Sparse Random Graphs-II

Souvik Dhara

Research Fellow, Simons Institute

Graph Limits and Processes on Networks:
From Epidemics to Misinformation Boot Camp

Recap: Erdős-Rényi subcritical phase

➢ Considered $\mathrm{ER}_n(\frac{\lambda}{n})$: Erdős-Renyi random graph with n vertices and edge probability $\frac{\lambda}{n}$

➢ Studied relation between exploration and branching processes, and showed that exploration can be dominated by a Poisson(λ) branching process

➢ For $\lambda < 1$: Showed $\mathbb{E}[C(v)] = O(1)$

Theorem: Subcritical $\mathrm{ER}_n(\frac{\lambda}{n})$

If $\lambda < 1$, then

$$\frac{\max_v C(v)}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{I_\lambda}, \qquad \text{where } I_\lambda = \lambda - 1 - \log \lambda$$

Recap: Erdős-Rényi supercritical phase

We proved

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Let $C_{(i)}$:= i-th largest component of $\mathrm{ER}_n(\frac{\lambda}{n})$. If $\lambda > 1$, then

$$\frac{C_{(1)}}{n} \xrightarrow{\mathbb{P}} \zeta_\lambda > 0 \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\mathbb{P}} 0$$

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➡ Was shown by growing two neighborhoods, and they must intersect when neighborhoods become large enough $\Omega(\sqrt{n})$

Plan today

- Consider other models with more realistic features, summarize results, and give heuristics for applying BP approximation technique

- Percolation, Epidemics: Use Path counting to prove results on general graphs and see whether we can apply these results to sparse graphs

- Using Stochastic Process Convergence in to find limits of component sizes of Random Graphs

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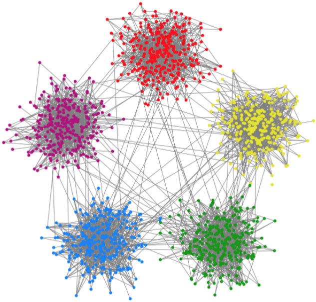
Stochastic Block Model

➢ Model with global community structure – popular model in CS/ML for *community detection problem*



Pic source: Abbe (2018)

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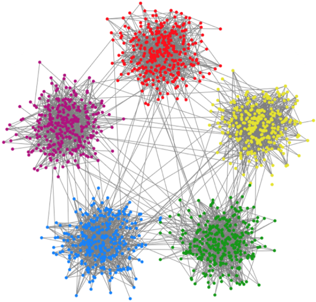

Model description:

1. $K \geqslant 2$ communities, size of community $i = n_i$, where $\frac{n_i}{n} \to \rho_i$, $\rho_i > 0$

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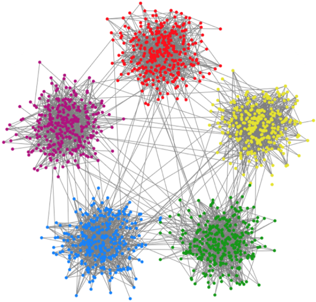

Model description:

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2. Edge between community i, j w.p. $\frac{P_{ij}}{n}$ ($P_{ij} \in (0, 1)$), independently

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Local neighborhoods of Stochastic Block Model

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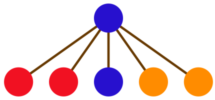

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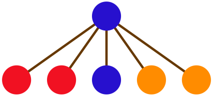

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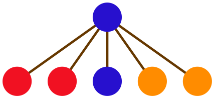

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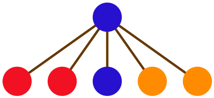

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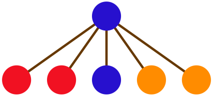

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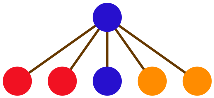

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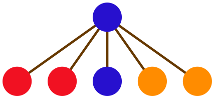

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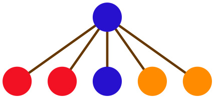

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➢ There is a more challenging and general models with continuum of colors
➡ See foundational work of Bollobás, Janson, Riordan (2007) on general inhomogeneous random graphs

