Extremely Deep Proofs

Noah Fleming, Toniann Pitassi and Robert Robere

UCSD

Columbia University
IAS

McGill University

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- Several other size/space tradeoffs for various proof systems [R17,BN20,R18]

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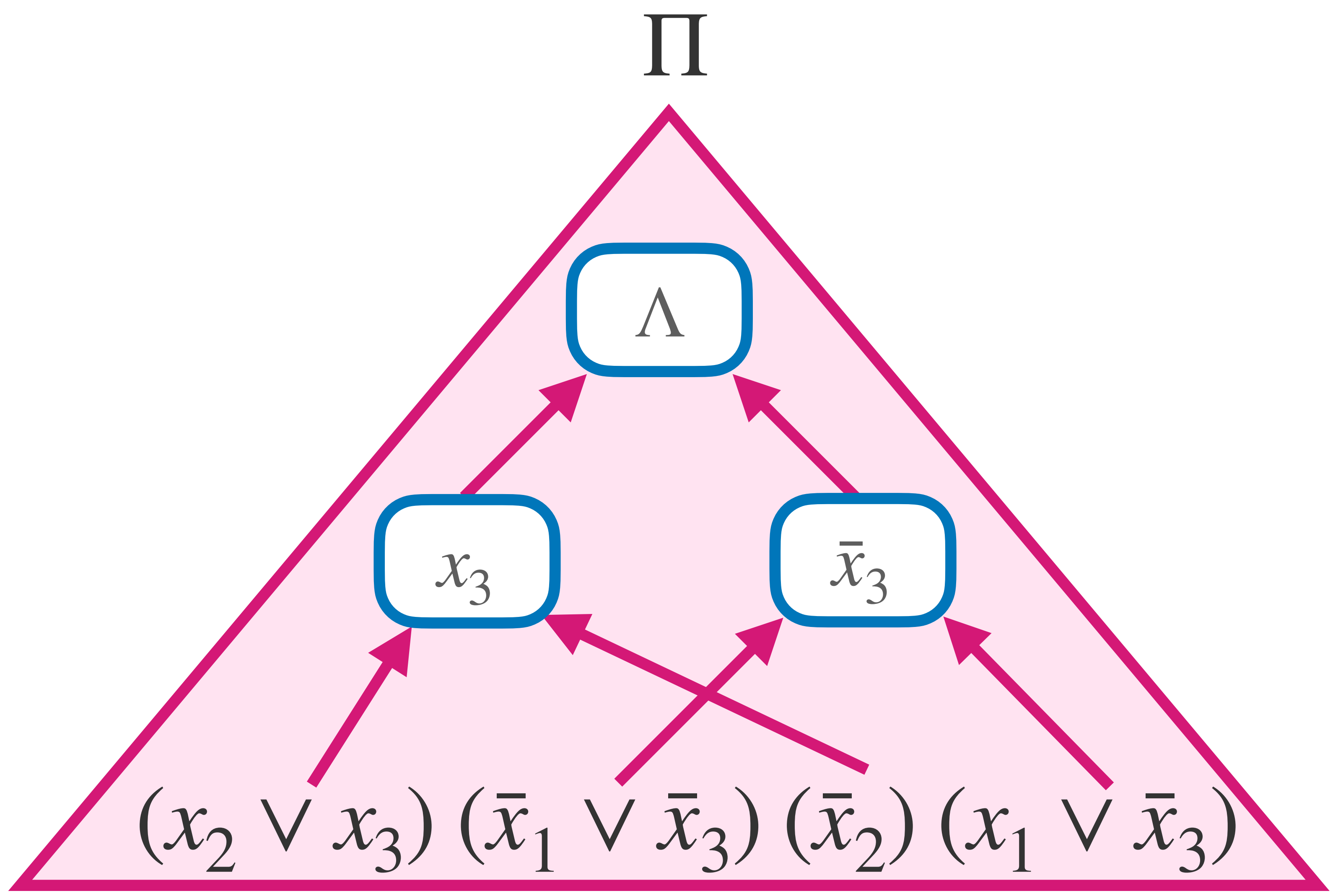

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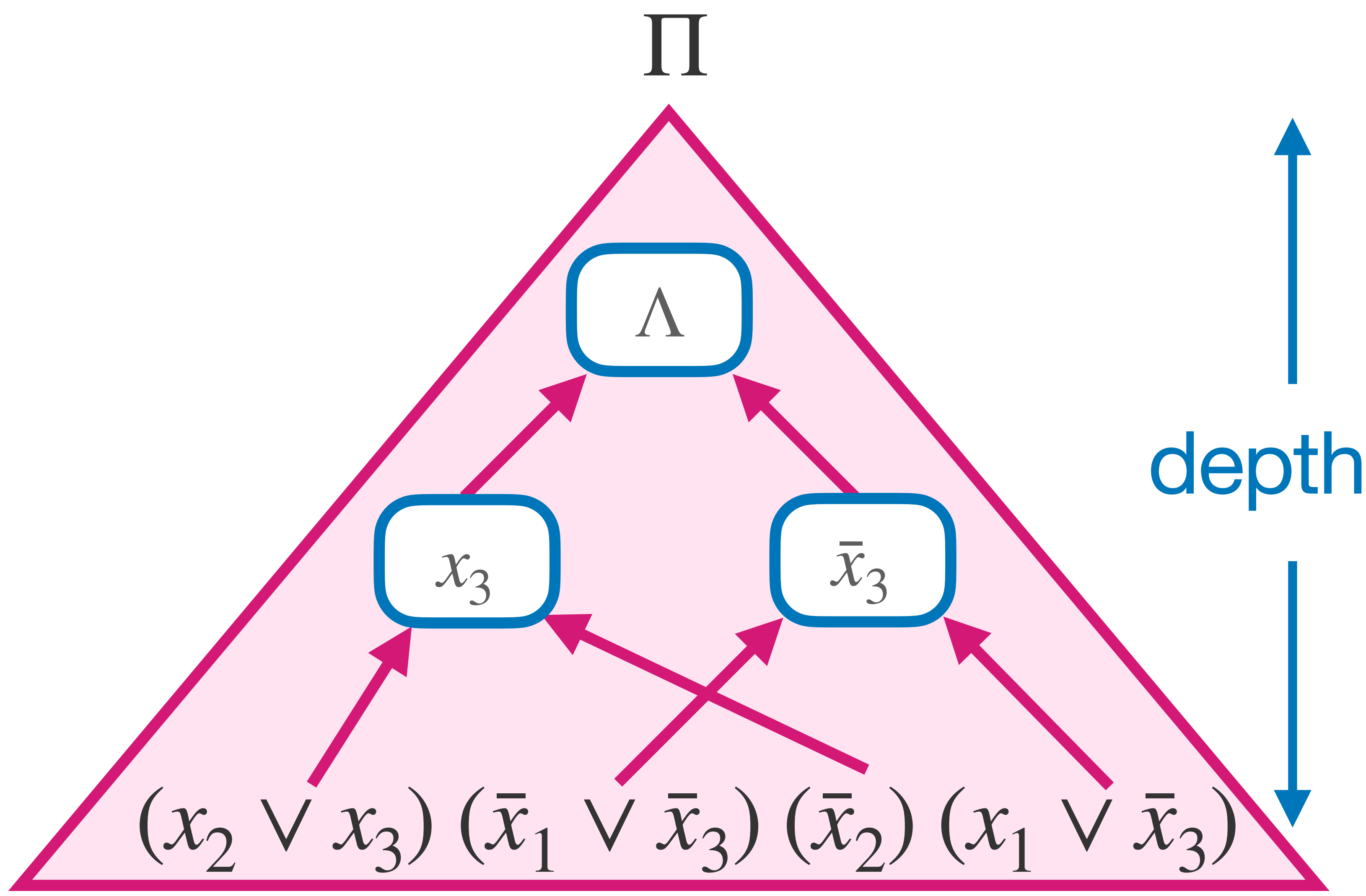
depth(Π): longest root-to-leaf path



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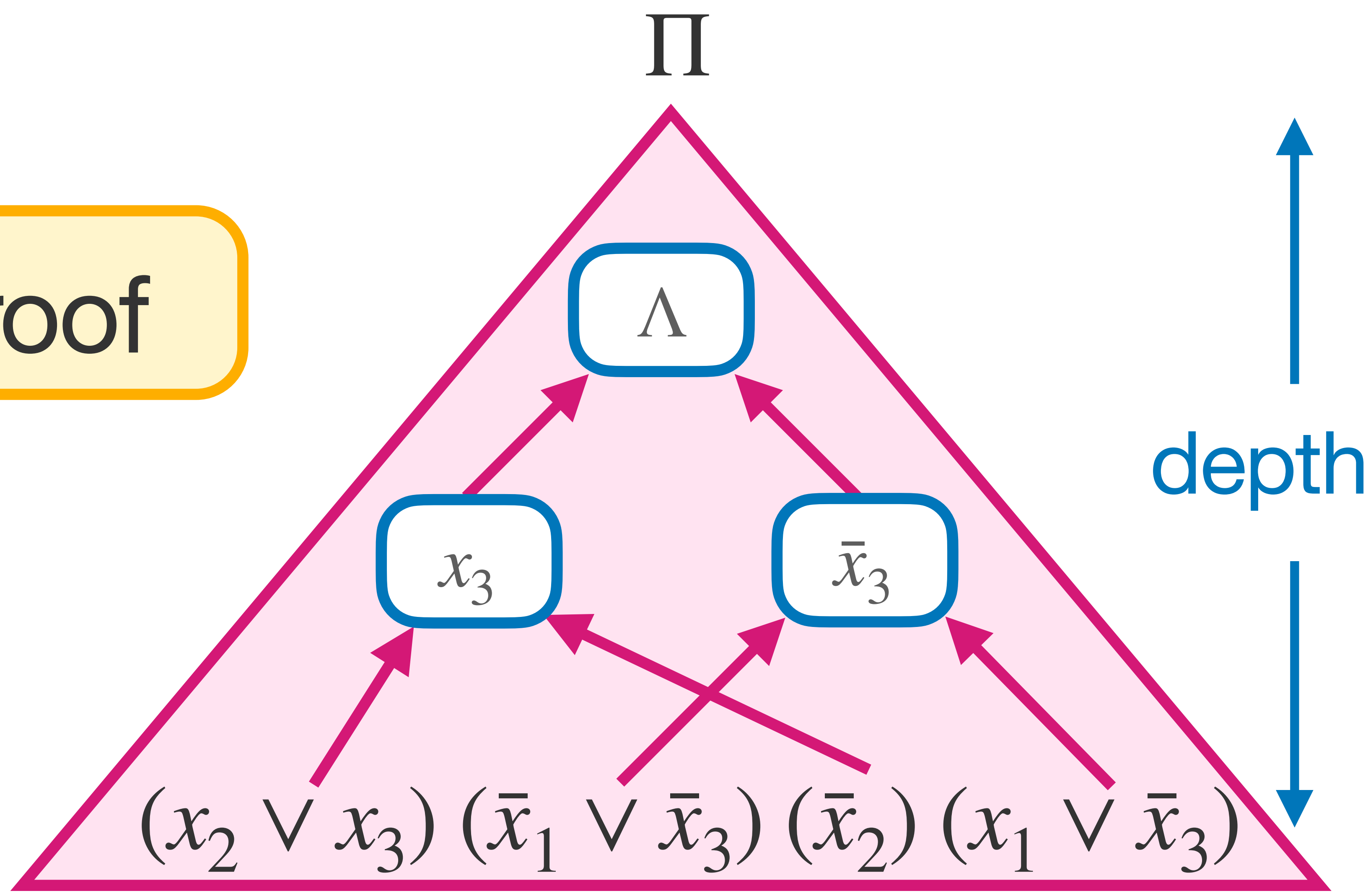
Like circuit depth, captures a notion of "parallelism" of a proof



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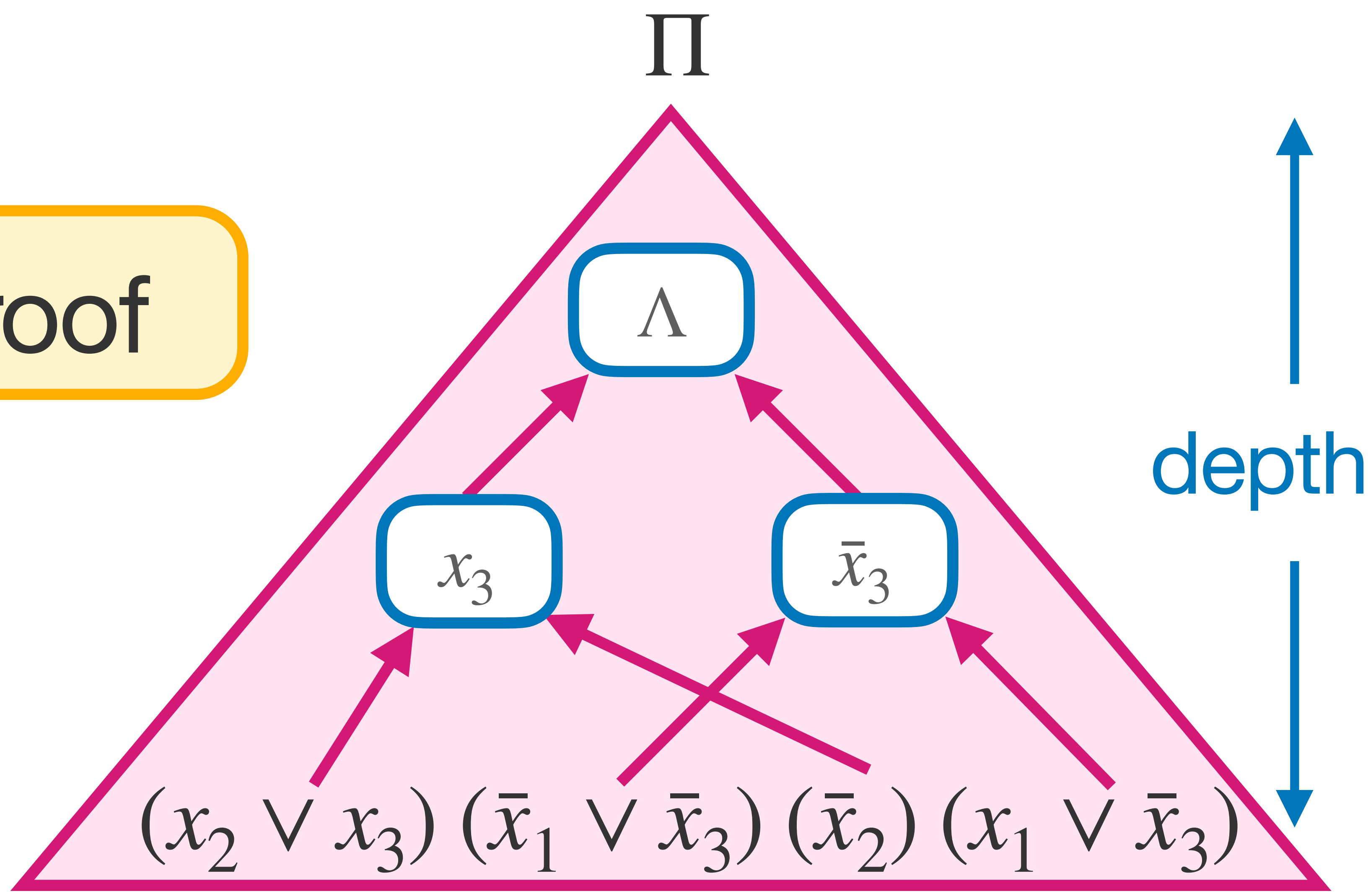

