Sparse Random Graphs-I

Souvik Dhara

Graph Limits and Processes on Networks:
From Epidemics to Misinformation Boot Camp

Lets start with a few questions...

How **Disease** becomes **Epidemic**?

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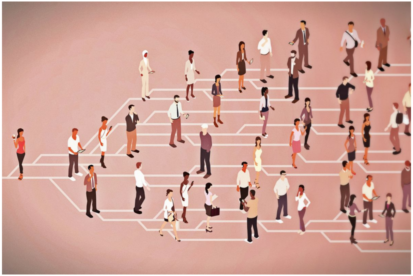

When does Misinformation reach a **large population**?

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These seemingly unrelated questions have a few commonalities:

1. There is an underlying *large* network with a *complex structure*
2. There is emergence of behavior having drastic impact, a.k.a. *phase transition*

➢ Random Graphs provide a *simplified probabilistic representation* to model these complex system.

- ➡ Capture structural properties (degree distribution, communities)
- ➡ Provide insight into emergence of different types of behavior such as phase transition

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Example: Will see how local neighborhood structure impact global properties like phase transition, typical distances

➢ Random Graphs serve to get provable guarantees for graph algorithms

Example: Heuristic algorithms for NP-hard problems such as graph partitioning, coloring

Plan

Today:

- Local Branching Process approximation technique on random graphs
- Explore its relation to *Giant Component Problem* on different models

Tomorrow:

- Applications to Percolation, Epidemics
- Using Stochastic Process convergence in Random Graphs

Let's start with the most elementary yet fundamental model...

Erdős-Rényi Random Graph

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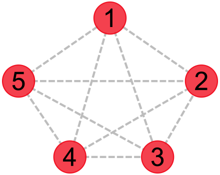

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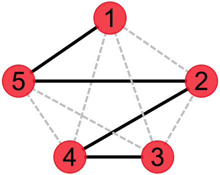

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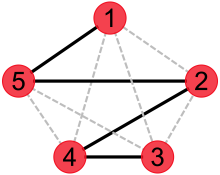

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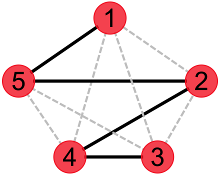

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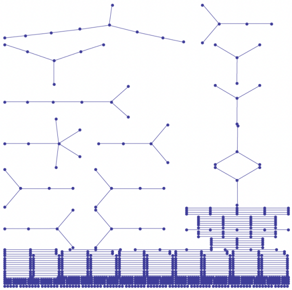
Historical note:

➢ This model was actually studied by Gilbert (1959) and heuristically by Solomonoff & Rapoport (1951)

➢ Erdős & Rényi (1959) initially worked with a slightly different model where fixed number of edges sampled uniformly. In a sequence of eight papers between 1959-1968 they laid the foundation of Random Graph theory

What are we after?

$\text{ER}_n(p)$ with $n = 1000$



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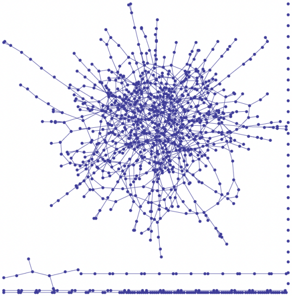

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Pic source: van der Hofstad (2017)

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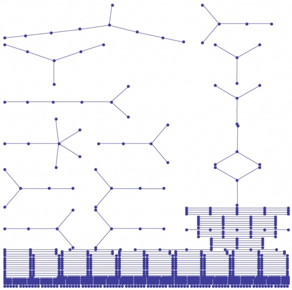

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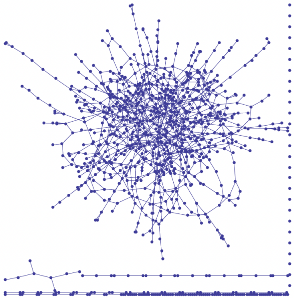

$p = \frac{2}{n}$

➢ If $p = \frac{\lambda}{n}$, then there is *phase transition around* $\lambda = 1$

➡ $\lambda < 1$: All components are small

➡ $\lambda > 1$: There is a unique *giant component*

Pic source: van der Hofstad (2017)

Local neighborhood structure of $ER_n(\frac{\lambda}{n})$

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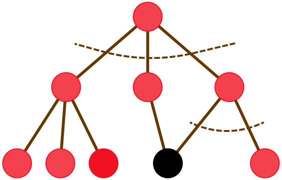

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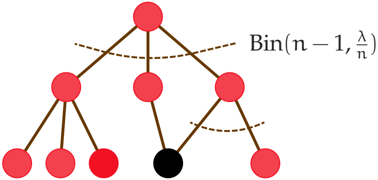

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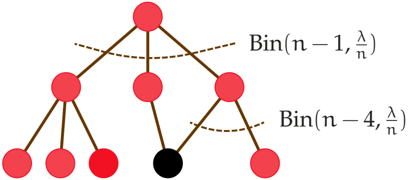