Up next: Model for degree-heterogeneous networks

Pic source: Wikimedia Commons

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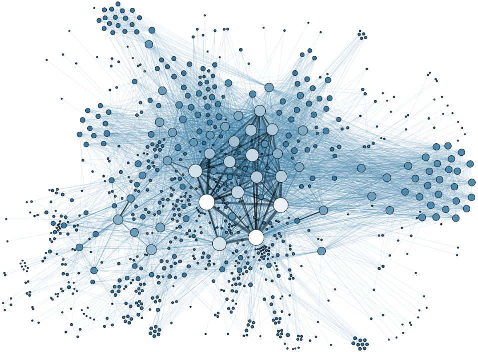

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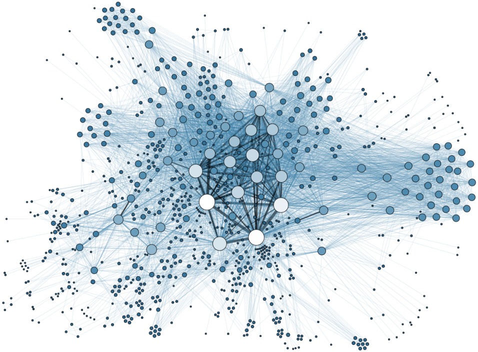

- Such degree-heterogenous networks with hubs are common occurrences
  - The degree distribution can be power-law, truncated power-law etc., but it is definitely quite far from Poisson
  - Need a simple, analytically tractable model – **Configuration Model**

Pic source: Wikimedia Commons

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Canonical model to generate graphs with given degrees $\mathbf{d} = (d_1, \ldots, d_n)$

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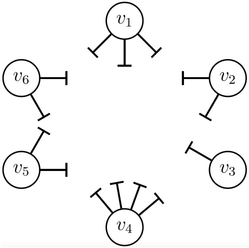

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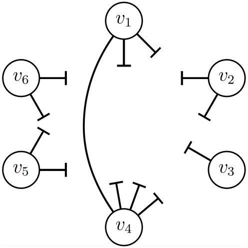

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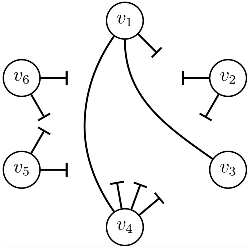

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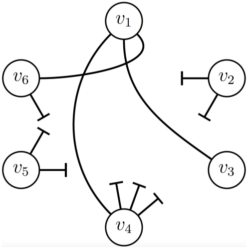

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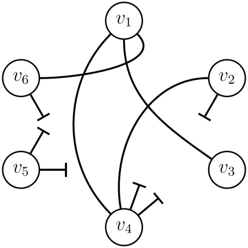

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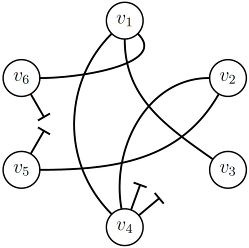

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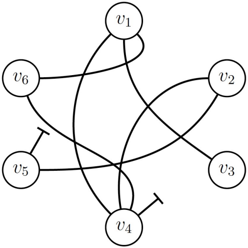

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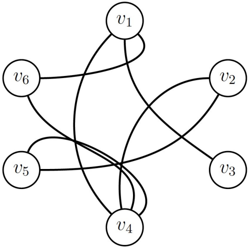

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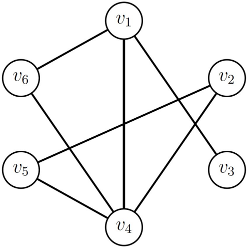

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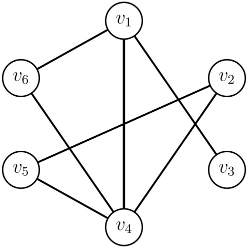

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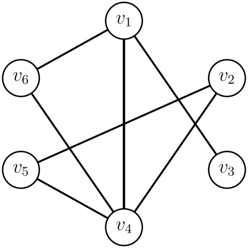

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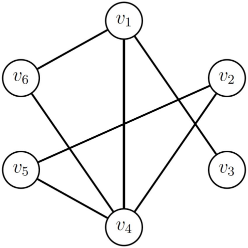

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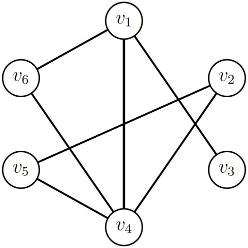

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Brief History:

➢ Introduced by Bender and Canfield (1978), Bollobás (1980) to study uniform random regular graphs