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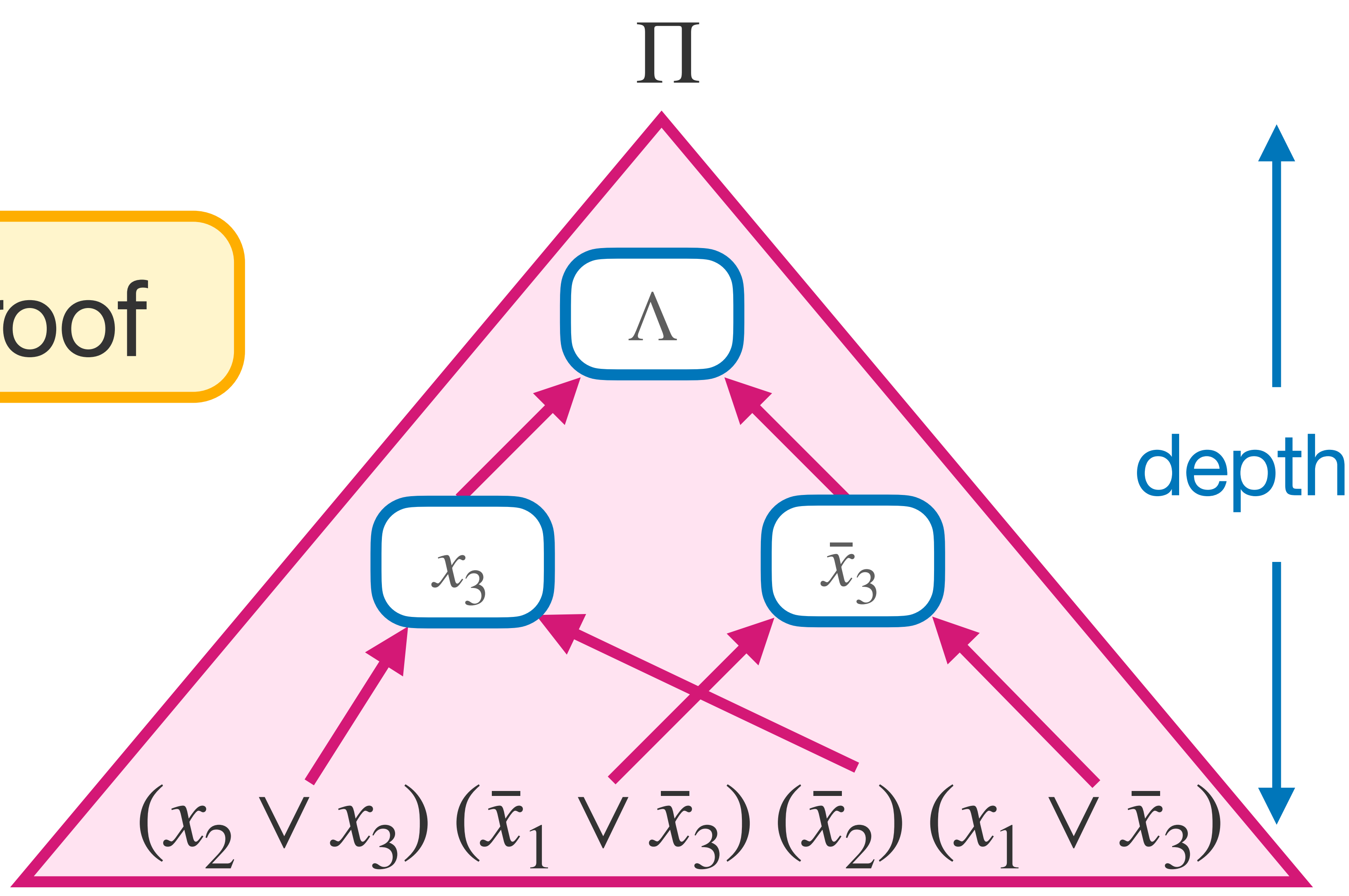

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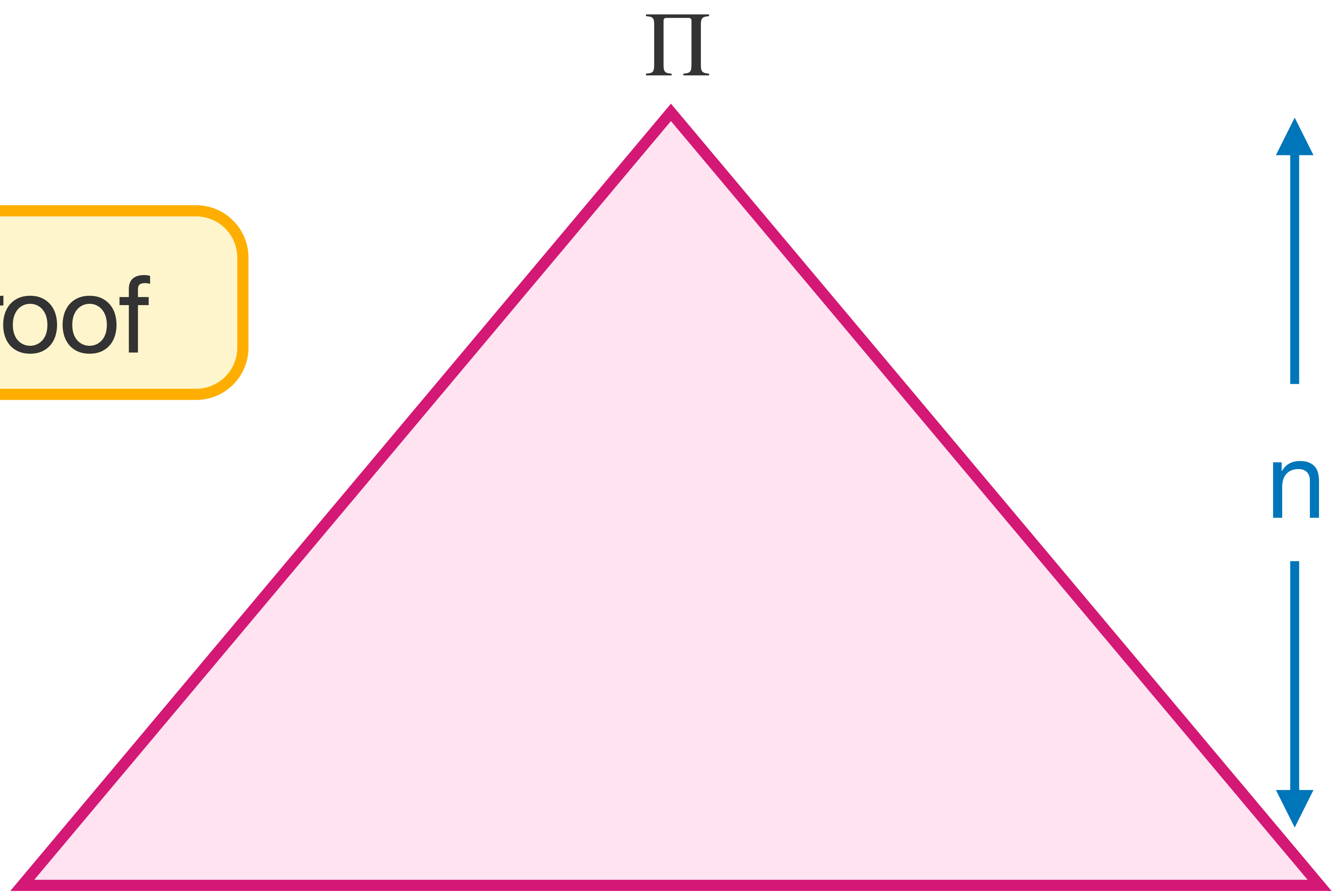

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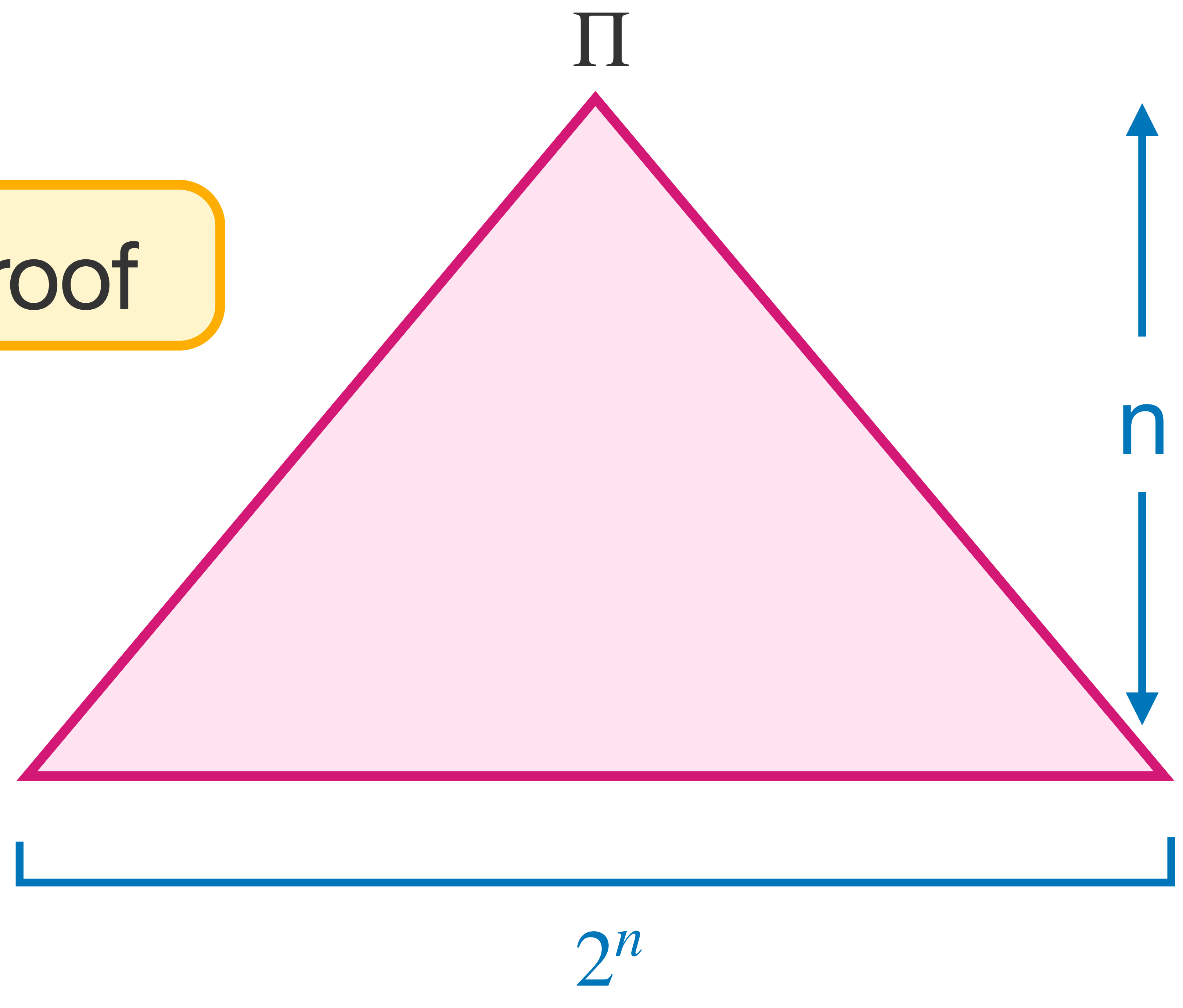

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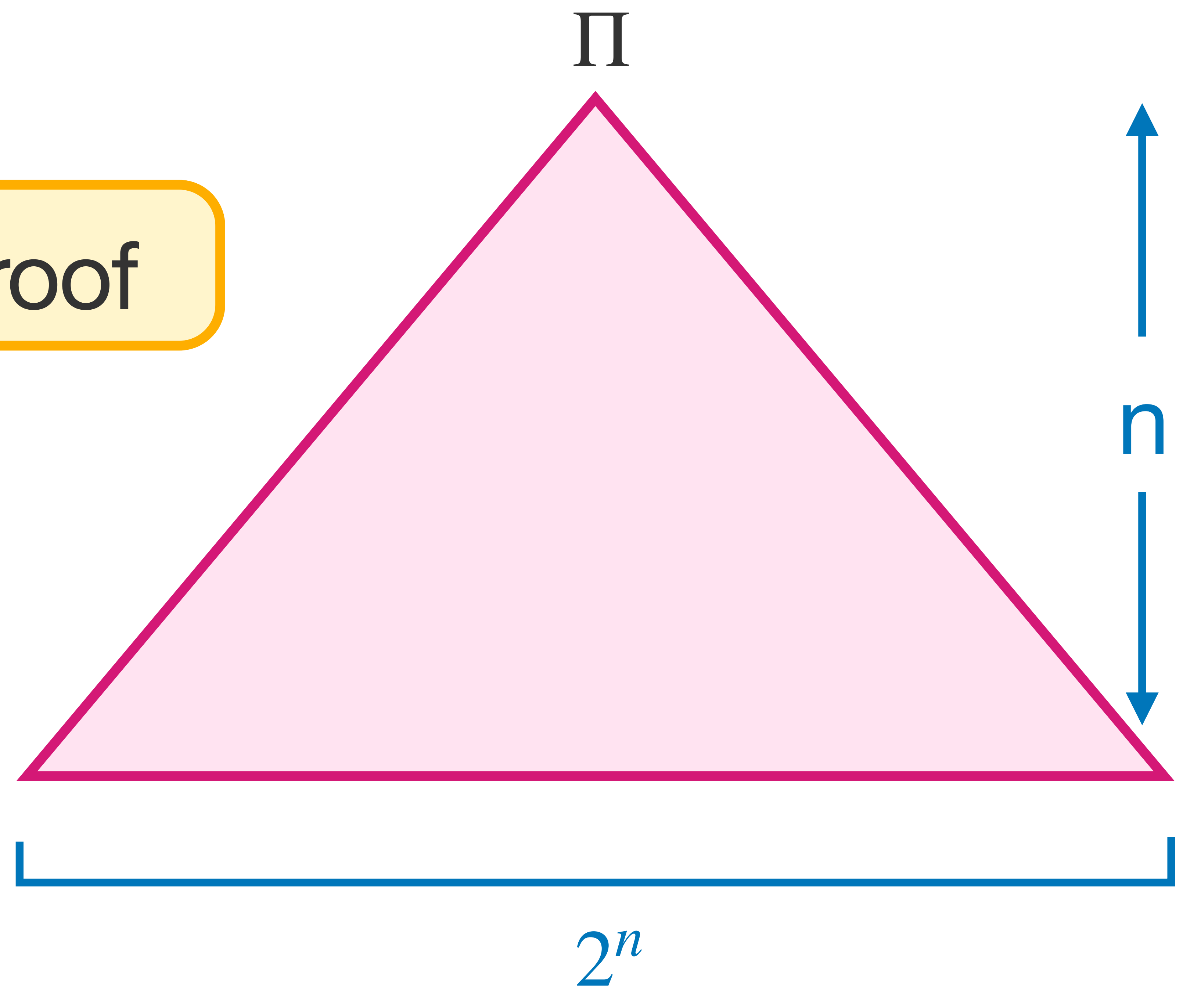

