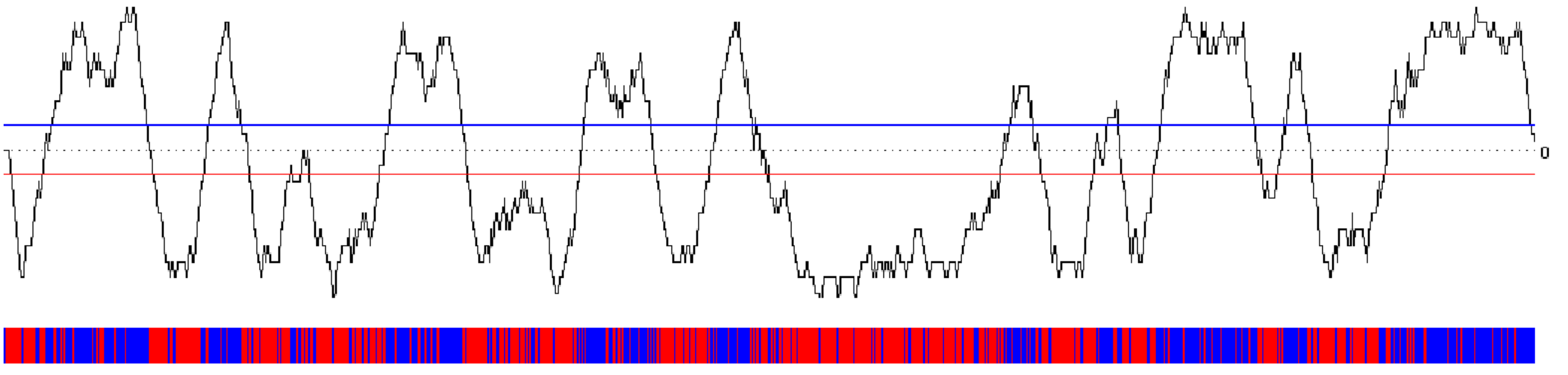
BIRTH OF FIREWALLS:



time $t = 80$

Simulation with $n = 10000, w = 10.$

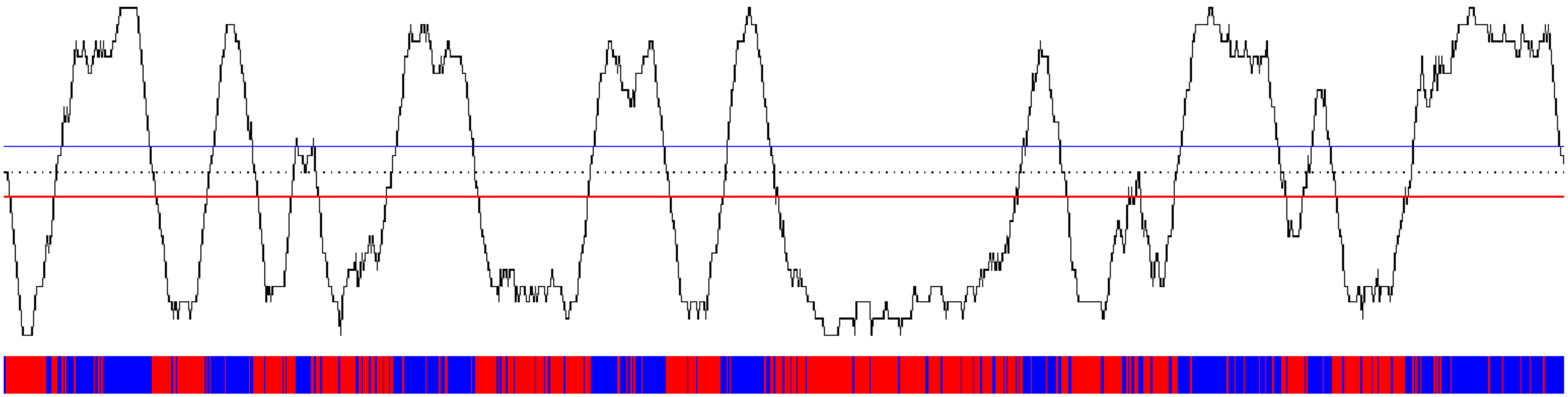
BIRTH OF FIREWALLS:



time $t = 120$

Simulation with $n = 10000, w = 10.$

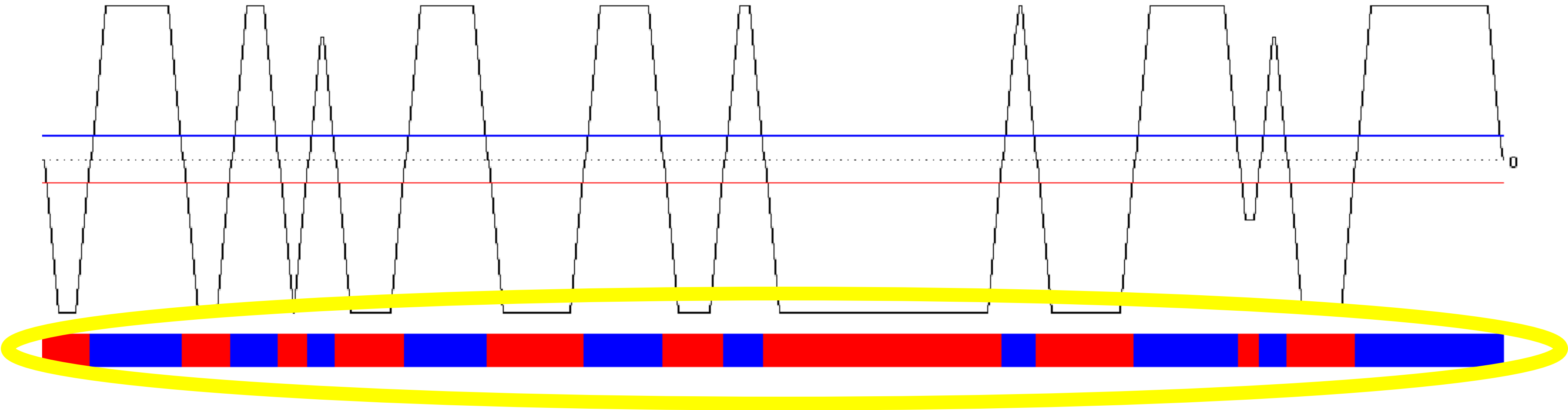
BIRTH OF FIREWALLS:



time $t = 160$

Simulation with $n = 10000, w = 10.$

EMERGENT STRUCTURE:



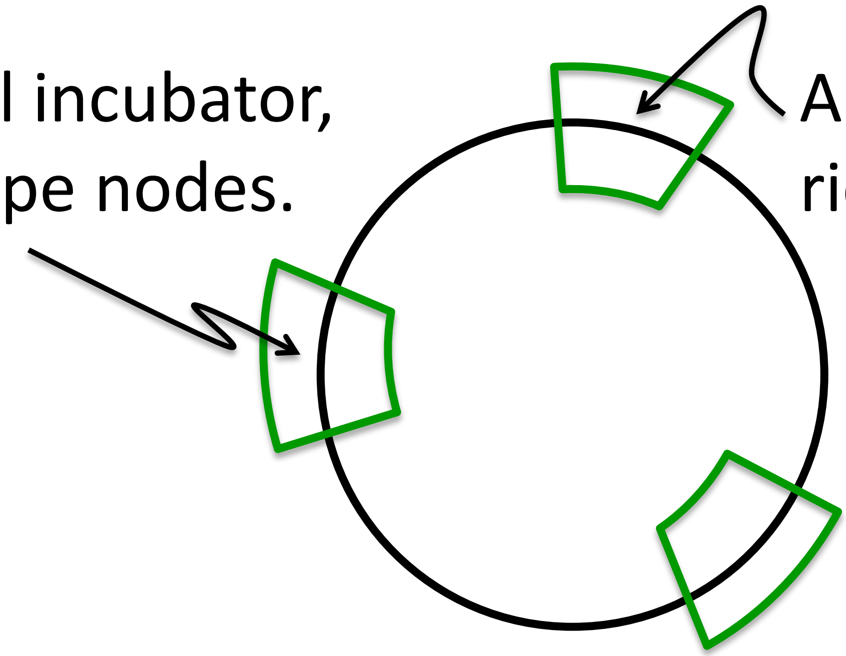
time $t = 260$ (final)

Simulation with $n = 10000, w = 10.$

TECHNIQUE:

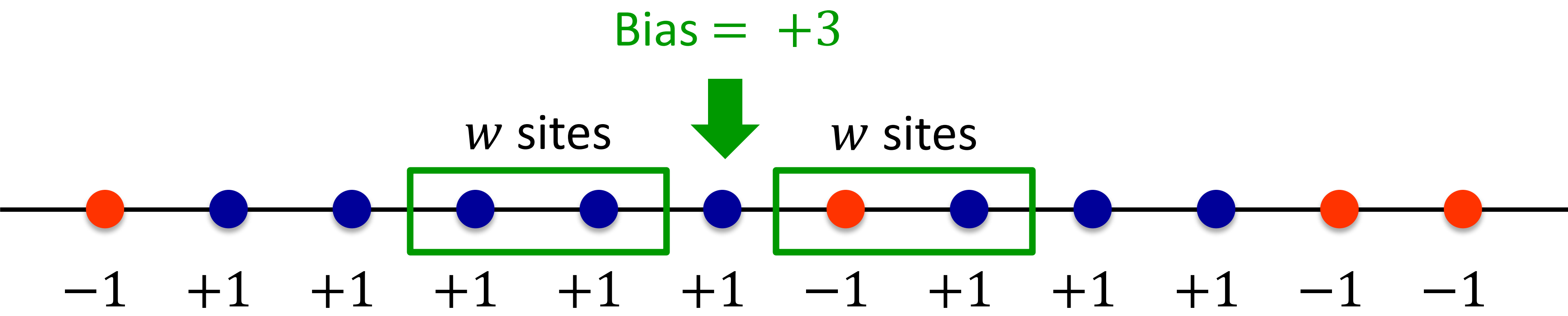
A ● firewall incubator, rich in ●-type nodes.

A ● firewall incubator, rich in ●-type nodes.



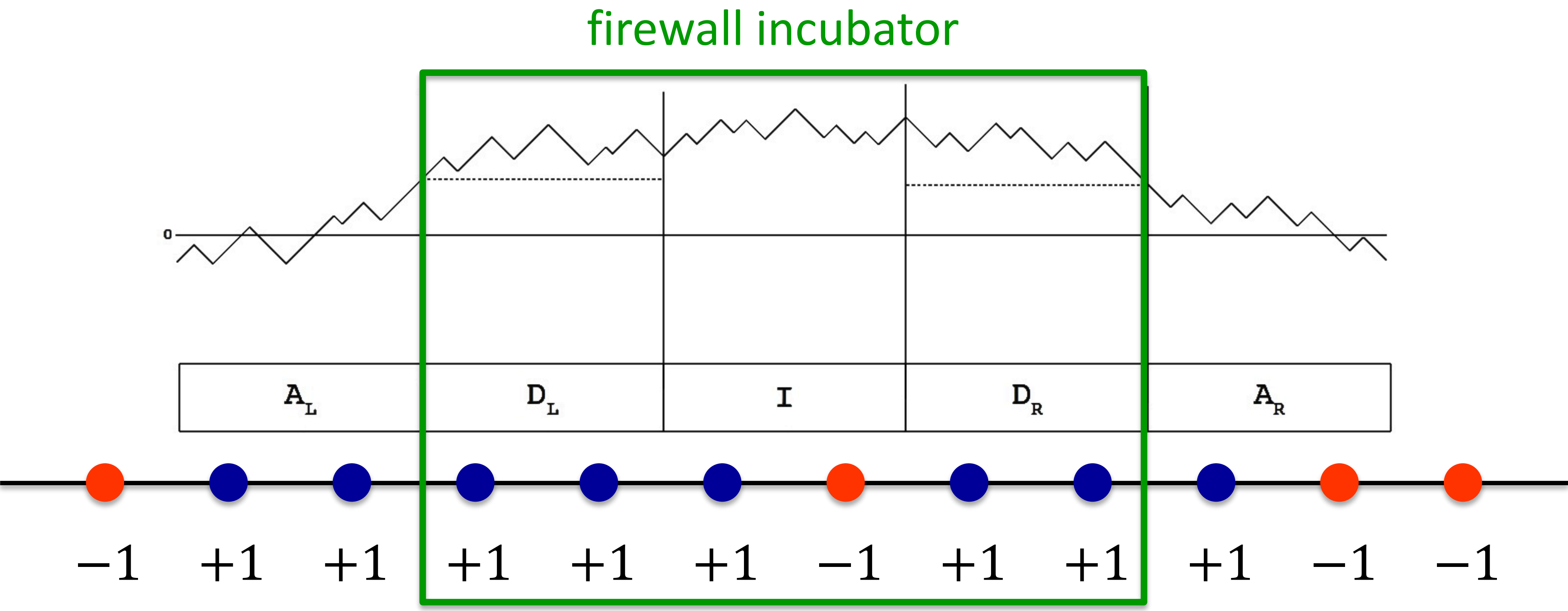
1. Define *firewall incubators*, frequent at initialization.
2. Show firewall incubators are likely to become firewalls.