➢ Giant emergence studied by Molloy & Reed (1995, 1998)

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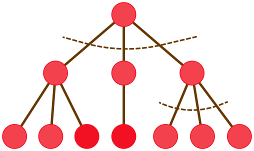
➢ Most often, one also assumes $\mathbb{E}[D_n^2] \to \mathbb{E}[D^2] < \infty$, which ensures

$$\liminf_{n\to\infty} \mathbb{P}(\mathrm{CM}_n(\mathbf{d}) \text{ is simple}) > 0 \qquad \text{Janson (2009)}$$

so that the results carry over to uniform graphs

Local neighborhoods of $\mathrm{CM}_n(\mathbf{d})$



BP approximation:

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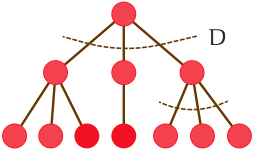

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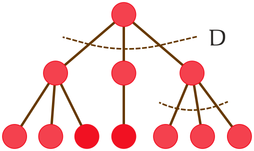

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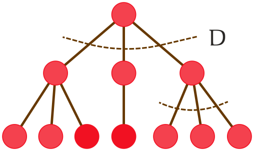

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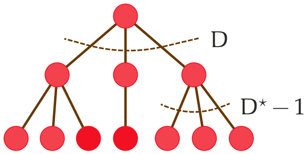

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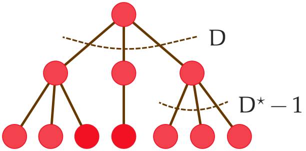

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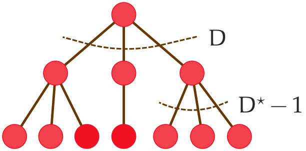

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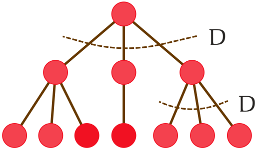

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Therefore,

$$\mathbb{P}(\text{BP survives}) = \zeta > 0 \quad \text{when } \nu := \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]} > 1$$

Emergence of Giant for $\mathrm{CM}_n(\mathbf{d})$

Let $\nu := \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}$

Theorem: Giant for $\mathrm{CM}_n(\mathbf{d})$

❶ For $\nu < 1$: $\frac{C_{(1)}}{n} \xrightarrow{\mathbb{P}} 0$ whp

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For the proof, there were two ingredients:

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- ➡ Will skip Two large components intersect

$$\mathbb{P}\left(C(u_1) \geqslant L, C(u_2) \geqslant L, u_1 \not\leftrightarrow u_2\right) \approx 0$$

Can be proved using similar ideas as ER, but is more complicated [a]

[a] see van der Hofstad (2021): *The giant in random graphs is almost local*

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➢ It may be that $\zeta = 1$, e.g., if $\mathbb{P}(D \geqslant 3) = 1$, then BP survives w.p. 1

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➢ Bollobás, Riordan, Spencer and Tusnády (2001) were to first study this model rigorously

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➢ If $m = 1$, this process produces a tree called preferential attachment tree

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- ➢ Graph is always connected, so no question of giant emergence here

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- Proof relies on Martingale arguments and Azuma-Hoeffding's inequality

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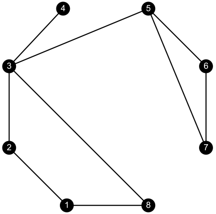
Dynamically evolving graphs: Preferential Attachment Model

- Vertices arrive sequentially and connects to vertices depending on degrees
- Leads to power-law degree distribution

Next lets study Percolation problem and its relation to Epidemic threshold

Percolation on finite networks

Percolation: Given graph G, keep each edge w.p. p independently



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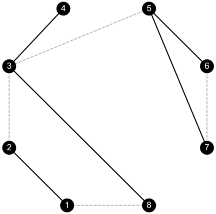

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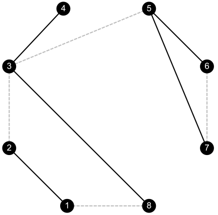

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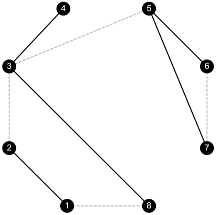

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Def: Percolation threshold

Let u be a uniform vertex. p_c called percolation threshold on $(G_n)_{n\geqslant 1}$ if for any $\varepsilon > 0$

- For $p < p_c(1-\varepsilon)$: $\frac{C(u)}{n} \xrightarrow{\mathbb{P}} 0$
- For $p > p_c(1+\varepsilon)$: $\frac{C(u)}{n} = \Theta(1)$ whp

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Finding epidemic threshold is same as finding percolation threshold...