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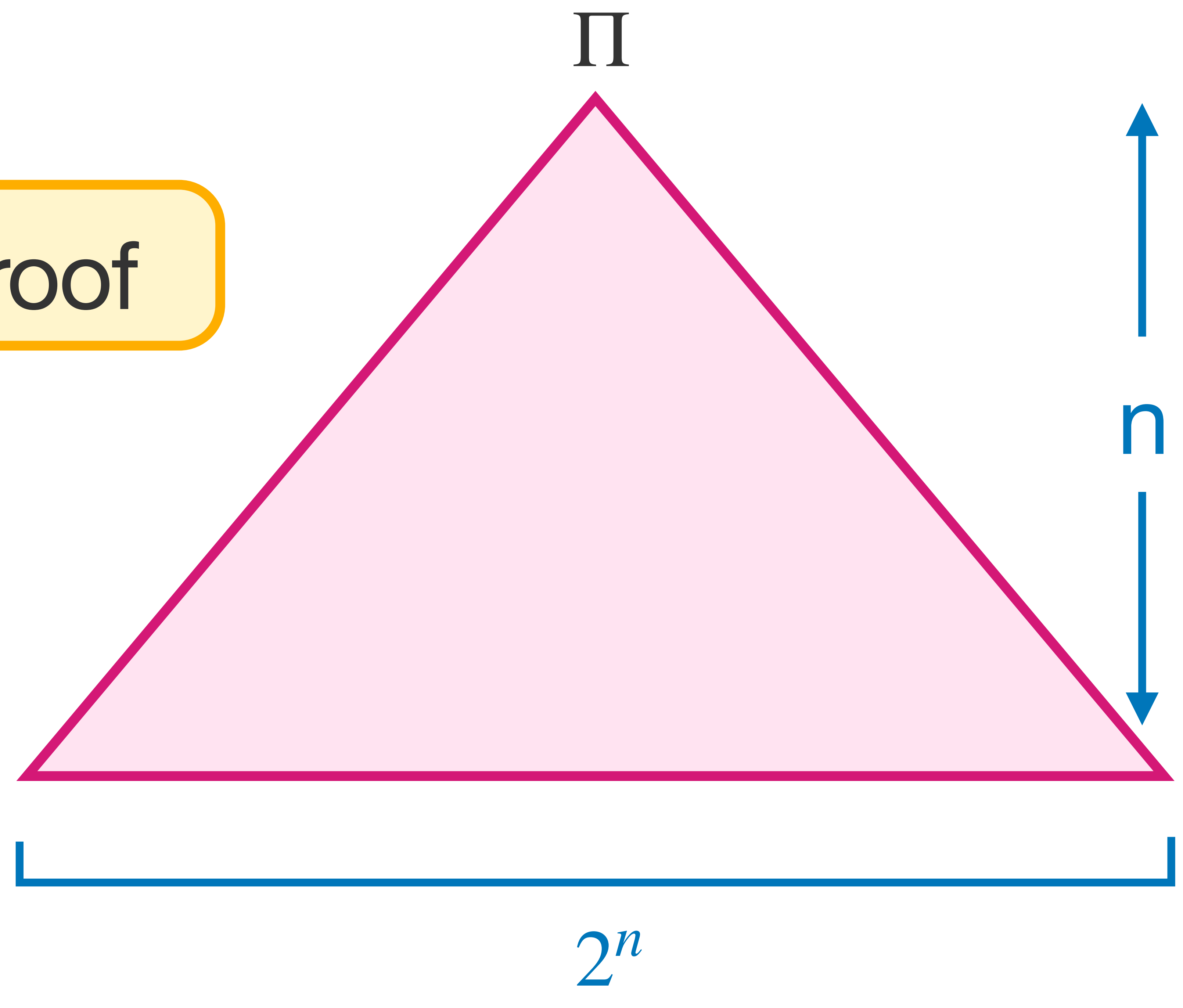

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— Resolution cannot

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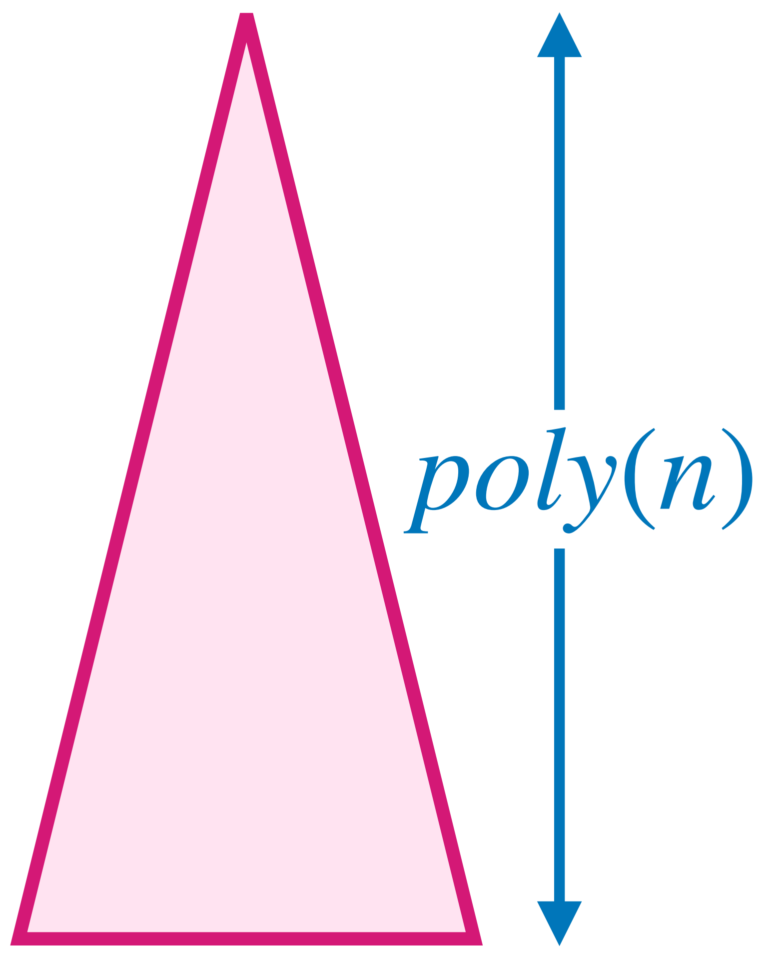
For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF F on n variables such that

- There is a polynomial size P-proof of F
- Any subexponential-size P-proof of F must have poly$(n) > n$ depth



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For any $P \in \{$Resolution, Res(k), Cutting Planes$\}$

There is a CNF F on n variables such that

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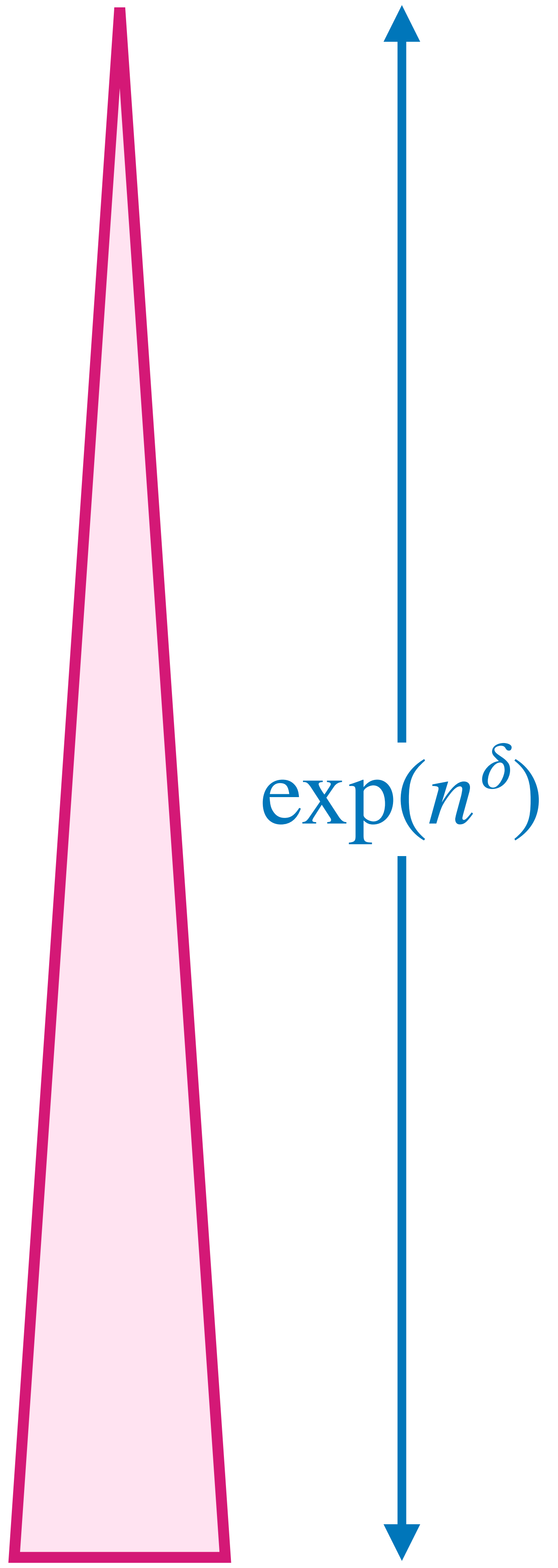
- Any subexponential-size P-proof of F has weakly exponential depth



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* Caveat: F has $n^{O(c)}$ many clauses — we'll come back to this!

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How do we do compression? Lifting!

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Let P, Q be two proof systems

A **lifting theorem** relates the complexity of

- P-proofs of F
- Q-proofs of $F \circ g$

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A width lower bound on F implies a size lower bound on $F \circ \mathsf{XOR}_2$!

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$\implies$ Naive simulation is essentially the best! (A theme of lifting theorems)

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Typically in a Lifting Theorem…

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$\rightarrow$ Q is a strong proof system

A lifting theorem shows that the most efficient Q-proof of $F \circ g$ is to simulate the most efficient P-proof of F (with extra overhead to handle g)

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i.e., it "lifts" lower bounds on weak proof systems to strong proof systems

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Find a gadget g such that

1. The number of variables n of $F \circ g$ will be much smaller than N
2. Any small-size Resolution proof of $F \circ g$ will require the same depth as proving F

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… In fact, we will compose with the Nisan-Wigderson generator!

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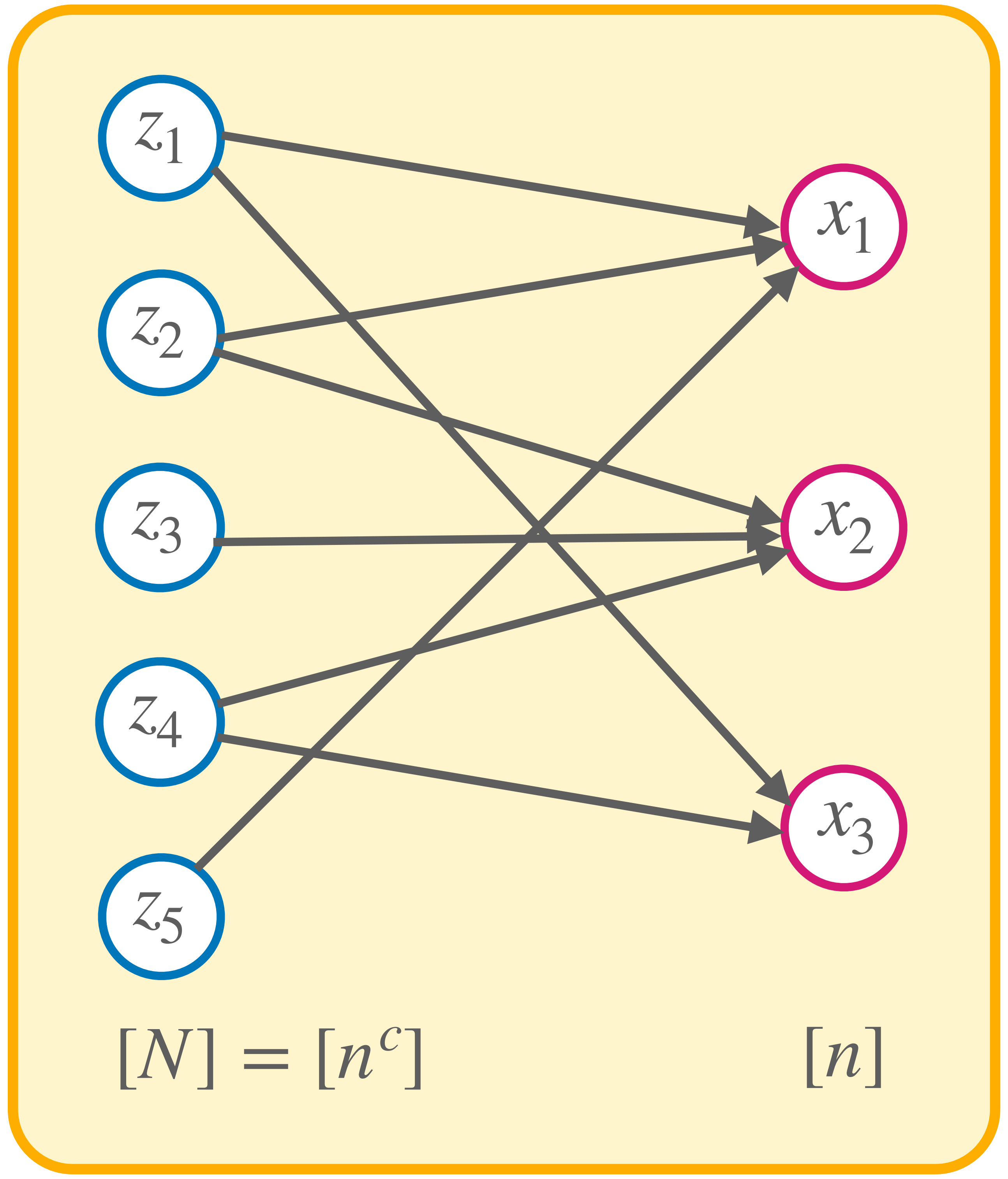

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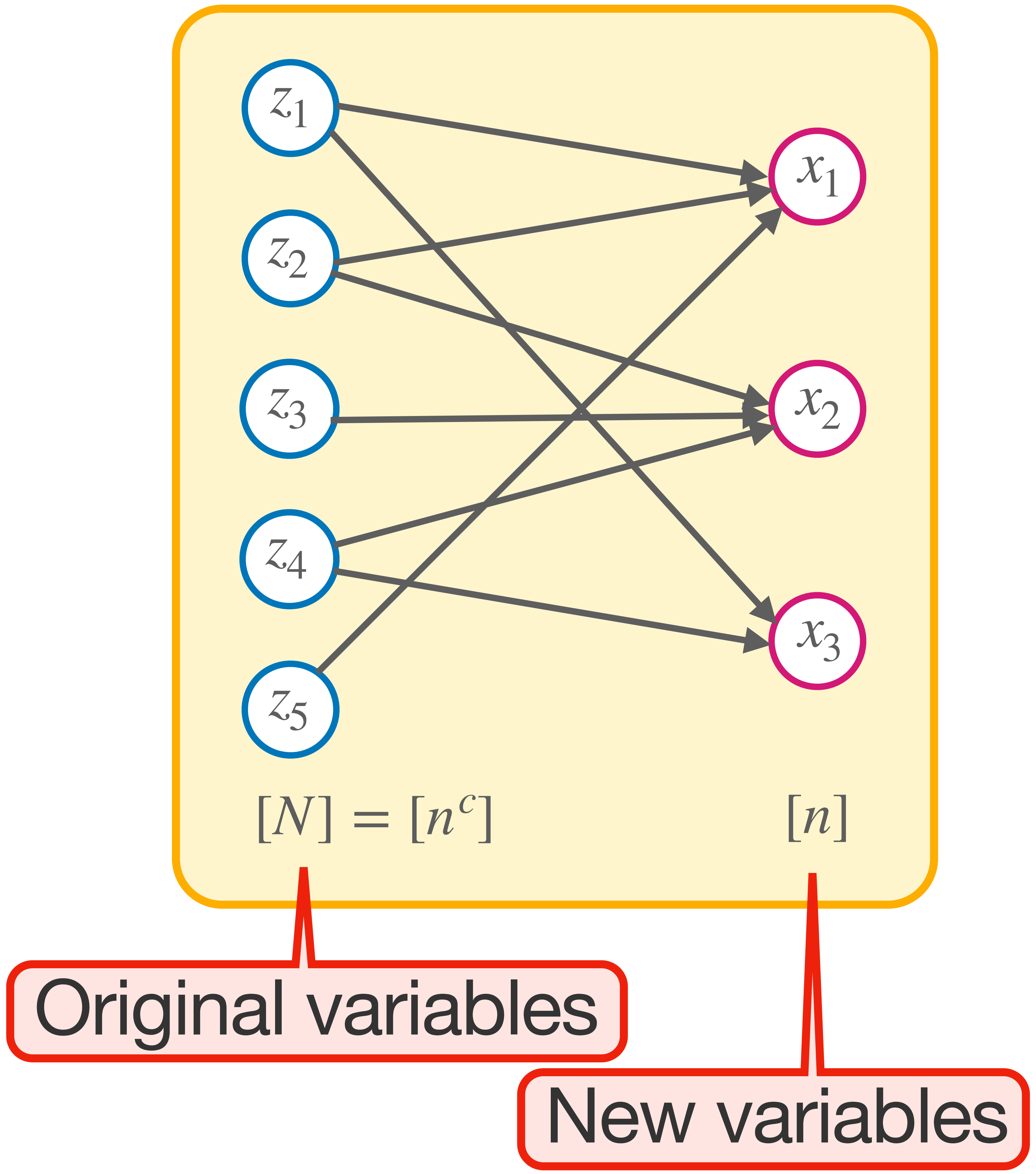