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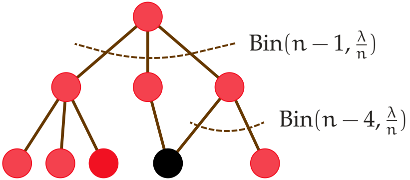

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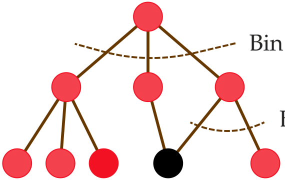

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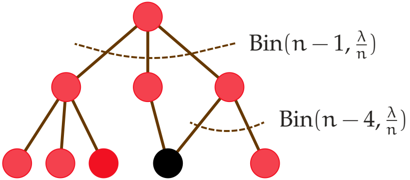

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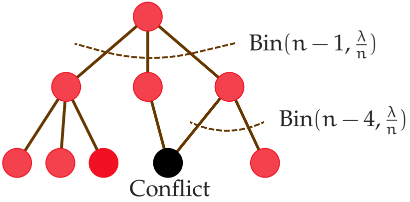

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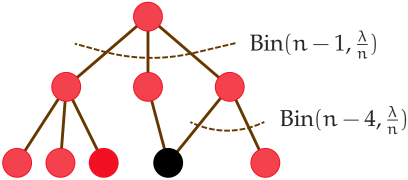
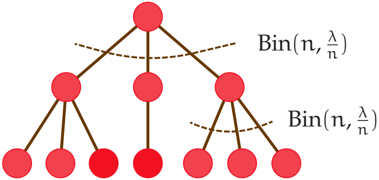
- ➡ Depletion of vertices
- ➡ Conflicts among new vertices

Domination by Branching Process

Let's consider a random process without Depletion and Conflicts





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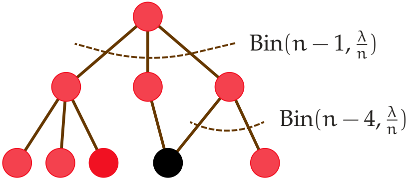

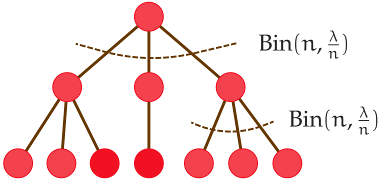

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Simple Fact

Let N_k = # vertices at depth k for $ER_n(\frac{\lambda}{n})$ exploration, and $\bar{N}_k$ denotes same for Branching process. There is a coupling such that w.p. 1

$$N_k \leqslant \bar{N}_k \quad \forall k \geqslant 0$$

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Theorem: Subcritical $ER_n(\frac{\lambda}{n})$

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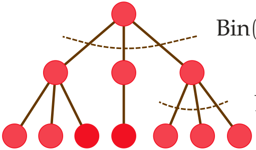
➢ Proof uses Large Deviation estimates for branching process survival prob

What happens to the BP for $\lambda > 1$

Lets again look at upper bounding Branching Process (BP)



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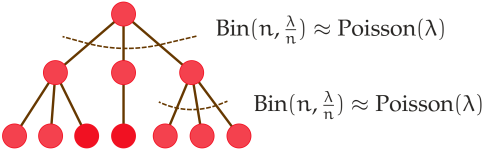

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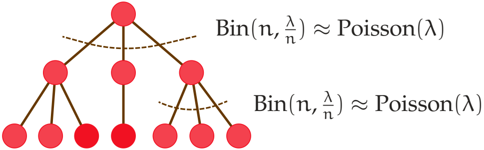

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➢ ζ_λ satisfies is a positive solution of $1 - \zeta = e^{-\lambda\zeta}$

Existence of a Giant for $\lambda > 1$

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Theorem: Supercritical $\mathrm{ER}_n(\frac{\lambda}{n})$

Let $C_{(i)}$:= i-th largest component of $\mathrm{ER}_n(\frac{\lambda}{n})$. If $\lambda > 1$, then as $n \to \infty$

$$\frac{C_{(1)}}{n} \xrightarrow{\mathbb{P}} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\mathbb{P}} 0$$

A unique giant component emerges...

Moments of large component sizes

van der Hofstad: *Random Graph and Complex Networks, Vol 2*

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Next, prove two lemmas but before that...

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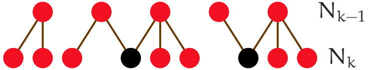

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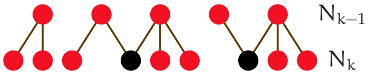

Fact: $\mathbb{E}[\#\text{conflicts}] \leqslant C\frac{\lambda^{2k}}{n}$

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Next, lets prove two lemmas

Proving first moment lemma

To prove: $\sum_{i \geqslant 1} \frac{C_{(i)}}{n} \mathbb{1}\{C_{(i)} \geqslant L\} \approx \zeta_\lambda$

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Key fact 1: Local neighborhood of u is approximately BP whp

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Key fact 1: Local neighborhood of u is approximately BP whp

➢ *Theory of approximating local neighborhood of graphs is called Local-weak convergence (Christian's talk)*

Proving second moment lemma

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Take two uniform vertices u_1, u_2. The second term equals

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Enough to show:

$$\lim_{L \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(C(u_1) \geqslant L, C(u_2) \geqslant L, u_1 \not\leftrightarrow u_2 \right) = 0$$

Two large components cannot be disjoint...