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➢ For general graphs: Draief, Ganesh, Massoulié (2006)

Theorem

Suppose G is a connected graph. Let $\lambda_1(A)$ denote largest eigenvalue of adjacency matrix A.

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Proof: Using Path counting. On Board

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However, for sparse graphs, $\frac{1}{\lambda_1(A)}$ is not the right threshold...

See: Bollobás, Borgs, Chayes, Riordan (2010)

Percolation threshold on sparse random graphs

Fact: For any connected graph G

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➢ For general sparse graphs, percolation on G is always viewed as a random graph. So, percolation threshold can be obtained by verifying

1. *The percolated graph converges locally weakly*
2. *Two large components intersect*

Finally, lets conclude with a fascinating technique that combines Random Graph theory and convergence of Stochastic Process

Back to Erdős-Rényi

Erdős-Rényi (1960) showed for $\mathrm{ER}_n(\frac{\lambda}{n})$:

- For $\lambda < 1$: $C_{(1)} = O(\log n)$ whp
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Theorem: Critical $\mathrm{ER}_n(\frac{\lambda}{n})$

For $\lambda = 1$:

$$n^{-2/3}(C_{(i)})_{i \geqslant 1} \xrightarrow{d} X \quad \text{in } \ell^2$$

Description of X will be clear soon...

Aldous (1997)

Exploration process method

Key Idea:

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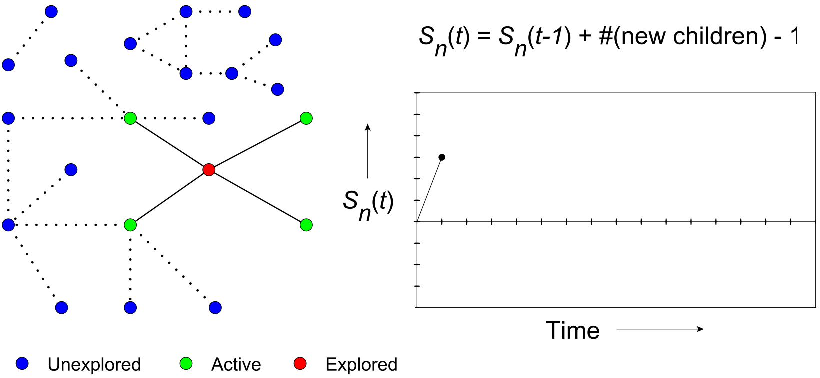

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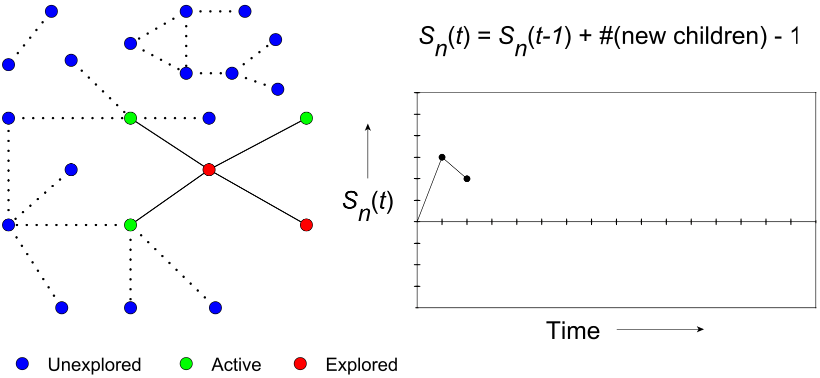