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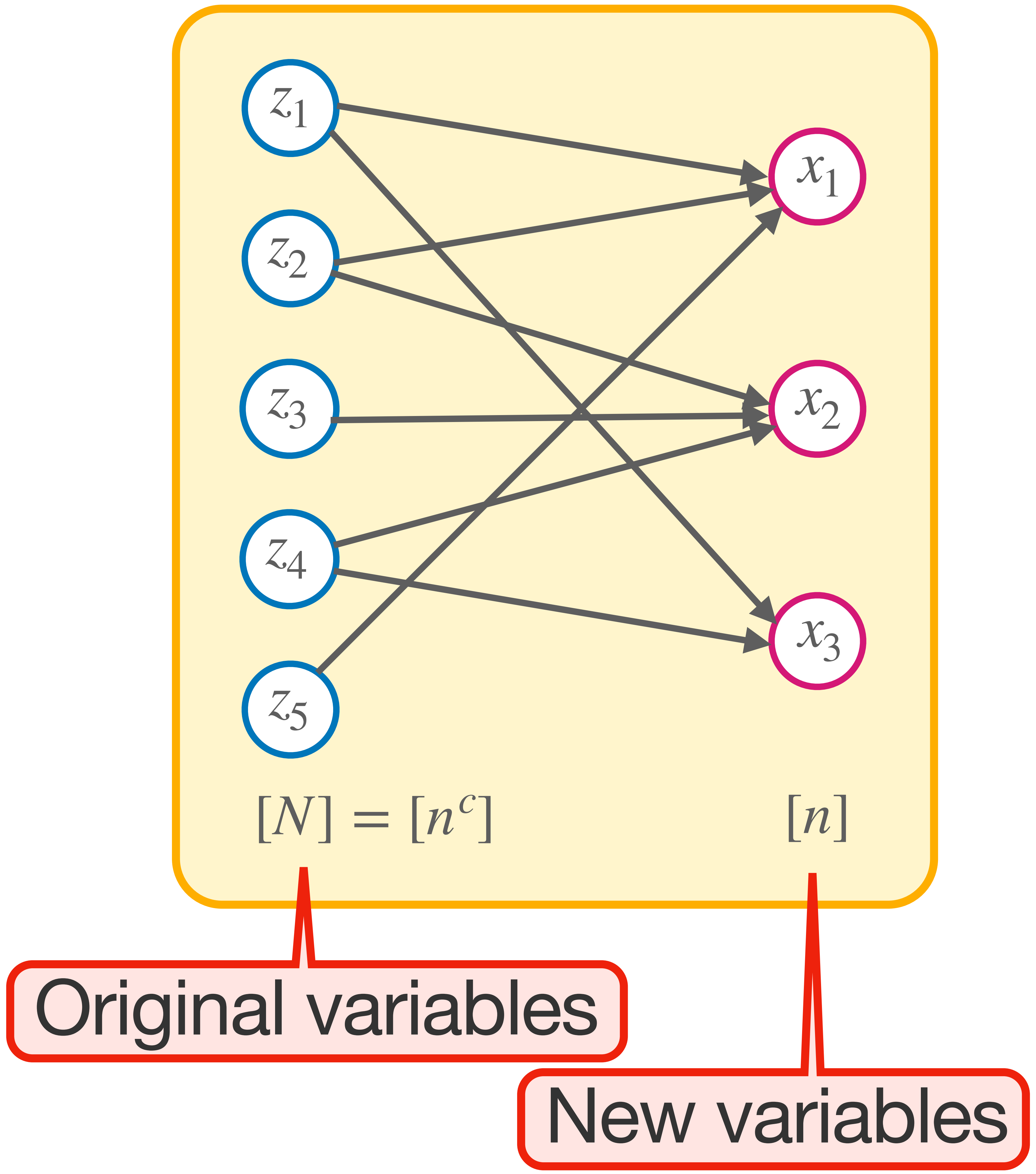

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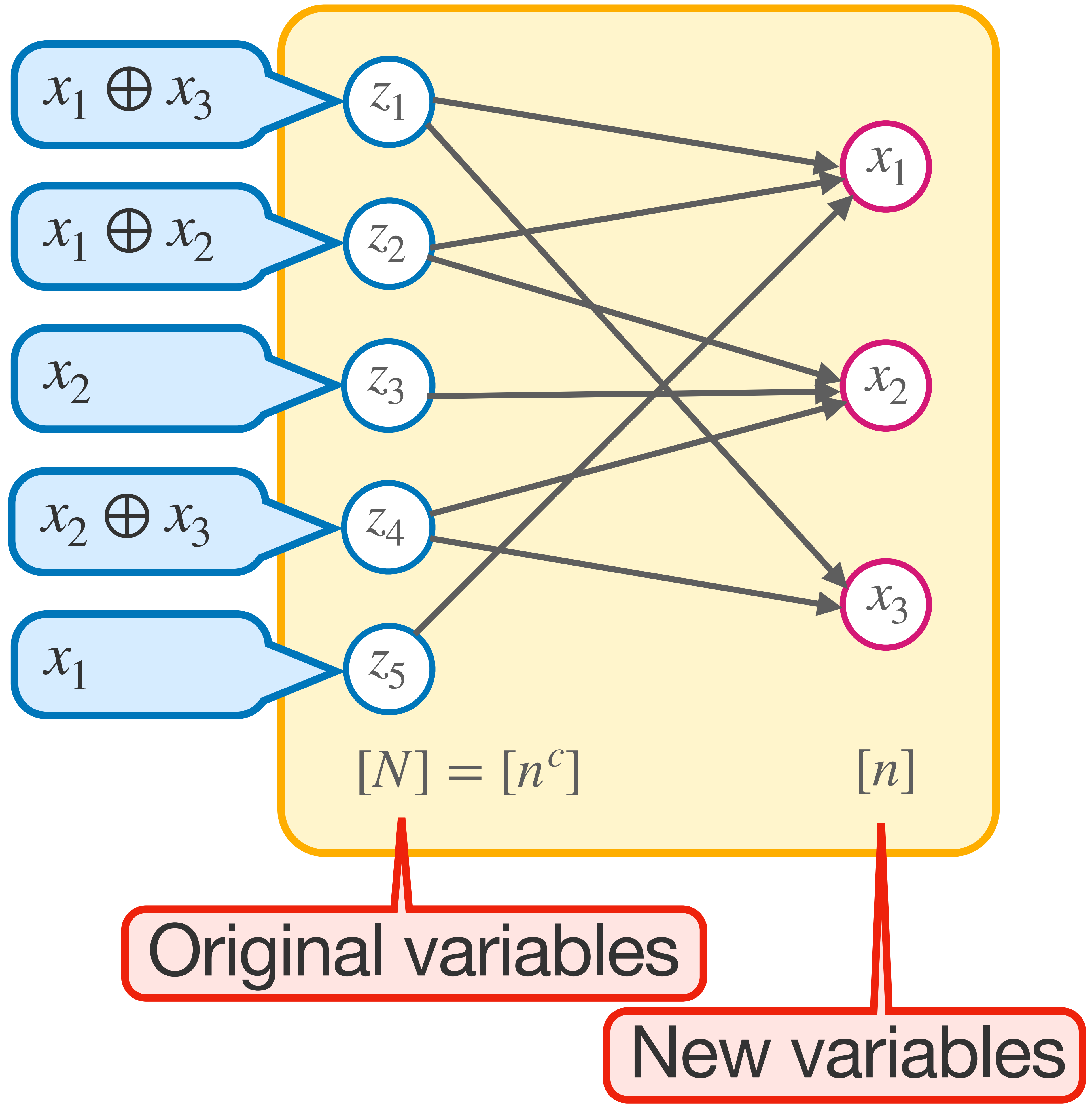

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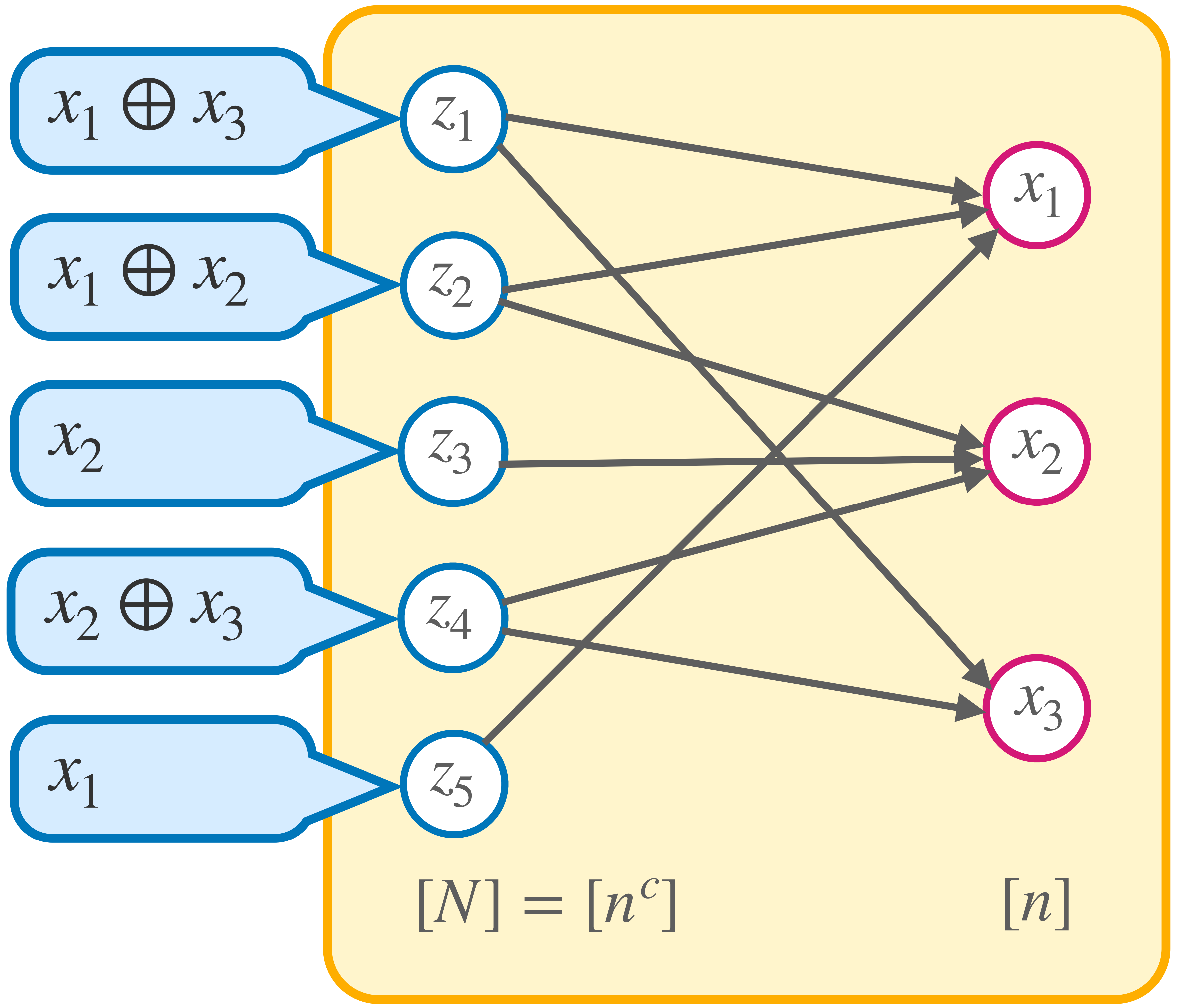

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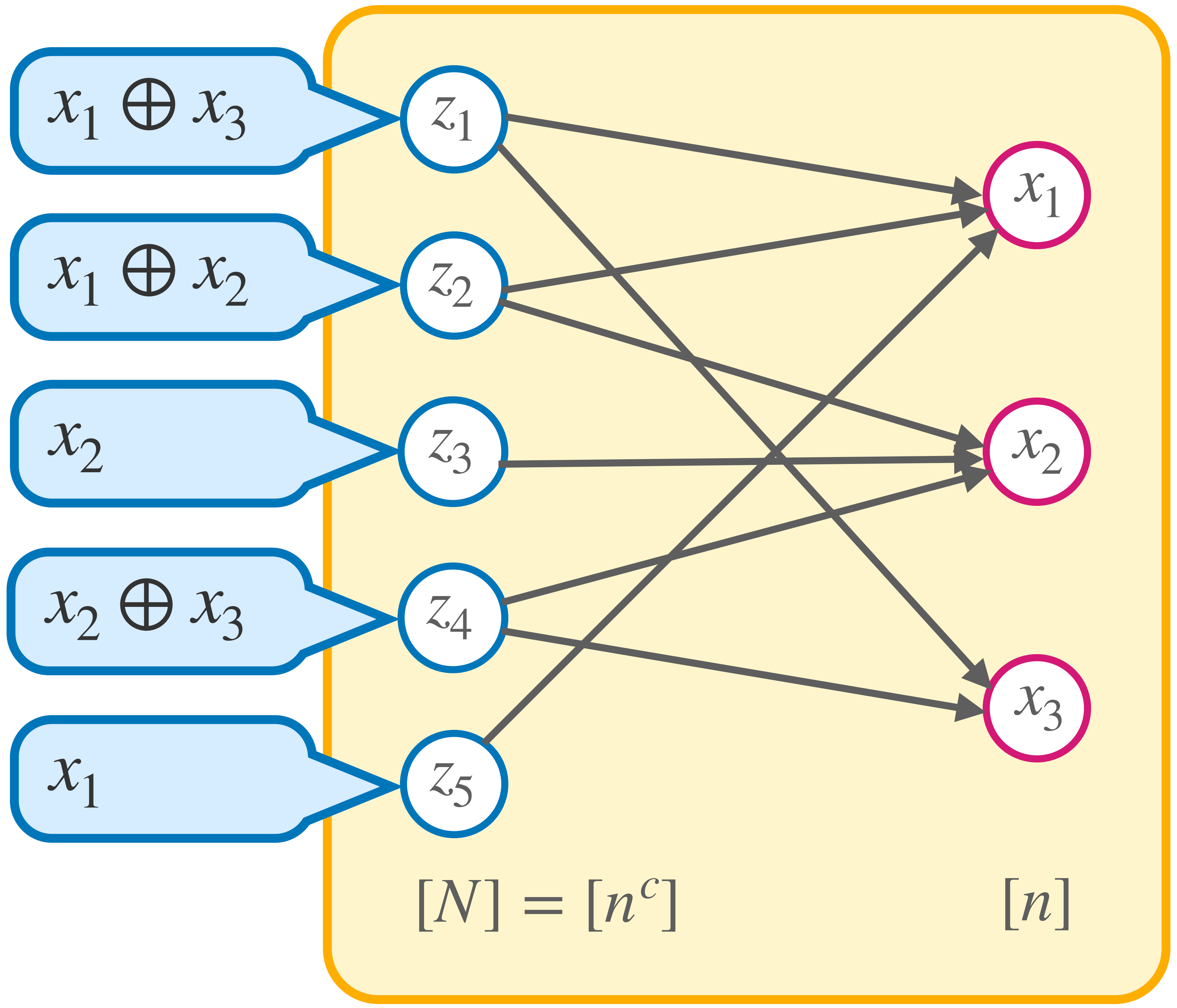