FIREWALL INCUBATORS:



Definition. The bias of a node at time t is the sum of the signs of sites in its neighborhood.

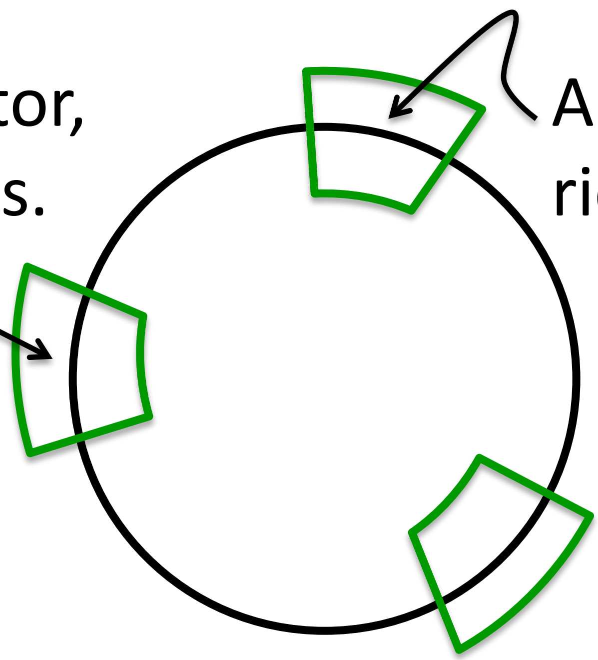
FIREWALL INCUBATORS:



Definition. A firewall incubator is a sequence of 3 sufficiently long and sufficiently biased blocks.

TECHNIQUE:

A ● firewall incubator,
rich in ●-type nodes.



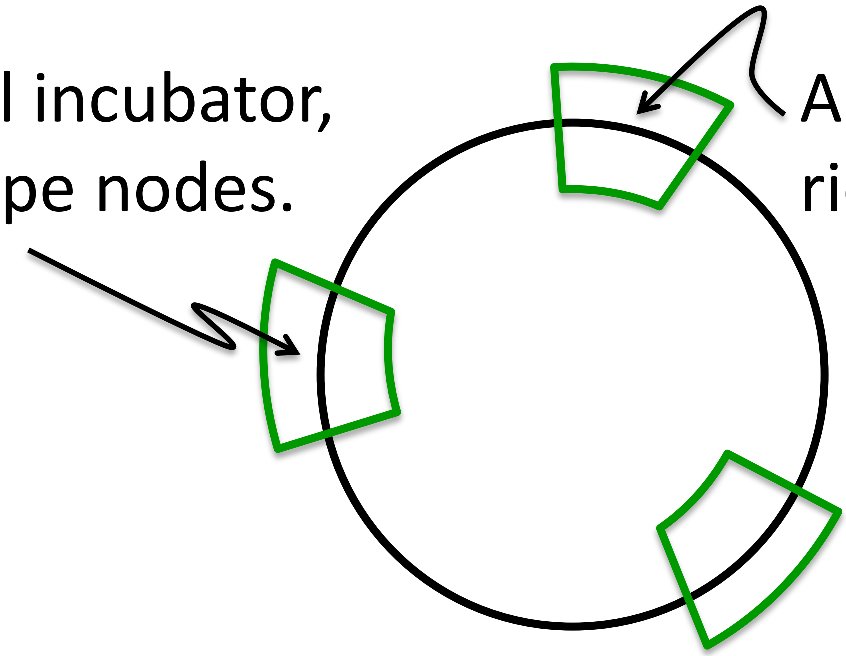
A ● firewall incubator,
rich in ●-type nodes.