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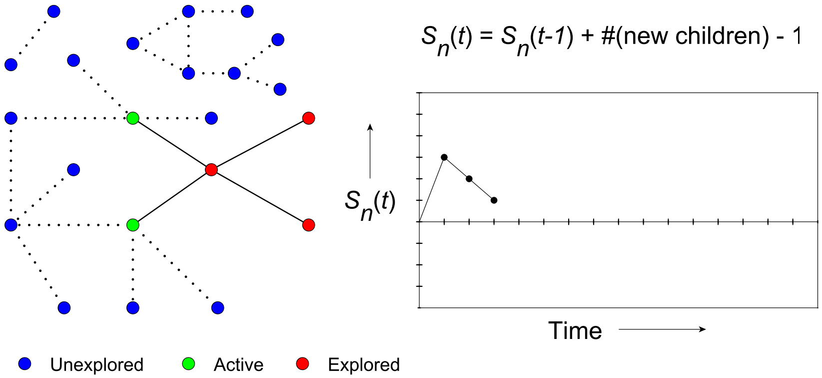

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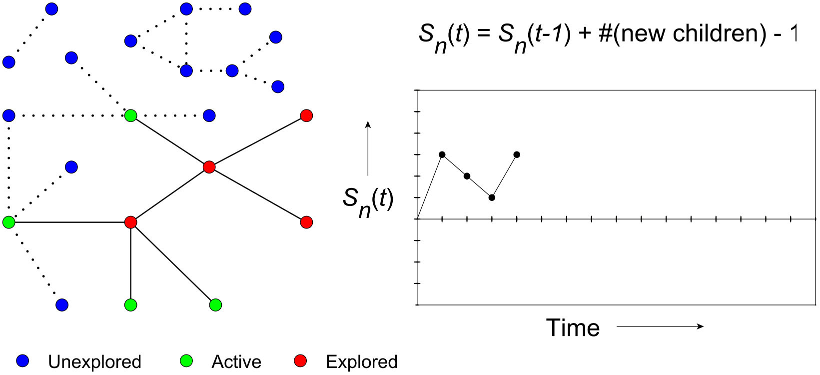

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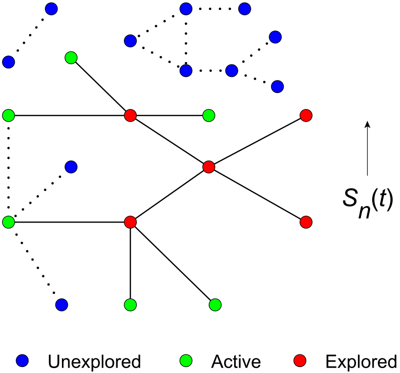
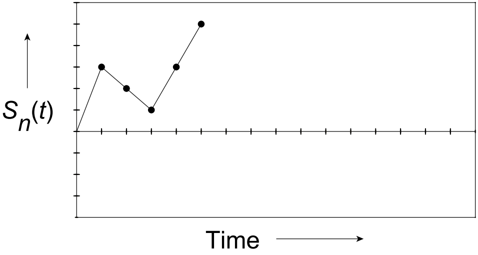


$S_n(t) = S_n(t\text{-}1)$ + #(new children) - 1



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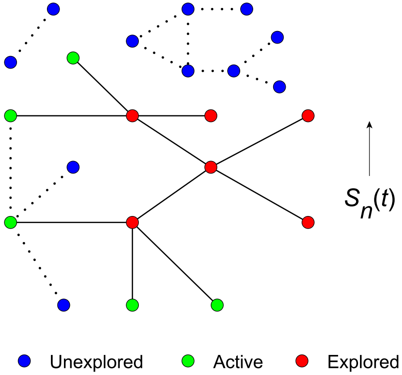

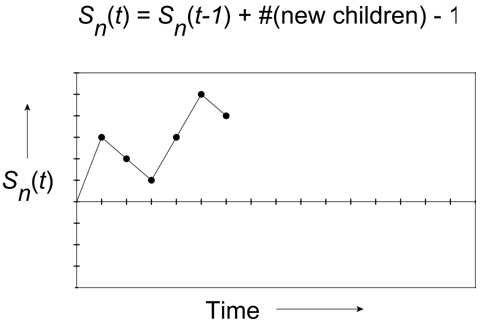

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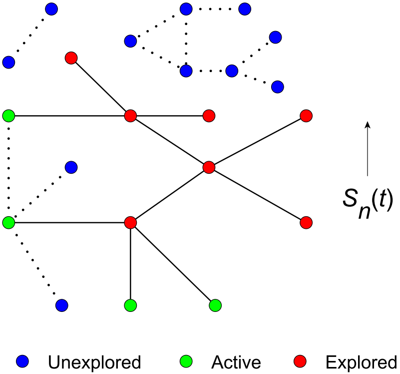