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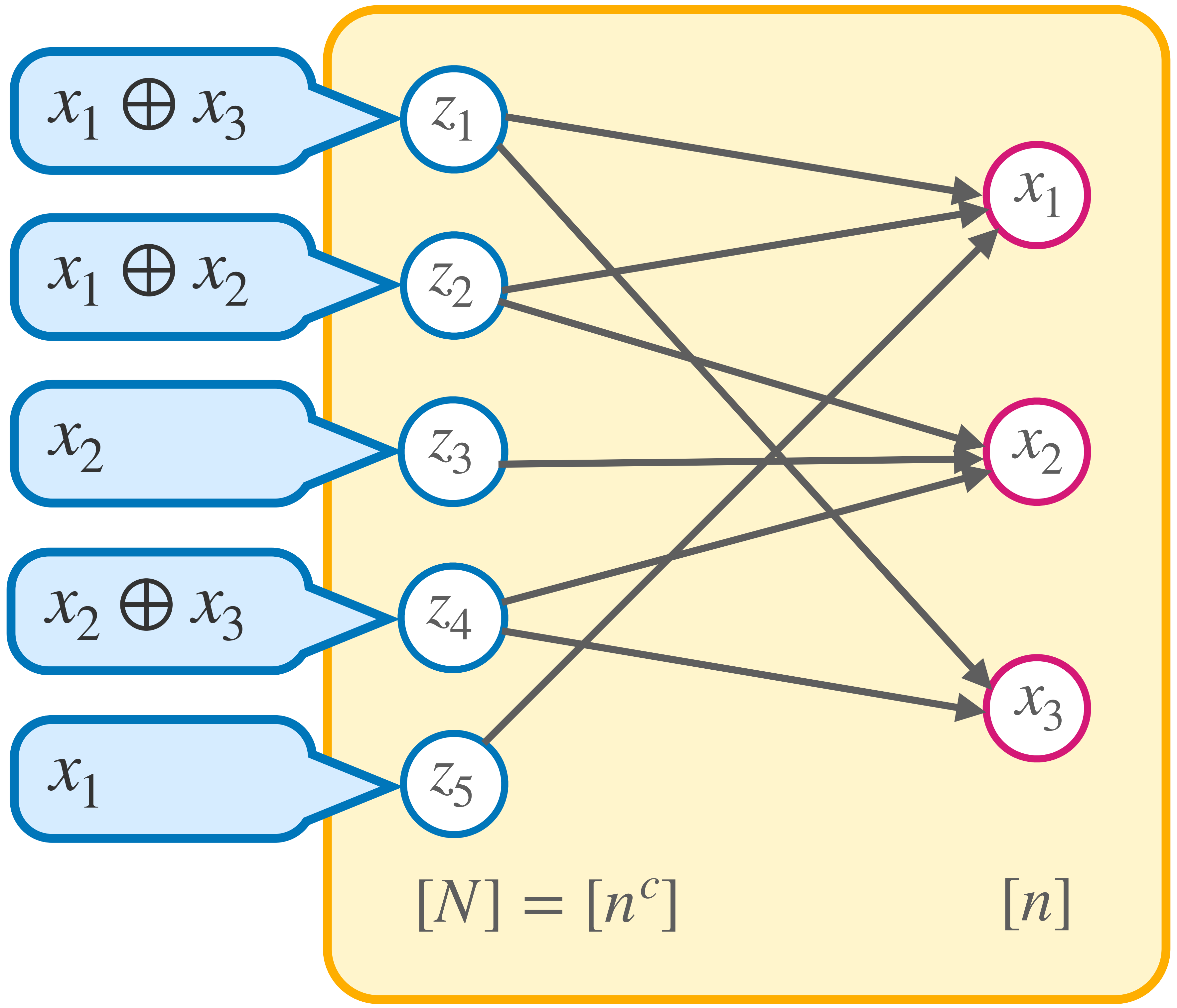

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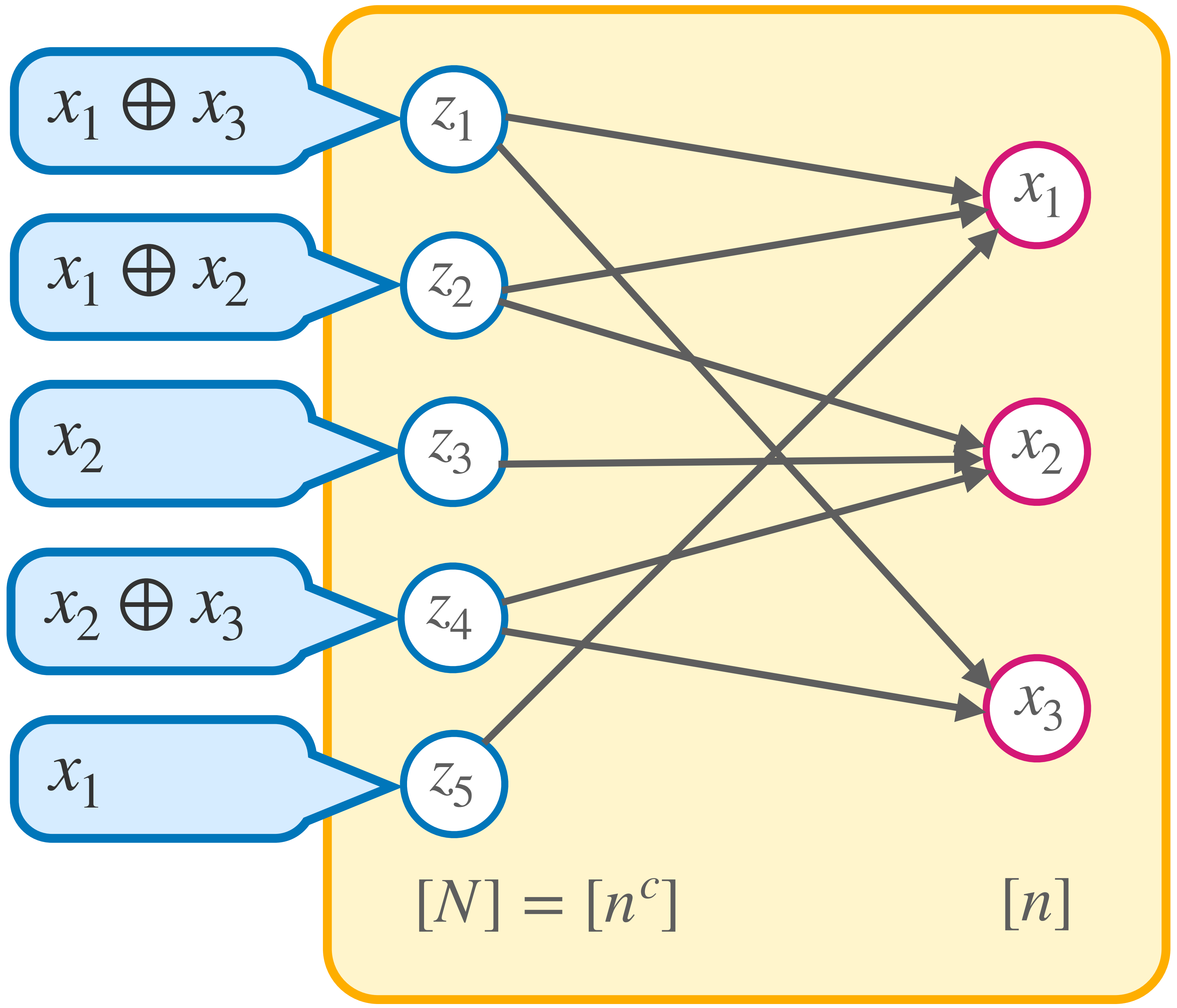

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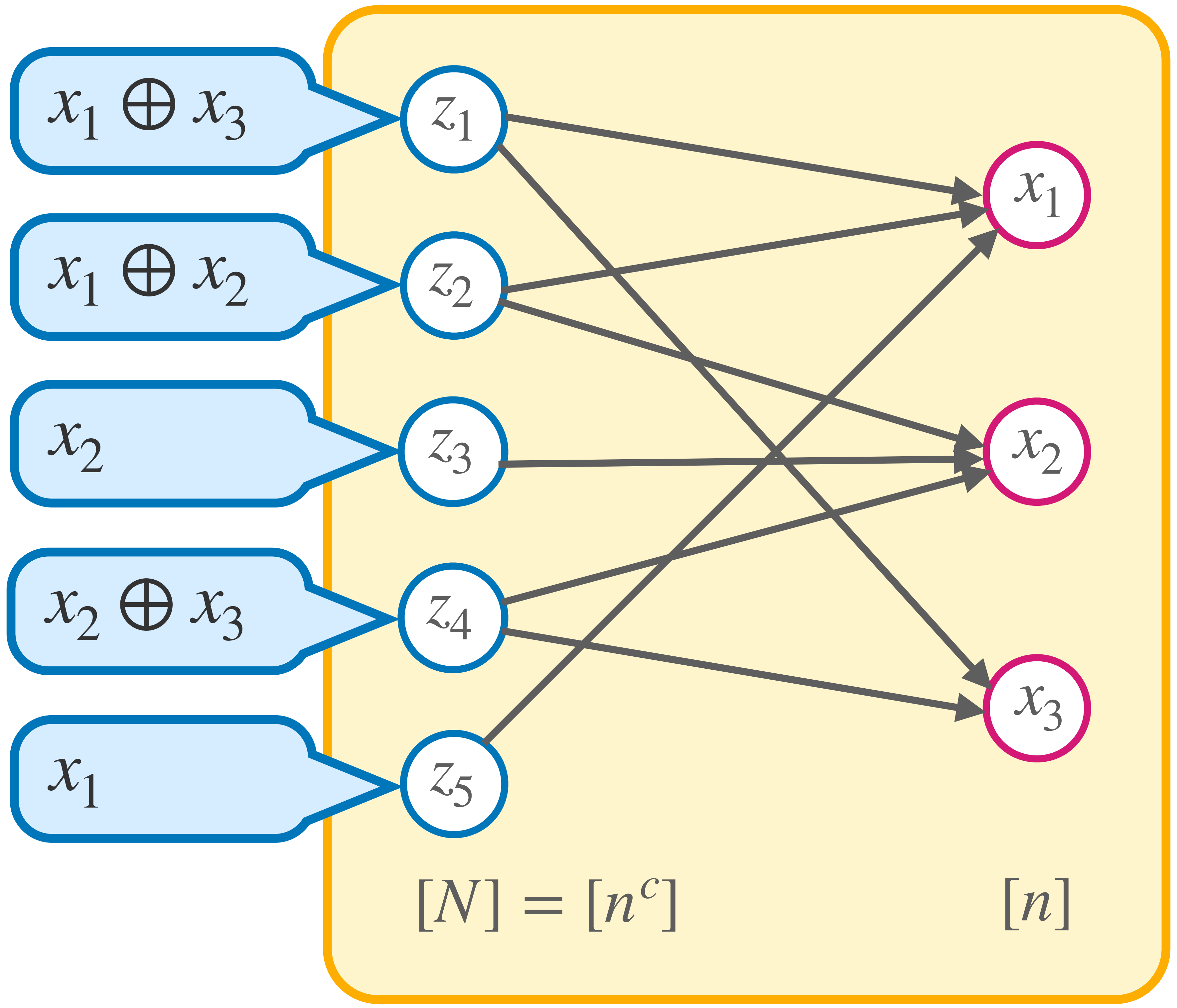
→ The best Resolution proof of $F \circ \mathrm{XOR}_G$ should essentially be to simulate the best proof of F



The Gadget

Let G be an $N \times n$ bipartite graph

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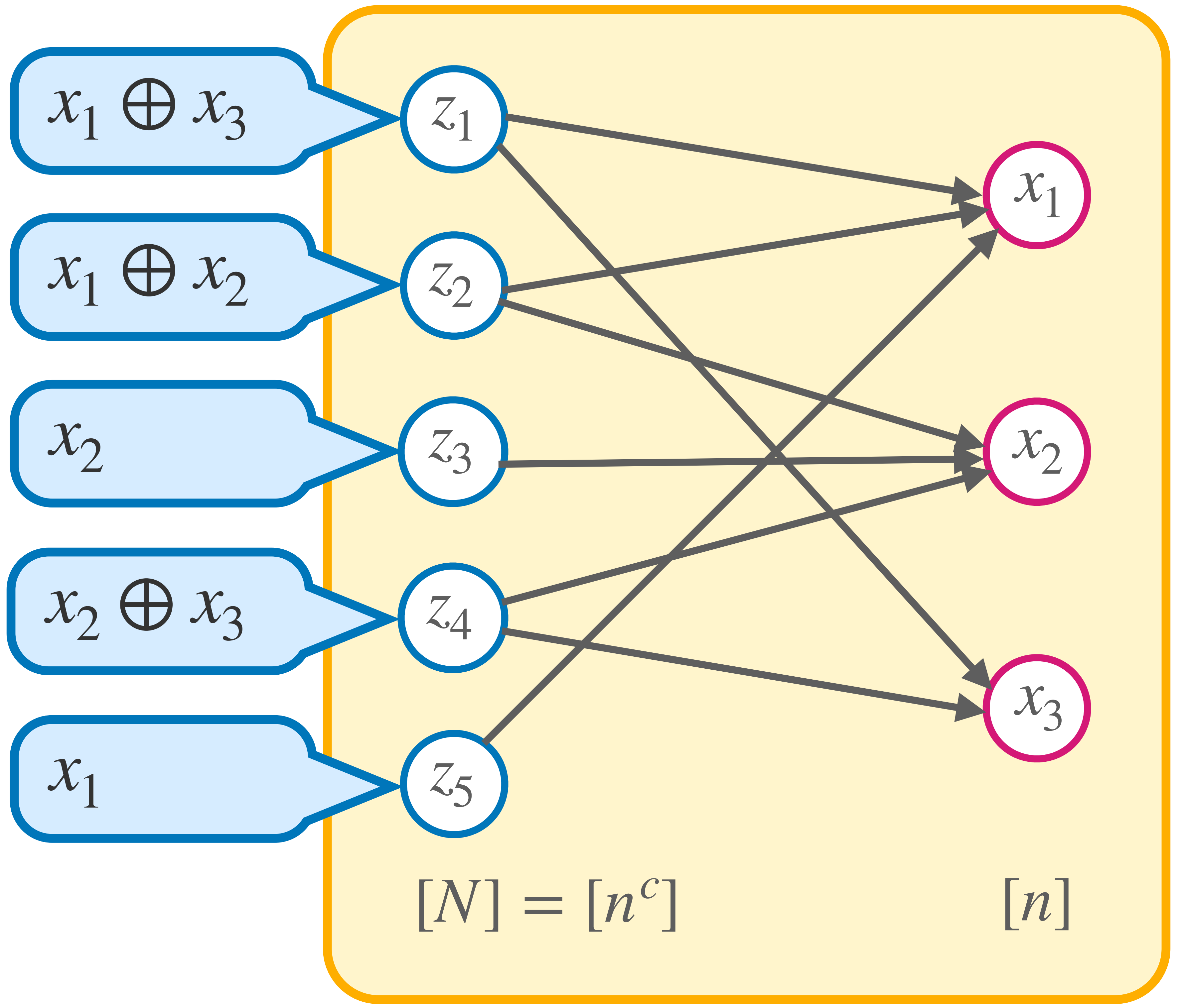

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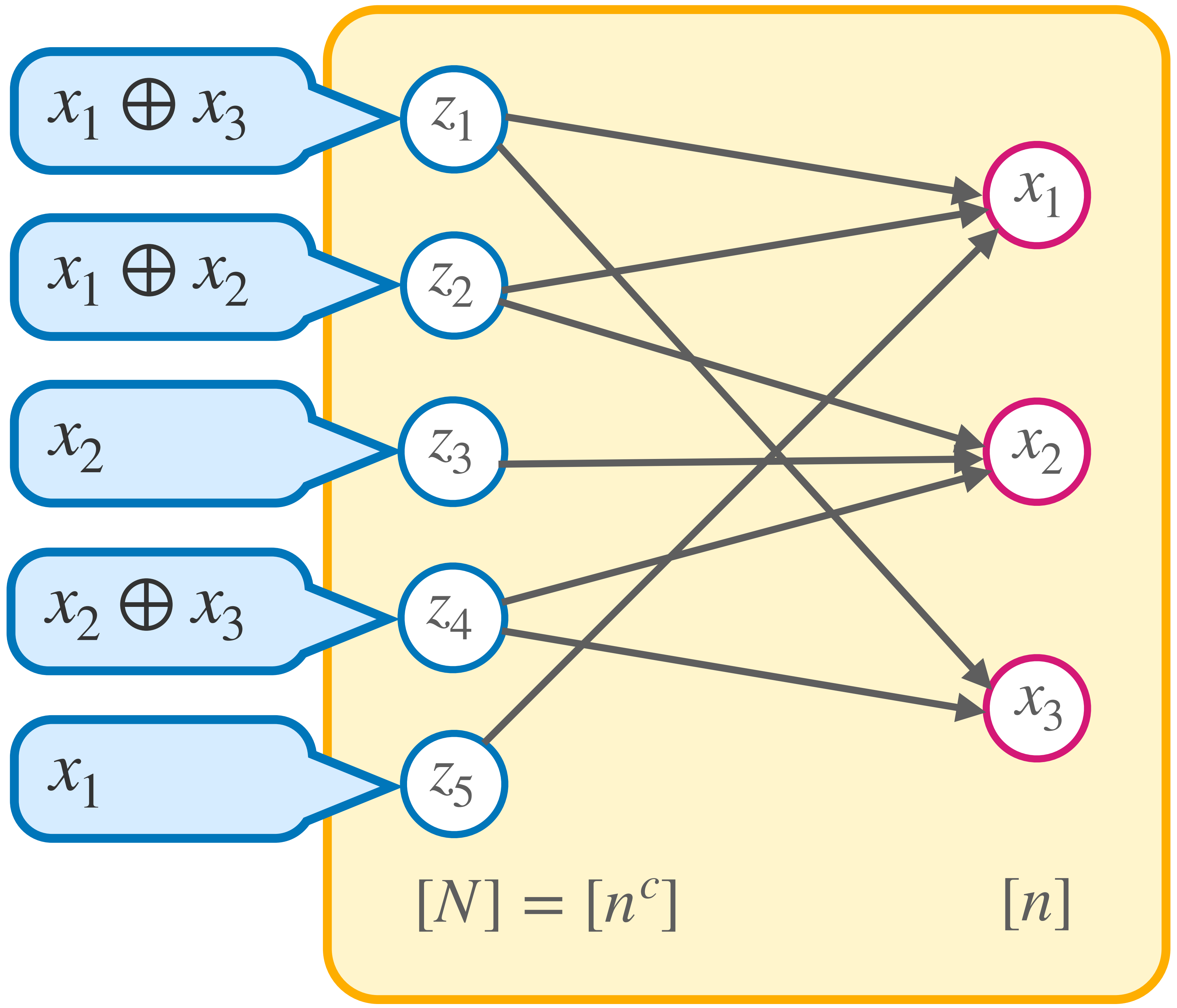