1. Define *firewall incubators*, frequent at initialization.
2. Show firewall incubators are likely to become firewalls.

TECHNIQUE:

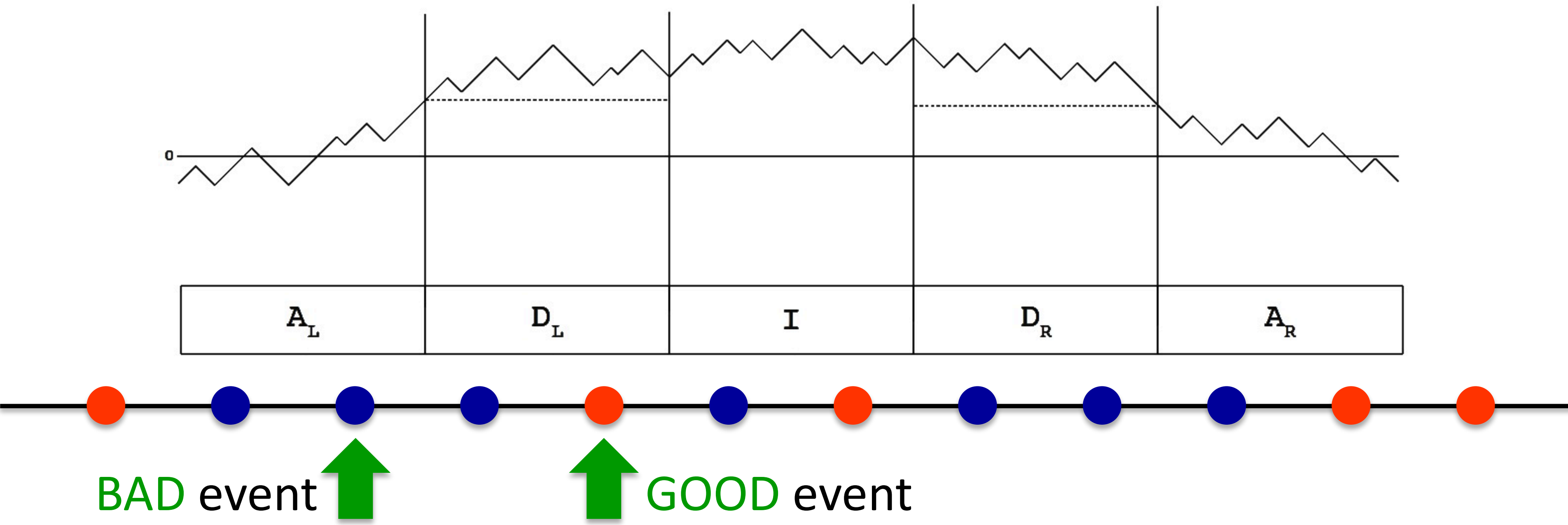
A ● firewall incubator,
rich in ●-type nodes.

A ● firewall incubator,
rich in ●-type nodes.



1. Define *firewall incubators*, frequent at initialization.
2. Show firewall incubators are likely to become firewalls.

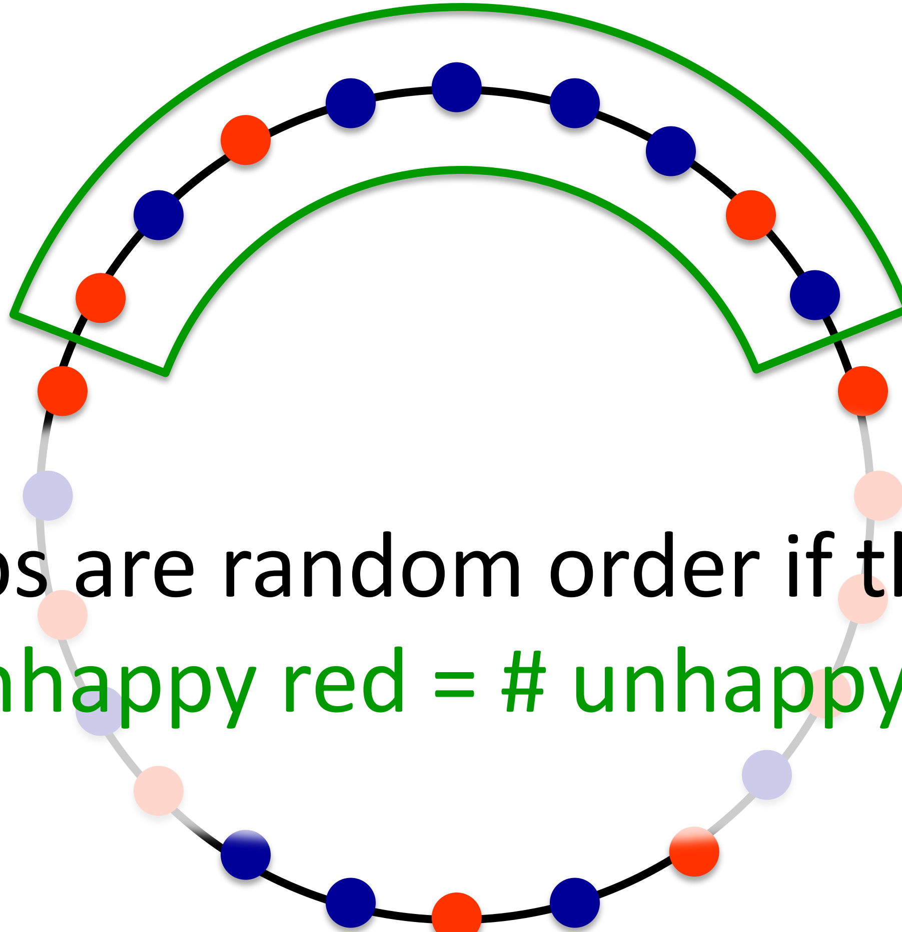
THE BIRTH OF A FIREWALL:



To show this incubator turns into a blue firewall:

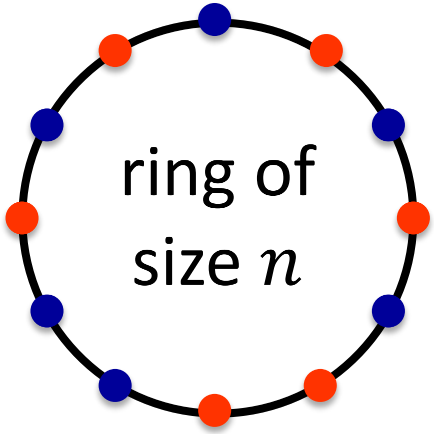
1. analyze random order of flips instead of swaps
2. argue due to bias all good events happen before too many bad events with probability $O(1/w)$.

FLIPS VERSUS SWAPS:



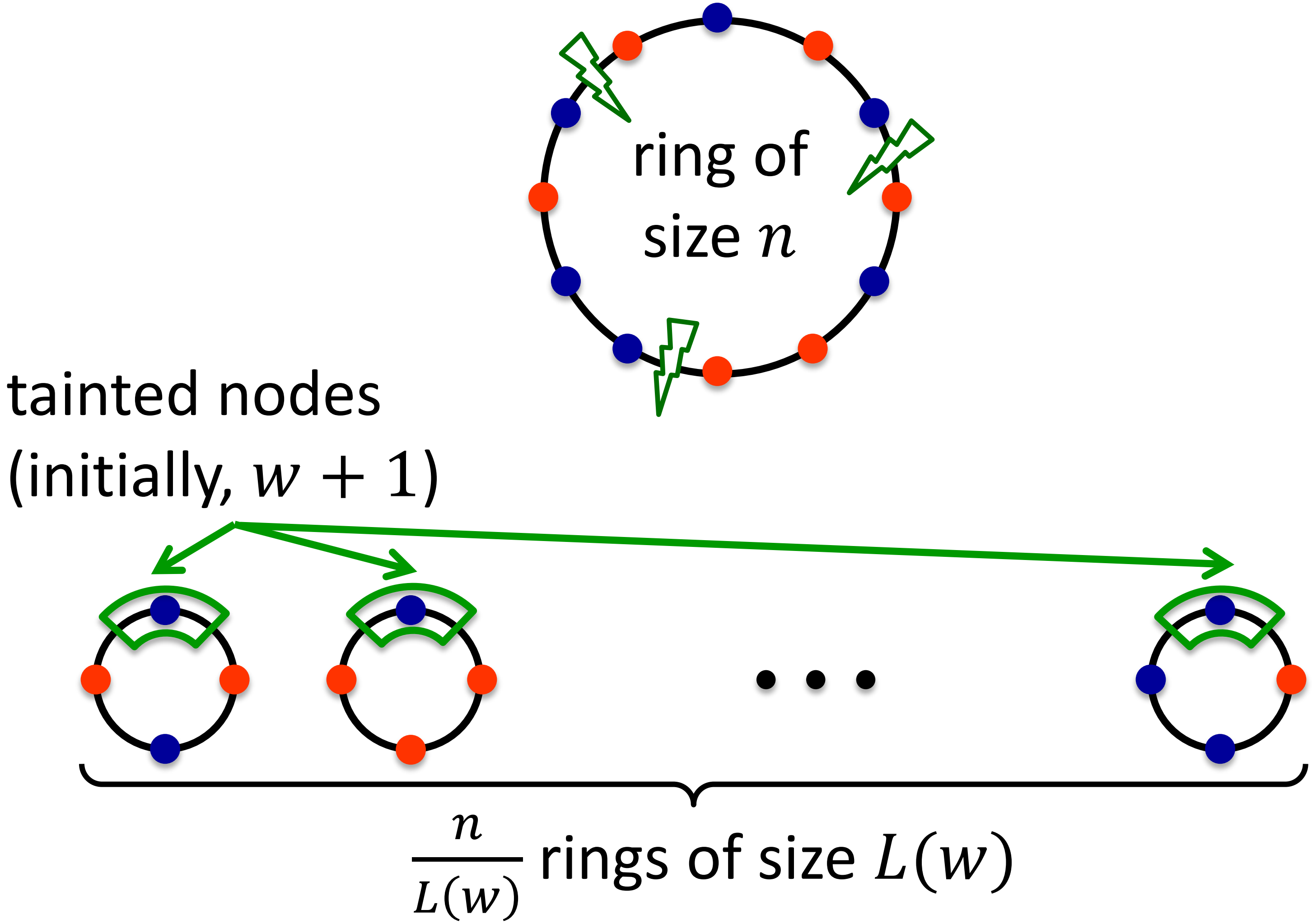
Swaps are random order if there is
unhappy red = # unhappy blue.