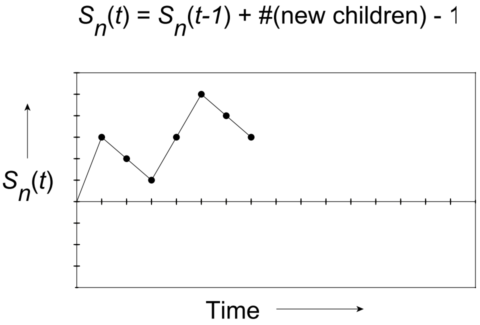

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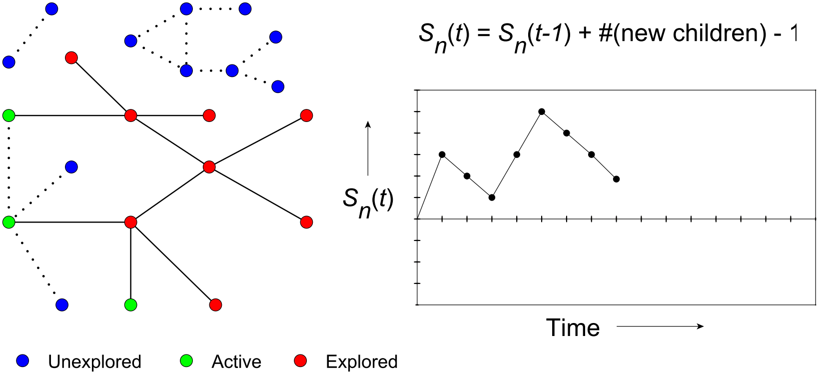

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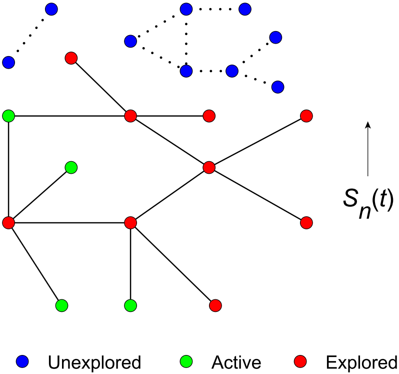

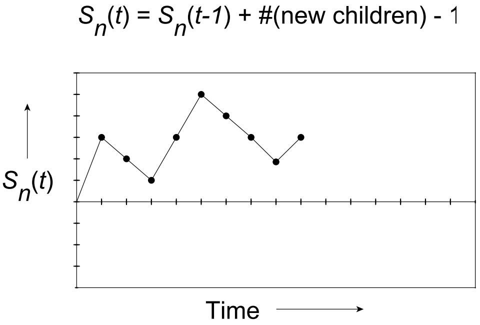

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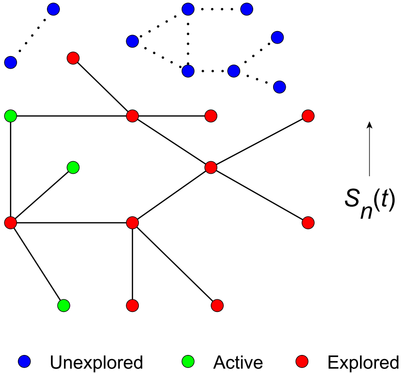

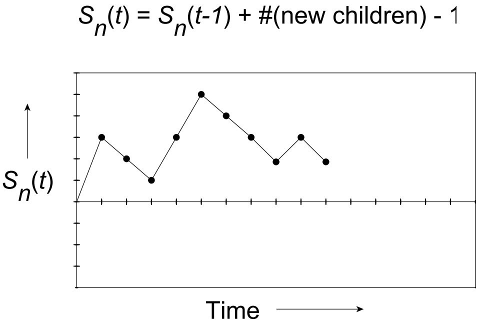

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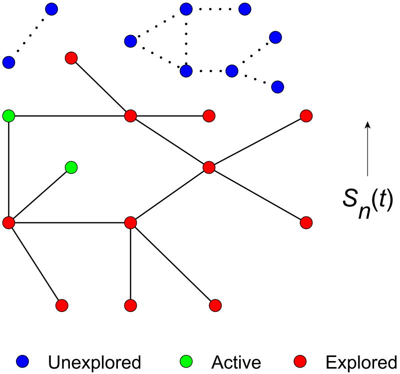