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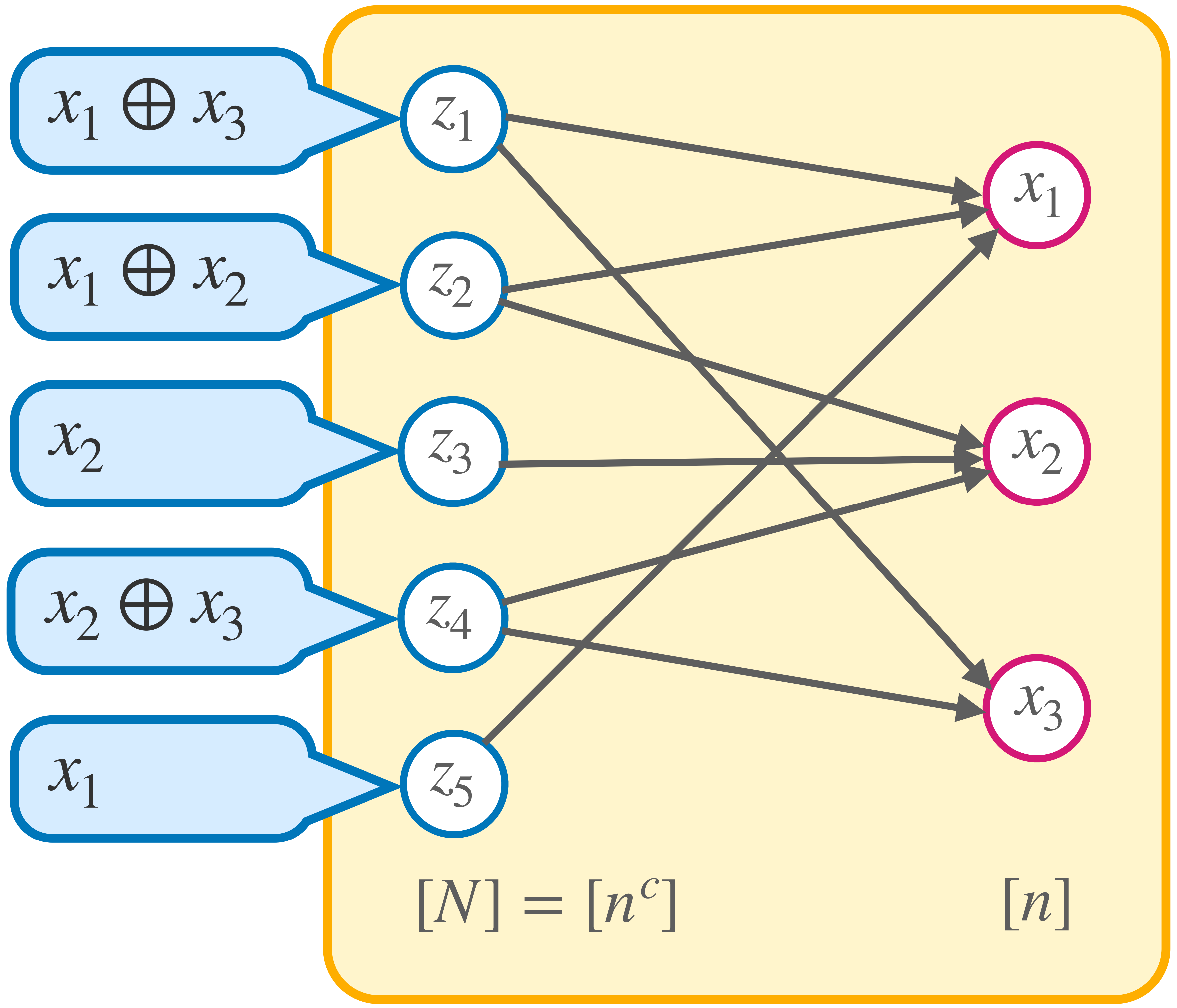

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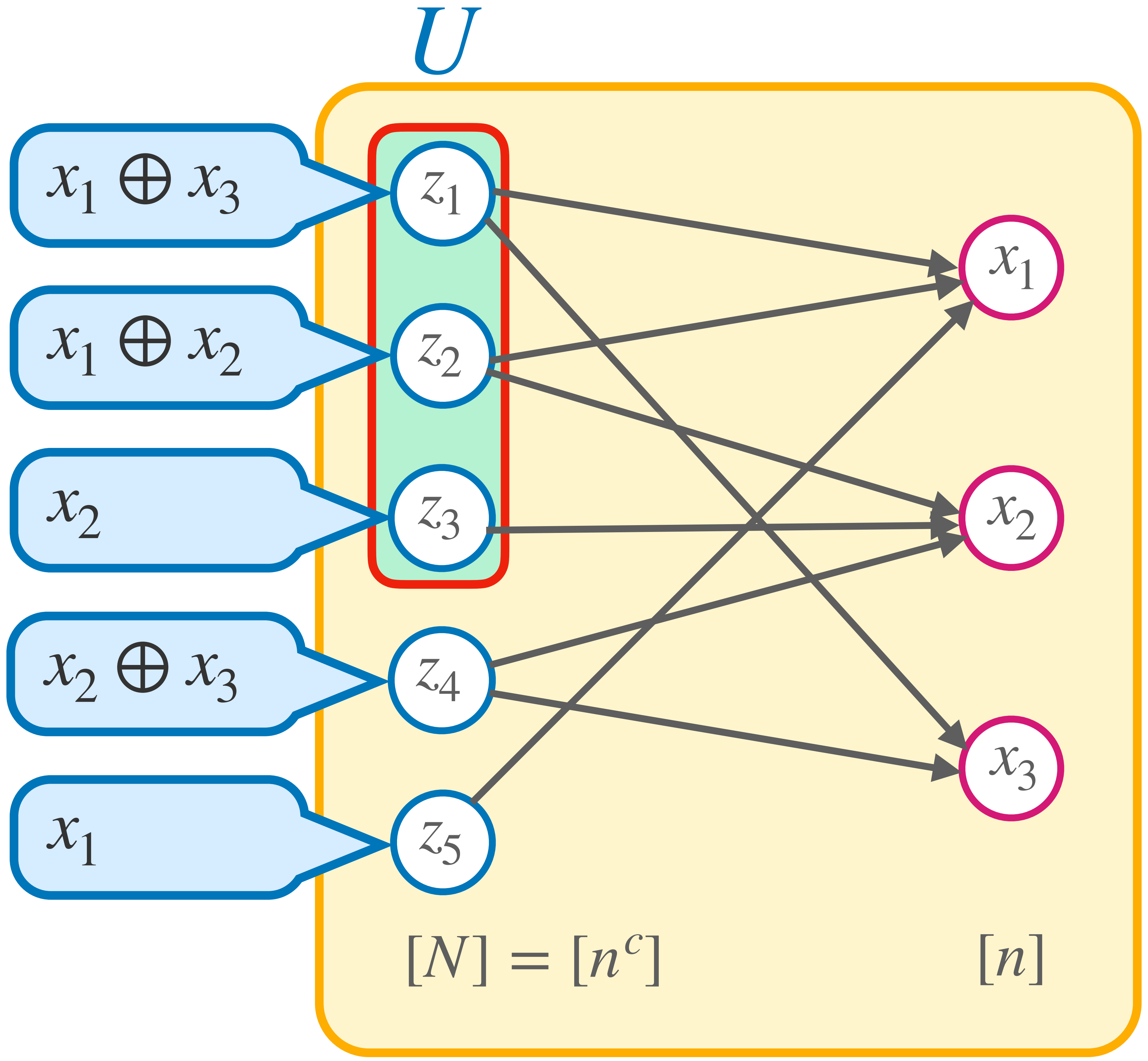

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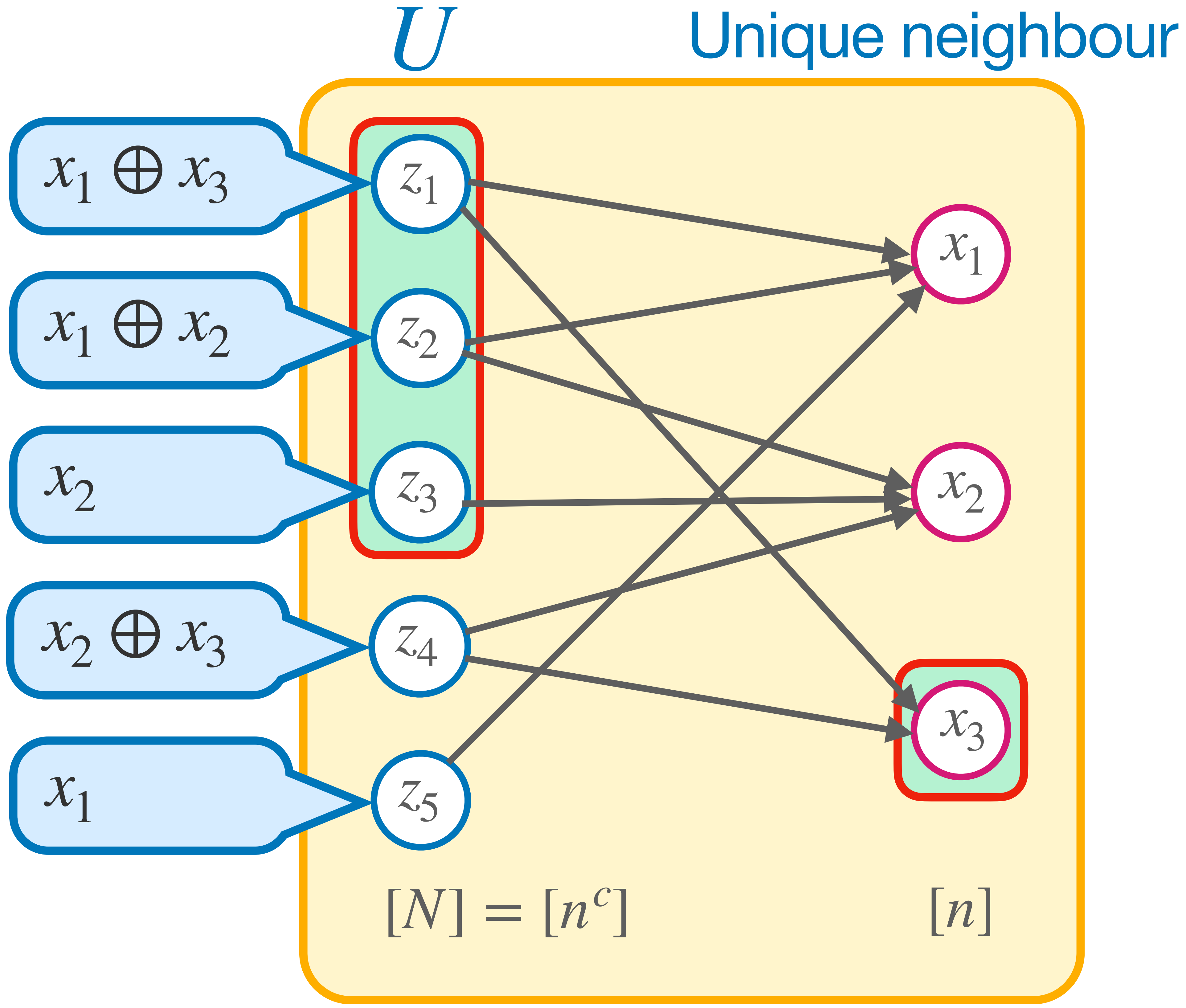

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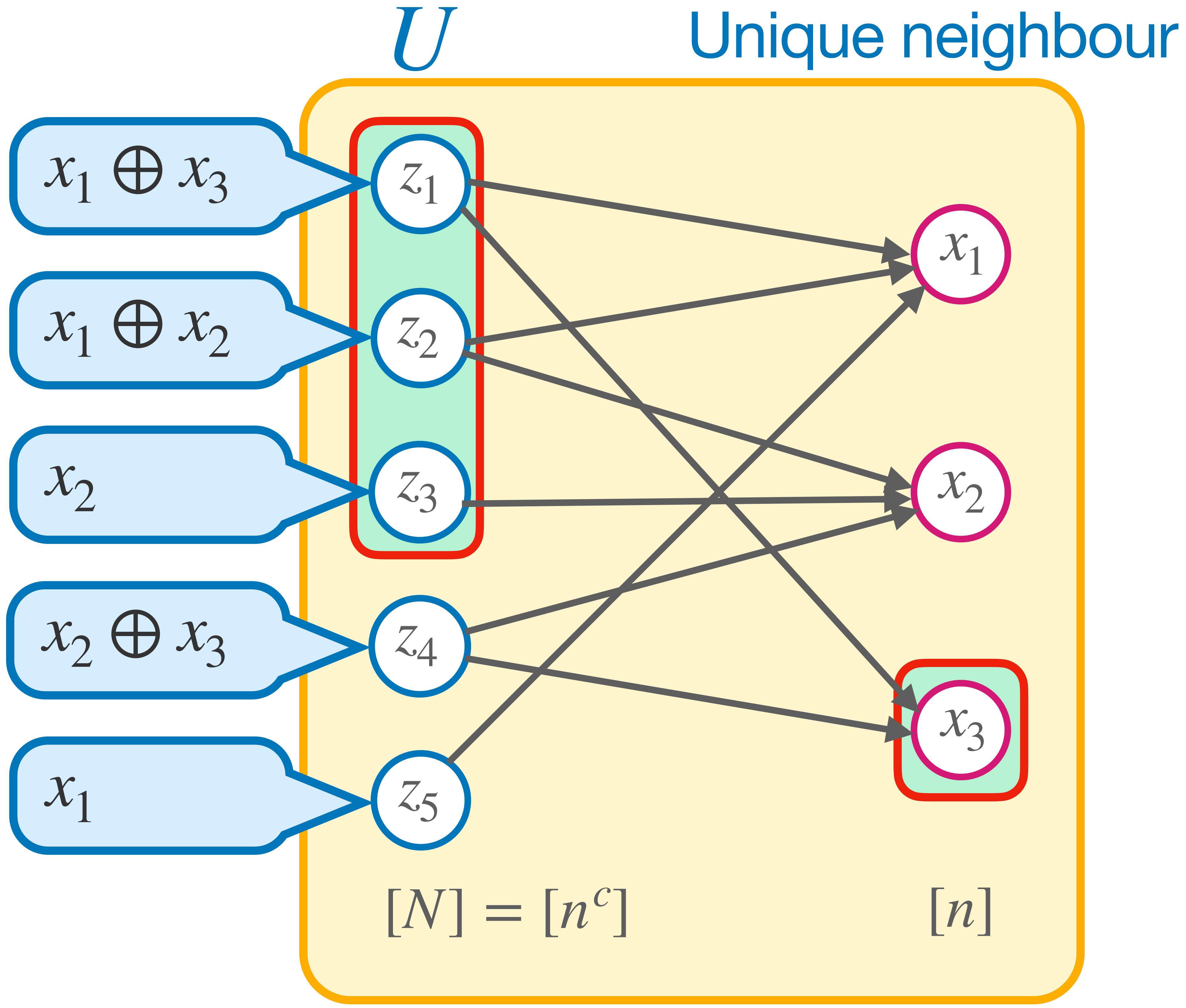

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→ Our gadget g will be XOR_G for expanding G

Depth Condensation

Main workhorse behind our tradeoff:

Depth Condensation Theorem: ([Razborov16] stated for tree-resolution)

Let G be r-expanding, F any unsatisfiable formula.