DEFINE STATE VARIABLES:

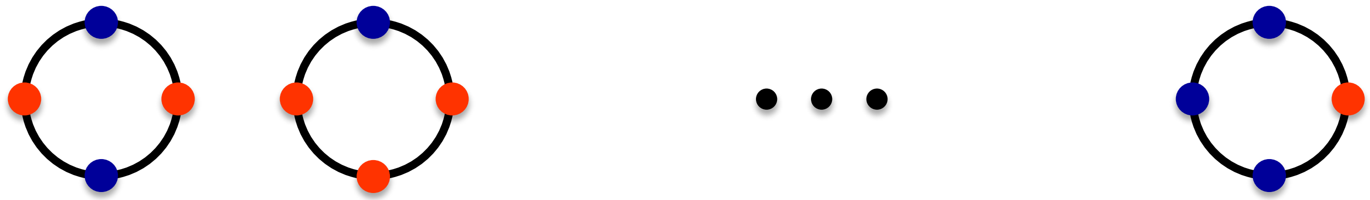


- $\sigma \in \{-1, +1\}^{w+1}$ is a labeling of a neighborhood
- $X_\sigma(t) =$ # of nodes with neighborhood labeling σ

DEFINE STATE VARIABLES:



DEFINE STATE VARIABLES:



- $\sigma \in \{-1, +1\}^{L(w)}$ is a labeling of a subring
- $X_\sigma(t)$ = number of nodes i such that subring containing i has label σ at time t (clockwise starting from i)

DEFINE STATE VARIABLES:

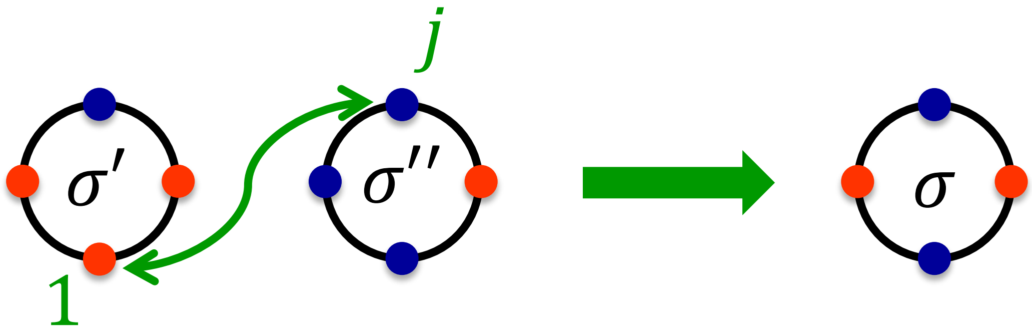
- $\sigma \in \{-1, +1\}^{L(w)}$: labeling of a subring
- $X_\sigma(t)$: label clockwise from i is σ at time t

Question. # unhappy red – # unhappy blue?

$$\Delta(\vec{X}) = \sum_{\sigma} u(\sigma)\, X_\sigma(t)$$

where $u(\sigma) = +1$ if starts with unhappy red,
and $u(\sigma) = -1$ if starts with unhappy blue

CALCULATE EXPECTATION:



For all j, σ', σ'', let $\mathrm{a}_\sigma(j, \sigma', \sigma'') = 1$ if swapping node 1 of σ' with node j of σ'' creates labeling σ
(and correspondingly for -1).