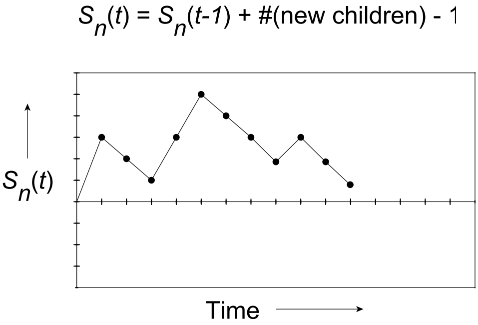

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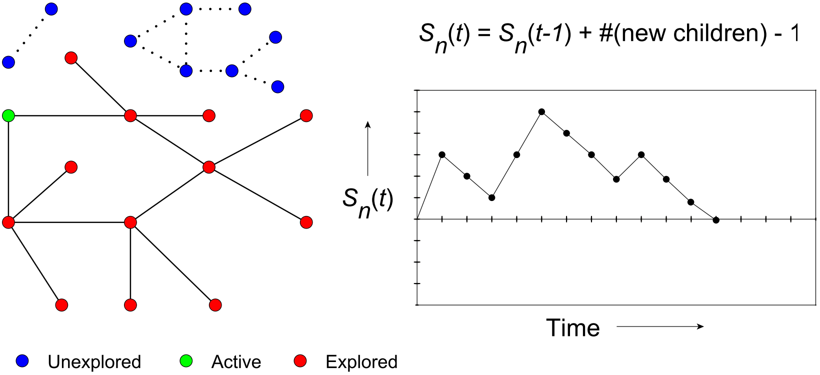

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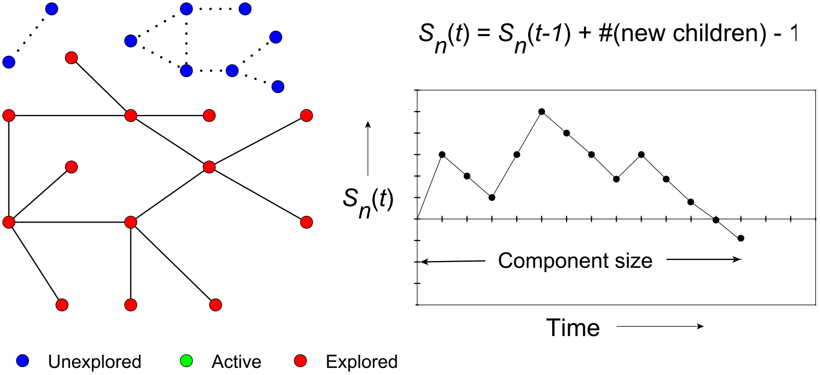

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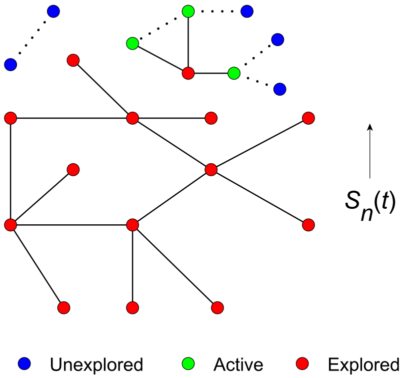

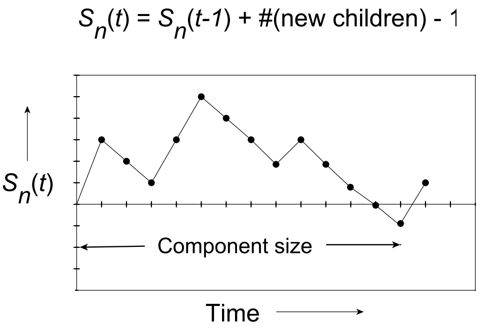

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Limit of exploration process gives limit of comp. sizes

Exploration process method contd.

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➢ Method also works for supercritical case. In that case the limit is deterministic. See Janson & Luczak (2007)

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- Used it to find non-degenerate limits of component sizes for $ER_n(\frac{\lambda}{n})$ with $\lambda = 1$

Further reading

Emergence of Giant and Random Graph Models

1. van der Hofstad: *Random graphs and complex networks Vol 1, Vol 2*
2. van der Hofstad: *The giant in random graphs is almost local*

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