If Π is a Resolution proof of $F \circ \mathsf{XOR}_G$ with $\mathsf{width}(\Pi) \leq r/4$ then

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Main Theorem (Res):

Let G be r-expanding.

If Π is a Resolution proof of $Peb \circ \mathsf{XOR}_G \circ \mathsf{XOR}_2$ with $\log \mathsf{size}(\Pi) \leq r/4$ then

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Let $\varepsilon > 0$, let $c \geq 1$ be real-valued parameter

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- For Res(k) we prove a Resolution width $\rightarrow$ Res(k) size lifting theorem with $g = XOR_2$, which uses the switching lemma of **[SBI04]**

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Our proof uses a characterization of resolution depth by Prover-Adversary games

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Claim: If there is a strategy for the Adversary such that the game always continues for at least d rounds, then any resolution proof of F requires depth $\geq d$

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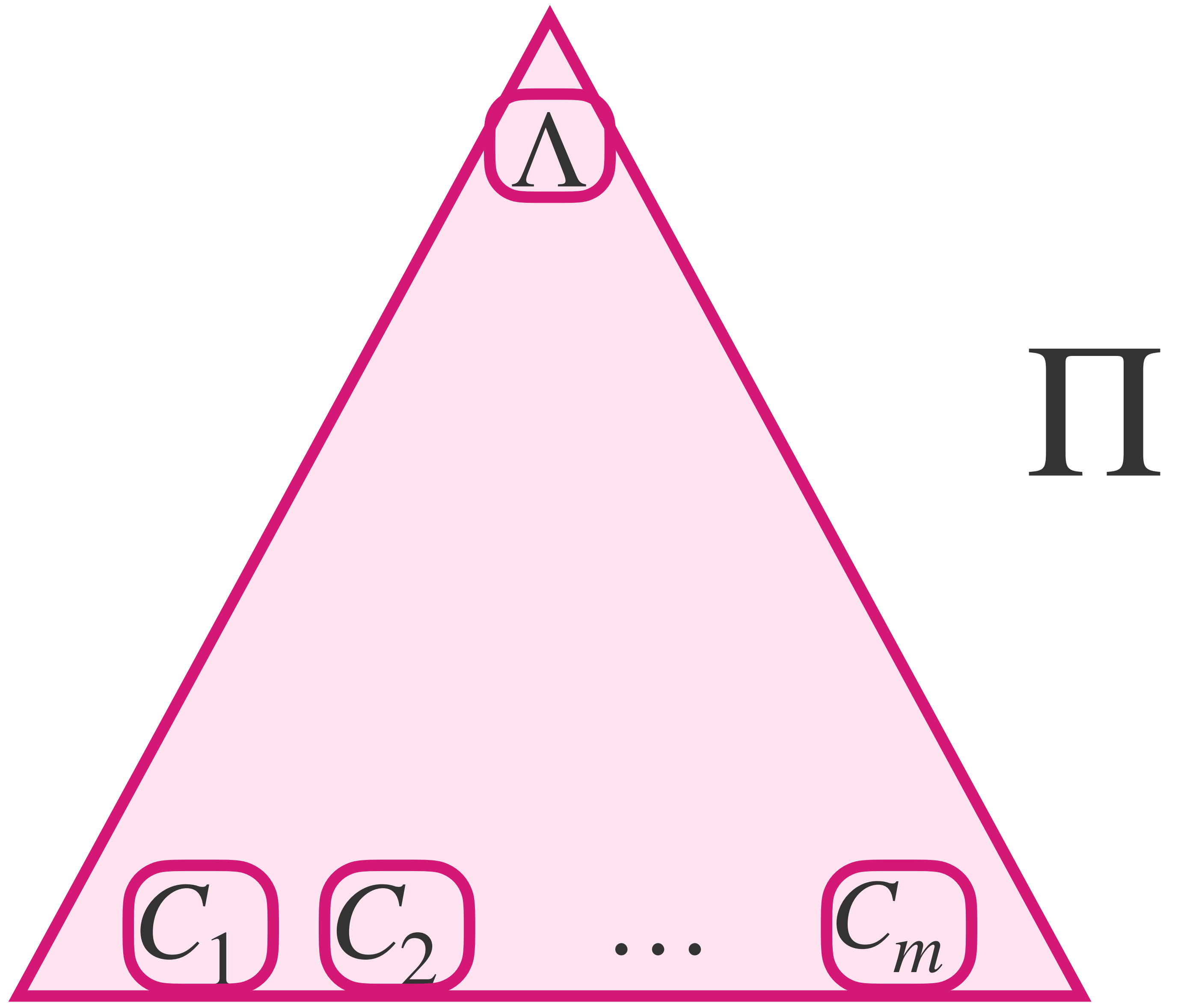

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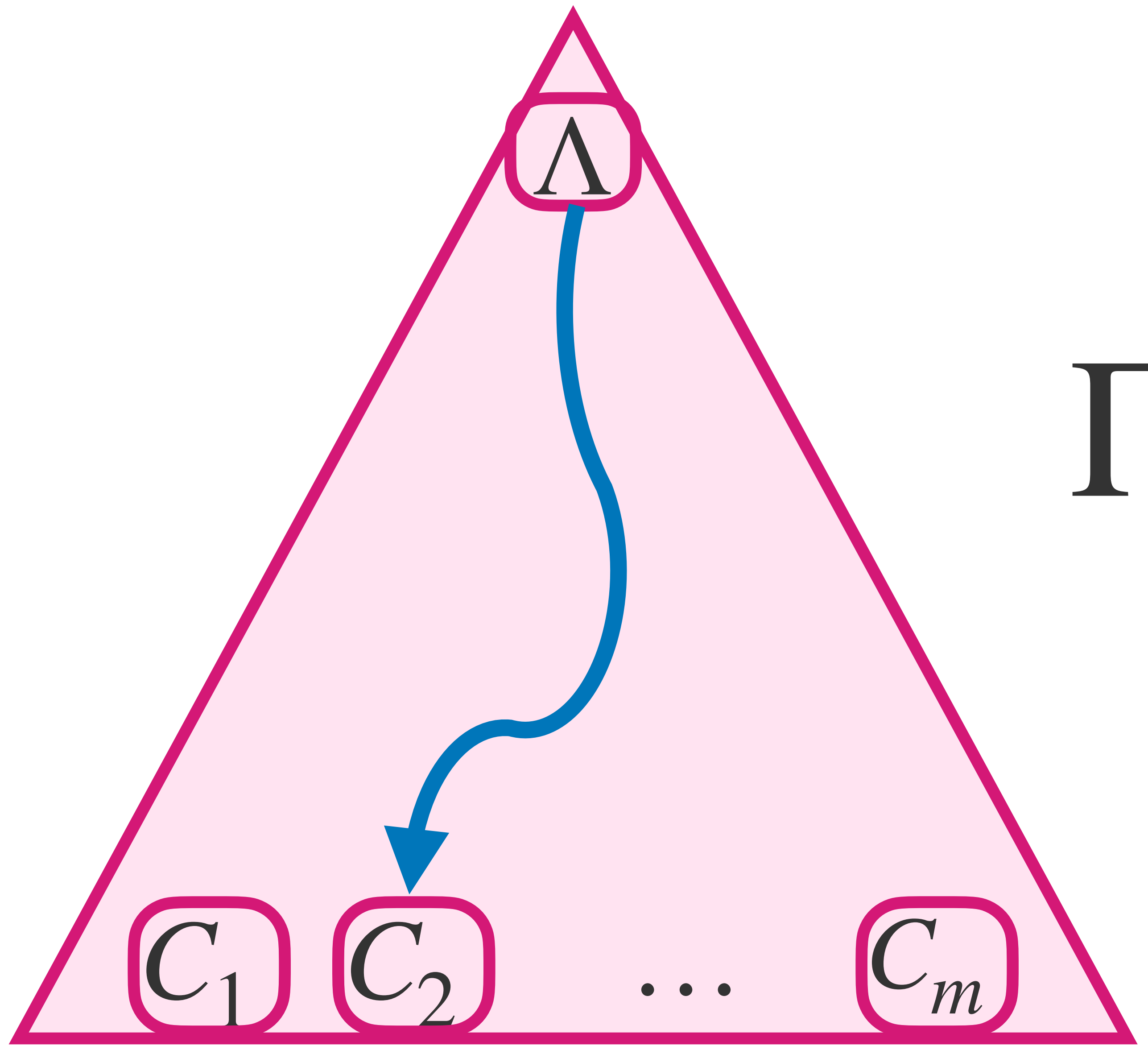

Π

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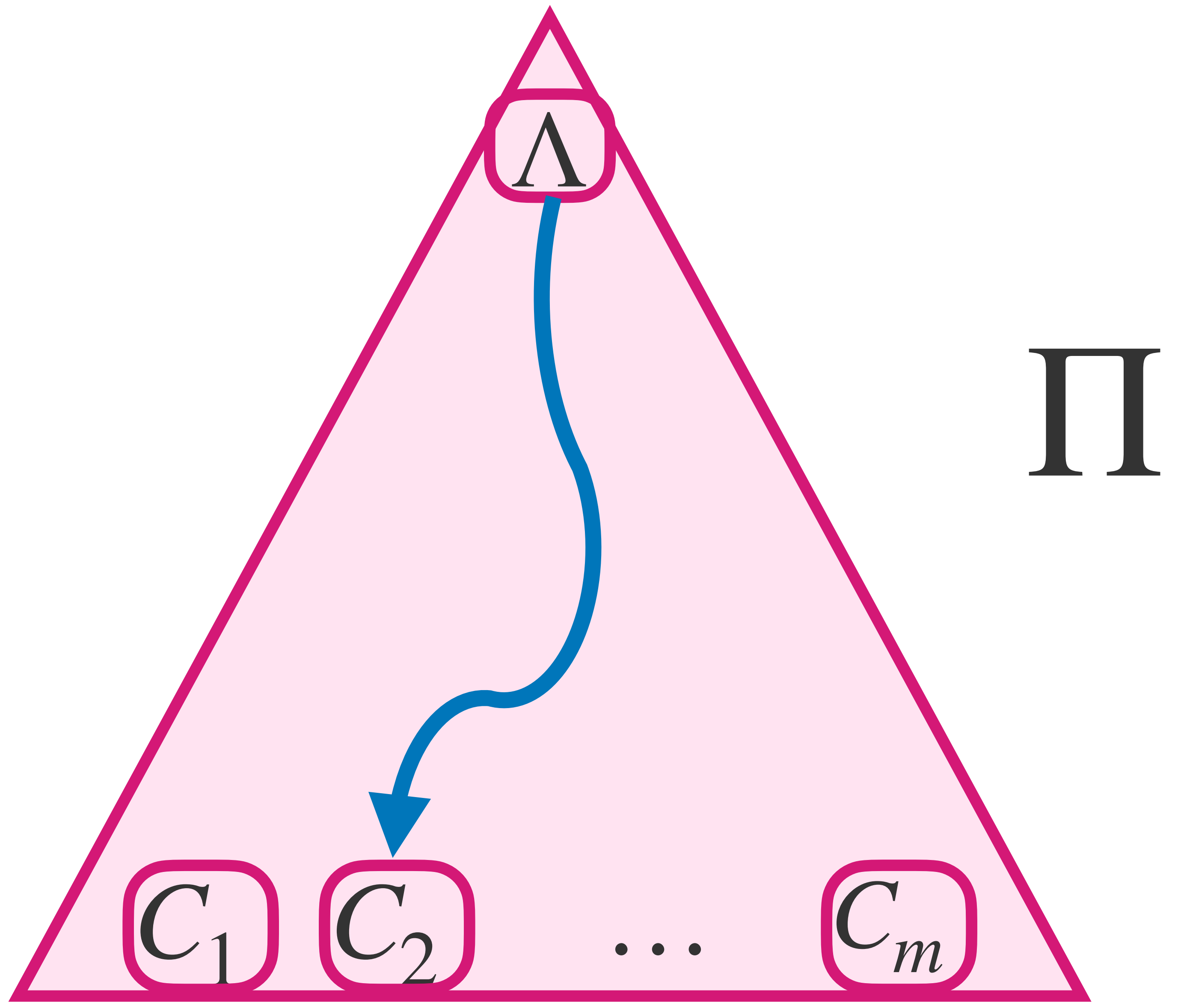

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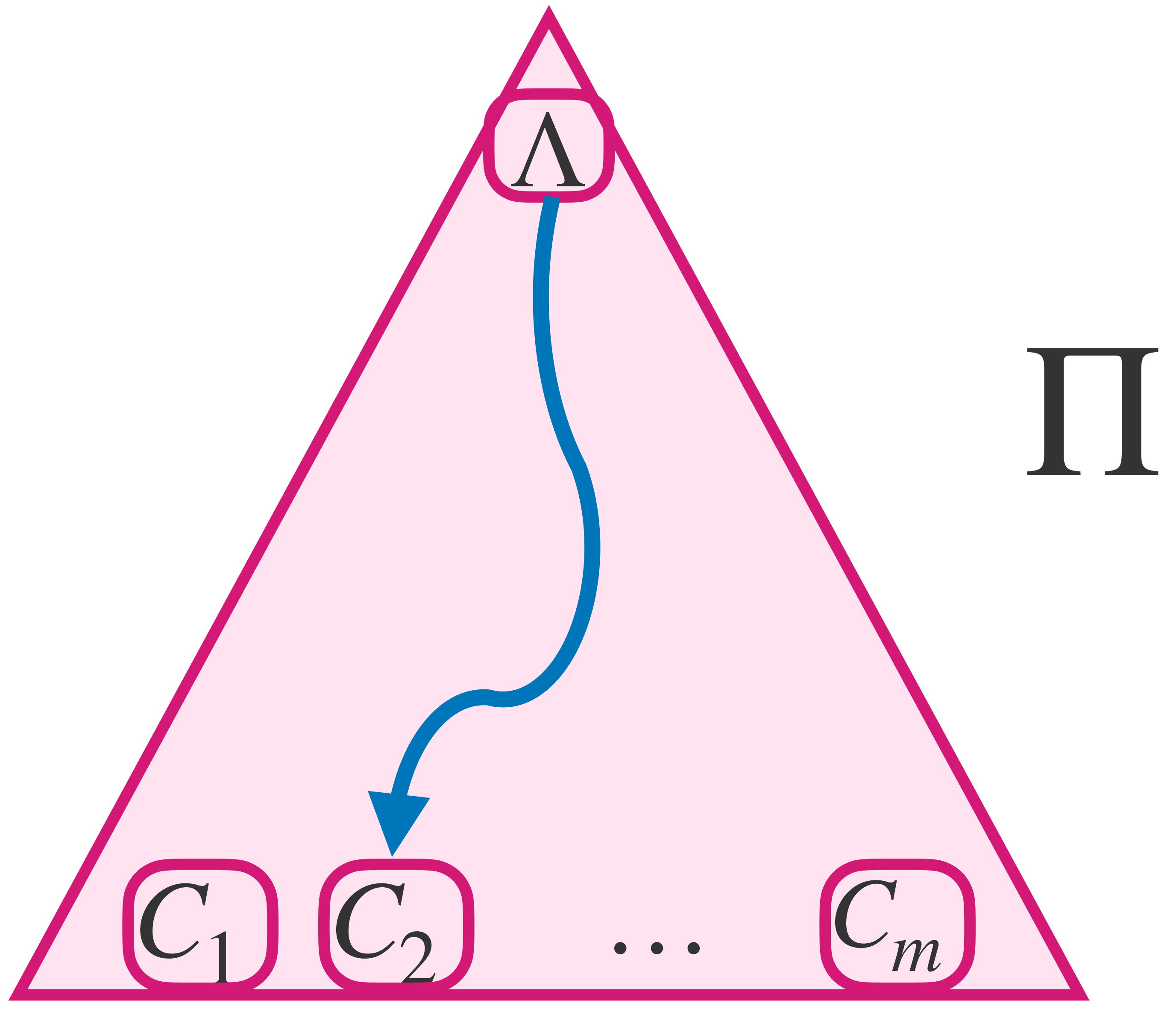