Two large components cannot be disjoint, why?

To show $\mathbb{P}\big(C(u_1) \geqslant L, C(u_2) \geqslant L, u_1 \not\leftrightarrow u_2\big) \approx 0$, suffices to prove

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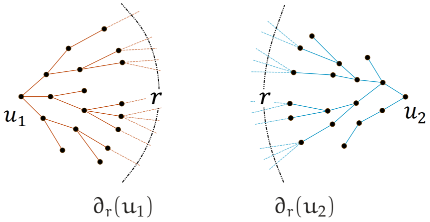

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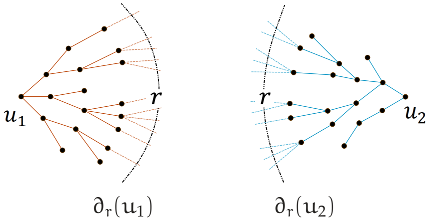

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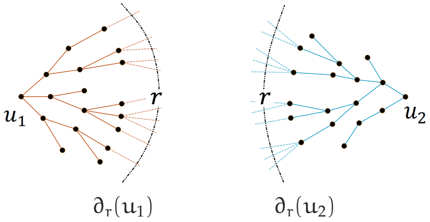

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➢ Advantage with conditioning on $\{\partial_r(u_1), \partial_r(u_2) \neq \varnothing\}$ is we can now explore rest of the graph

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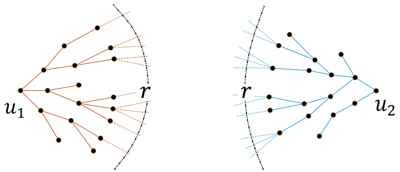

➢ Condition on $\{\partial_r(u_1), \partial_r(u_2) \neq \varnothing\}$ for large r

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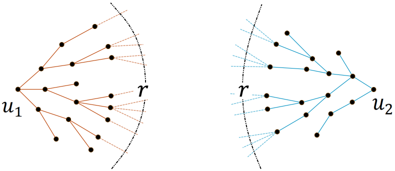

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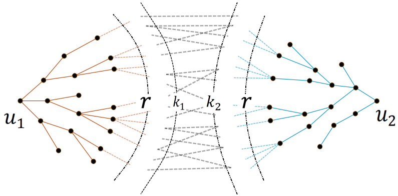

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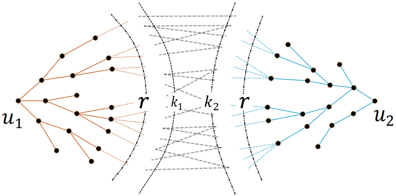

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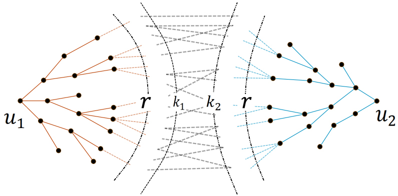

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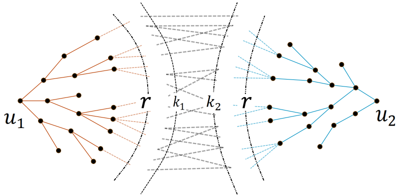

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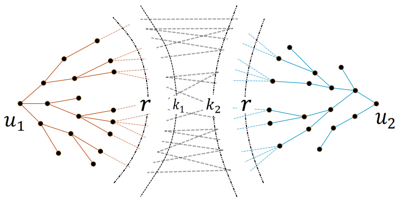

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Takeaway: If there are two components with large boundary, we can grow them until boundary has size $\sqrt{n}$ and then they intersect

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To prove: $\sum_{i \geqslant 1} \frac{C_{(i)}^2}{n^2} \mathbb{1}\{C_{(i)} \geqslant L\} \approx \zeta_\lambda^2$

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Summary: Giant for Erdős-Rényi

We proved

Theorem: Supercritical $\mathrm{ER}_n(\frac{\lambda}{n})$

Let $C_{(i)}$:= i-th largest component of $\mathrm{ER}_n(\frac{\lambda}{n})$. If $\lambda > 1$, then

$$\frac{C_{(1)}}{n} \xrightarrow{\mathbb{P}} \zeta_\lambda \quad \text{and} \quad \frac{C_{(2)}}{n} \xrightarrow{\mathbb{P}} 0$$

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1. **Local neighborhood approximation:** Local neighborhood of u is approximately BP whp and when BP survives, $C(u)$ is large

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➢ van der Hofstad (2021) proved this for general graphs that converge in local-weak convergence sense

Before moving on to other models, lets see another useful application of the above ideas...

Typical distances in Erdős-Rényi

Typical distances in the giant of Erdős-Rényi

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