Differential Equation Method

Nicole Immorlica

Microsoft Research

Simons Graph Limits Bootcamp

Outline

- Define method
- Apply to easy example
- Sketch application to hard example

Differential Equation Method

(Many) discrete time processes $\approx$ continuous time counterparts (solved with differential eqns).

Theorem (Wormald)

Fix RVs $X_1, \ldots, X_k$ for const. k defined by discrete time Markov process with states G_t of size n s.t.

- $|X_i(t+1) - X_i(t)| \leq \beta$ *for some constant* β
- $E[X_i(t+1) - X_i(t) | G_t] \leq f_i(\frac{t}{n}, X_1(t)/n, \ldots, X_k(t)/n) + o(1)$ *for some Lipschitz funcs* f_i.

Then

- $\frac{dx_i}{dy} = f(y, x_1, \ldots, x_k)$ *has unique soln satisfying init. conditions*
- $X_i(t) = nx_i(t/n) + o(n)$ *a.a.s. where* $x_i(0) = \frac{1}{n}X_i(0)$.

Balls and Bins

Model:

- n bins, initially empty
- $m = \alpha n \log n$ balls
- process G_t places one ball in random bin at each time t
- $X_i(t) = \#$ bins w/exactly i balls at time t, for $0 \leq i \leq k$

Question

What is dist. of $X_i(t)$?

Balls and Bins

Easy soln:

- n ways to choose bin
- $\binom{t}{i}$ ways to choose i balls for bin
- $(1/n)^i$ prob. bin gets chosen balls
- $(1-1/n)^{t-i}$ prob. bin doesn't get other balls

Hence:

$$X_i(t) = n\binom{t}{i}\left(\frac{1}{n}\right)^i\left(1-\frac{1}{n}\right)^{t-i} \sim n\frac{(t/n)^i}{i!}e^{-t/n}.$$

Wormald Approach

Change in X_i at $t+1$ if ball lands in bin with i balls or $i-1$ balls:

Difference Equation

$$E[X_i(t+1) - X_i(t)|G_t] = -\frac{X_i(t)}{n} + \frac{X_{i-1}(t)}{n}$$

where $X_{-1} := 0, X_0(0) = n$, $X_i(0) = 0$ for $1 \leq i \leq k$.

Wormald Approach

From Wormald Thm., choose $f_i(y, x_1, \ldots, x_k) = -x_i + x_{i-1}$. Then:

Differential Equation

$$\frac{dx_i}{d_y} = -x_i(y) + x_{i-1}(y)$$

where $x_{-1} := 0$, $x_0(0) = 1, x_i(0) = 0$ for $1 \leq i \leq k$.

Wormald Approach

For $i = 0$, simple to solve:

$$\frac{dx_0}{dy} = -x_0(y) \to e^{-y}$$

More generally,

Differential Equation Solution

$$x_i(y) = \frac{y^i}{i!}e^{-y}$$

Wormald Approach

Answer

Distribution of $X_i(t)$ is $nx_i(t/n) + o(n)$, i.e.,

$$X_i(t) = n\frac{(t/n)^i}{i!}e^{-t/n} + o(n)$$

asymptotically almost surely.