$$E[\, X_\sigma(t+1) - X_\sigma(t) | G_t \,]$$
$$= \sum_{j,\sigma',\sigma''} 2a(j, \sigma', \sigma'') \left(\frac{X_{\sigma'}(t)}{n}\right)\left(\frac{X_{\sigma''}(t)}{n}\right) + O\left(\frac{L(w)}{n}\right)$$

CHECK WORMALD CONDITONS:

$$E[\, X_\sigma(t+1) - X_\sigma(t) | G_t \,]$$

$$= \underbrace{\sum_{j,\sigma',\sigma''} 2a(j,\sigma',\sigma'') \cancelto{x_{\sigma'}(y)}{\left(\frac{X_{\sigma'}(t)}{n}\right)} \cancelto{x_{\sigma''}(y)}{\left(\frac{X_{\sigma''}(t)}{n}\right)}}_{f(y,\{x_\sigma\})} + O\left(\frac{L(w)}{n}\right)$$

- Bounded? $|X_\sigma(t+1) - X_\sigma(t)| \le 2L(w)$
- Lipschitz? $f(y,\{x_\sigma\})$ bounded quadratic

SOLVING DIFF EQ:

$$\frac{dx_\sigma(y)}{dy} = \sum 2a(\dots)\, x_{\sigma'}(y) x_{\sigma''}(y)$$

We actually only care about balance:

$$\Delta(\vec{x}) = \sum_\sigma u(\sigma) x_\sigma$$

where $u(\sigma) = +1$ if first node is unhappy blue, and $u(\sigma) = -1$ otherwise.

AVOIDING DIFF EQS:

Flipping labels. Let $\bar{\sigma}$ be "flip" of labeling σ and $\iota(\vec{x})$ be vector whose σ^{th} component is $x_{\bar{\sigma}}$.

1. vector $\iota(\vec{x})$ is permutation of state vector $\vec{x}$
2. fixed point set $\{\vec{x} | \iota(\vec{x}) = \vec{x}\}$ is
   - balanced as $x_\sigma = x_{\bar{\sigma}}$ and $u(\sigma) = -u(\bar{\sigma})$
   - invariant under diff eq by symmetry
3. initial state close to fixed point set w.h.p. and all close points have high balance

CONCLUDING BALANCE:

Theorem. For all sufficiently large n, with high probability the # of unhappy red and blue are approximately balanced for sufficiently long.

Proof Sketch. This is true of differential equation and therefore true of discrete process for as long as diff eq tracks discrete process closely.

BOUNDING SEGREGATION:

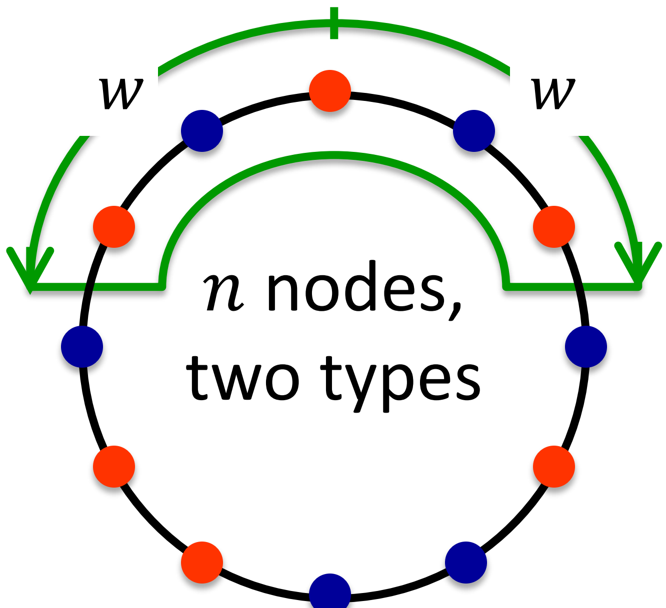
Theorem. Average segregation is $O(w^2)$.

Proof. A block of length $O(w)$ contains an incubator with constant probability. This incubator turns into a firewall with probability $\Omega({}^{1}/_{w})$.

Linear result: define a stronger incubator.

EPILOGUE:

On preaching tolerance…



happy if at least $\tau = 0.5$ neighbors of like-type

Segregation exponential in w for $\tau = 0.5 - \epsilon$!

[Barmpalias, Elwes, Lewis-Pye, 2014]