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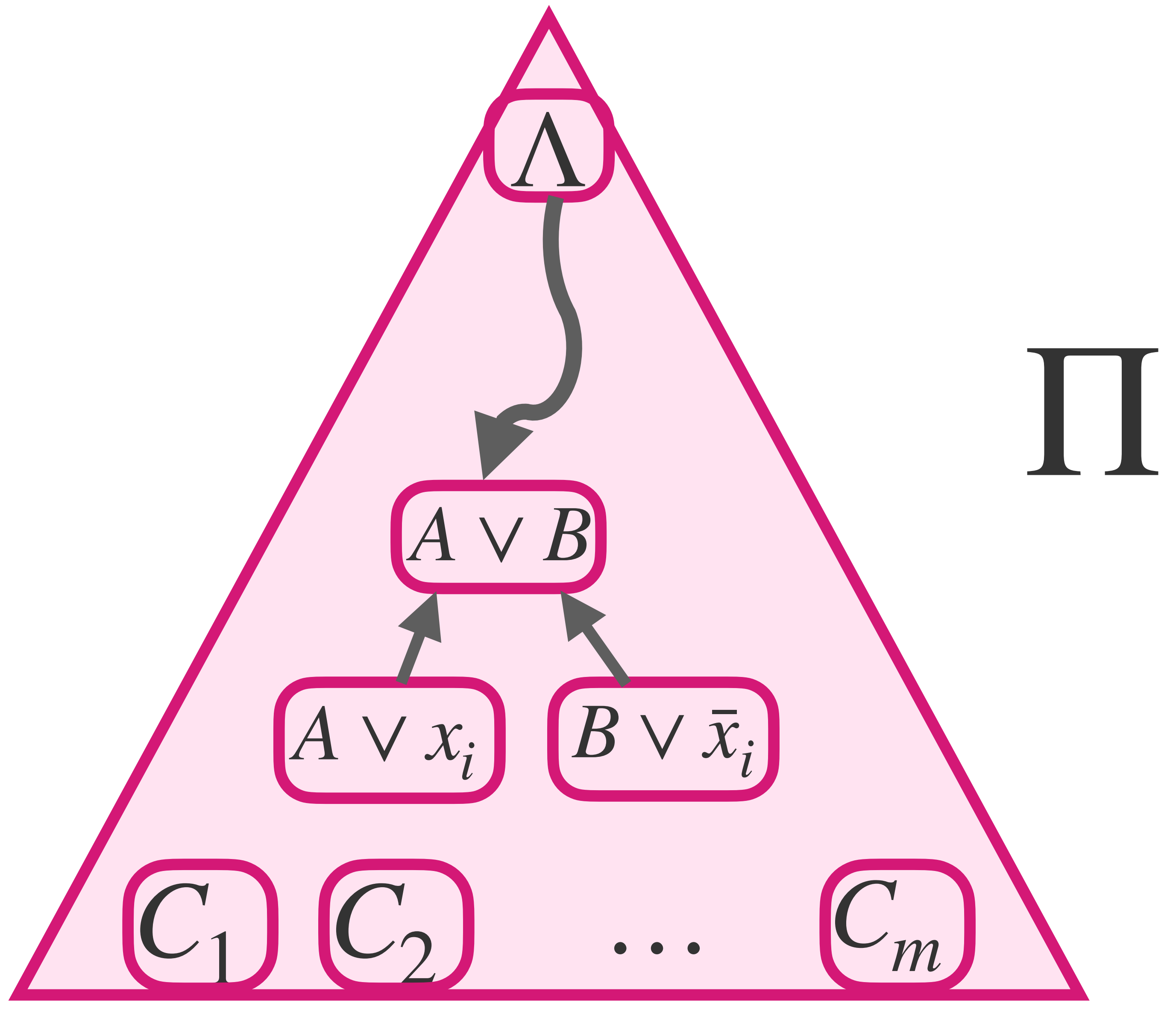

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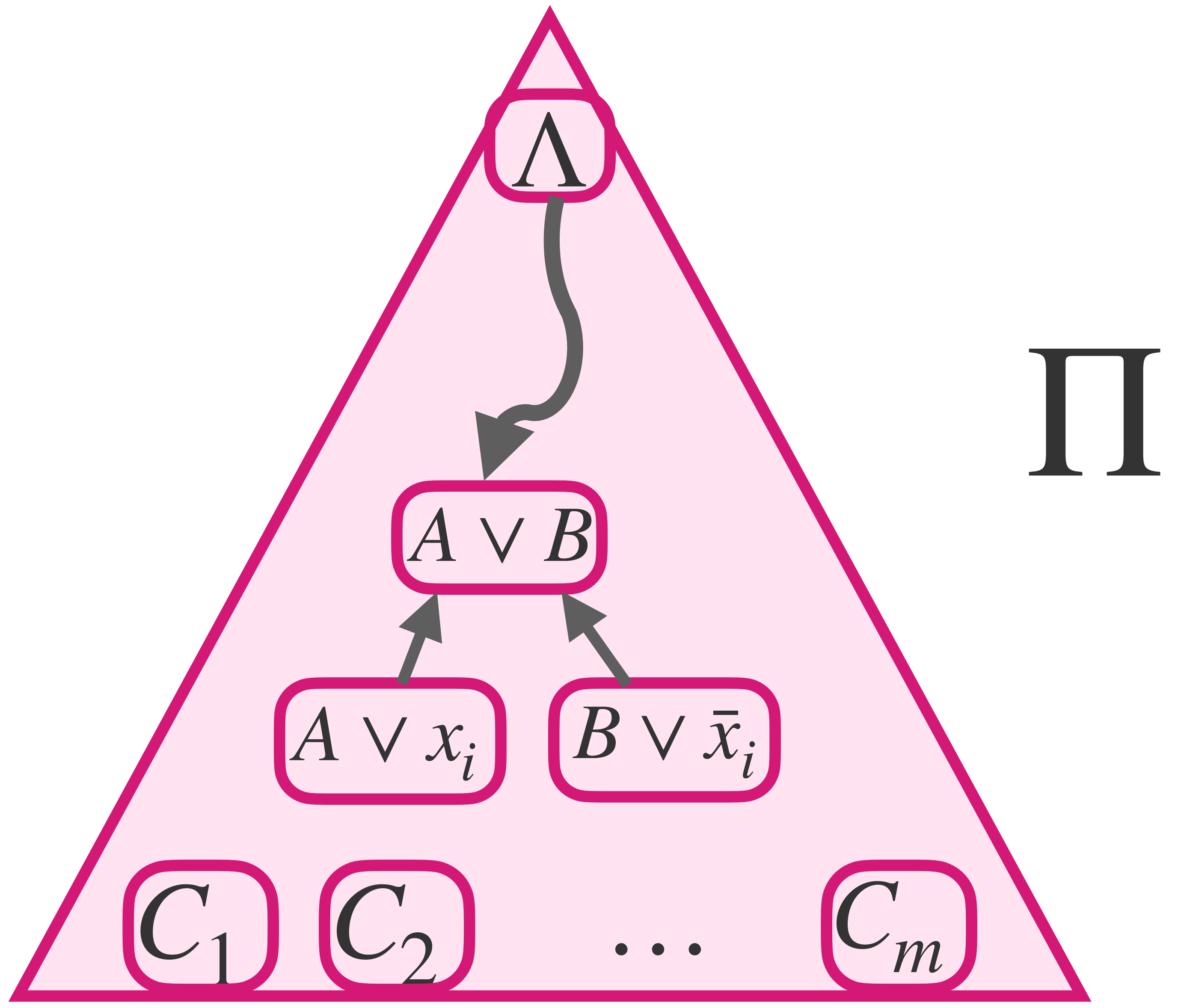

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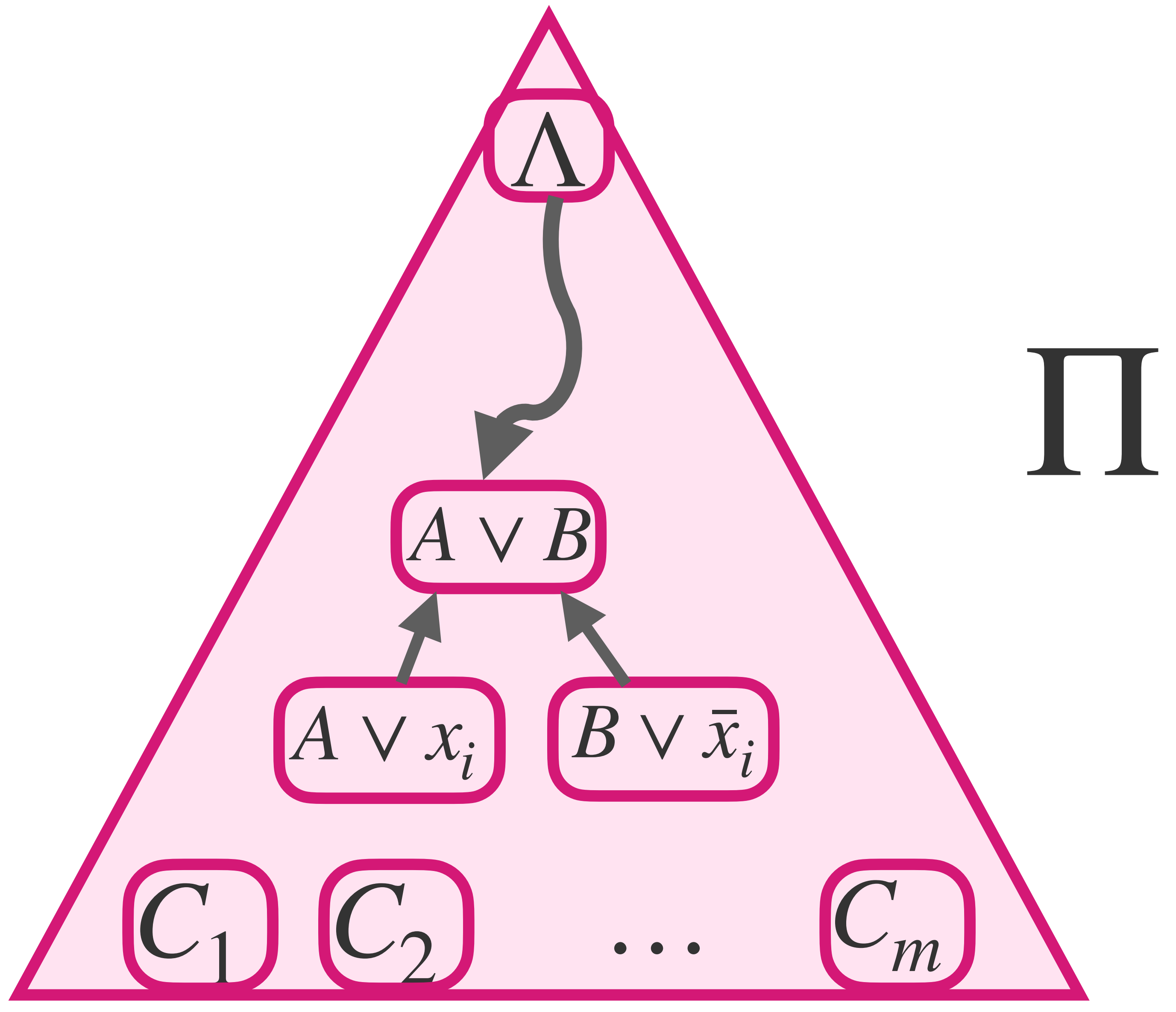

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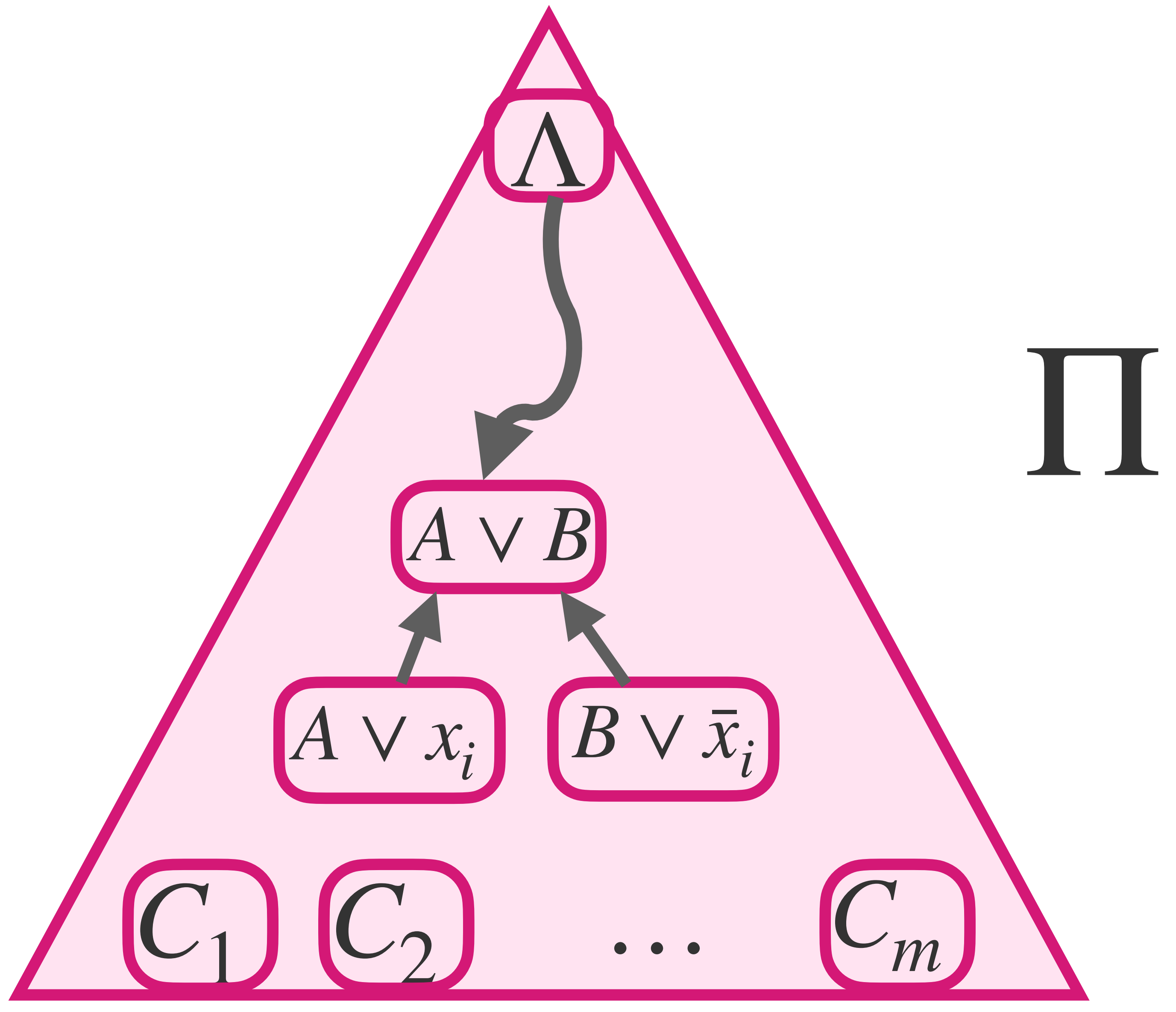
- Otherwise, move to $B \vee \bar{x}_i$. Forget $A \backslash B$



(New) Proof of Depth Condensation

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Let G be an r-boundary expander, F any unsatisfiable formula.

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$\rightarrow$ Use A to construct an Adversary Strategy for the w-bounded game on $F \circ XOR_G$ to survive $\Omega(d/w)$ rounds, for any $w \leq r/4$.

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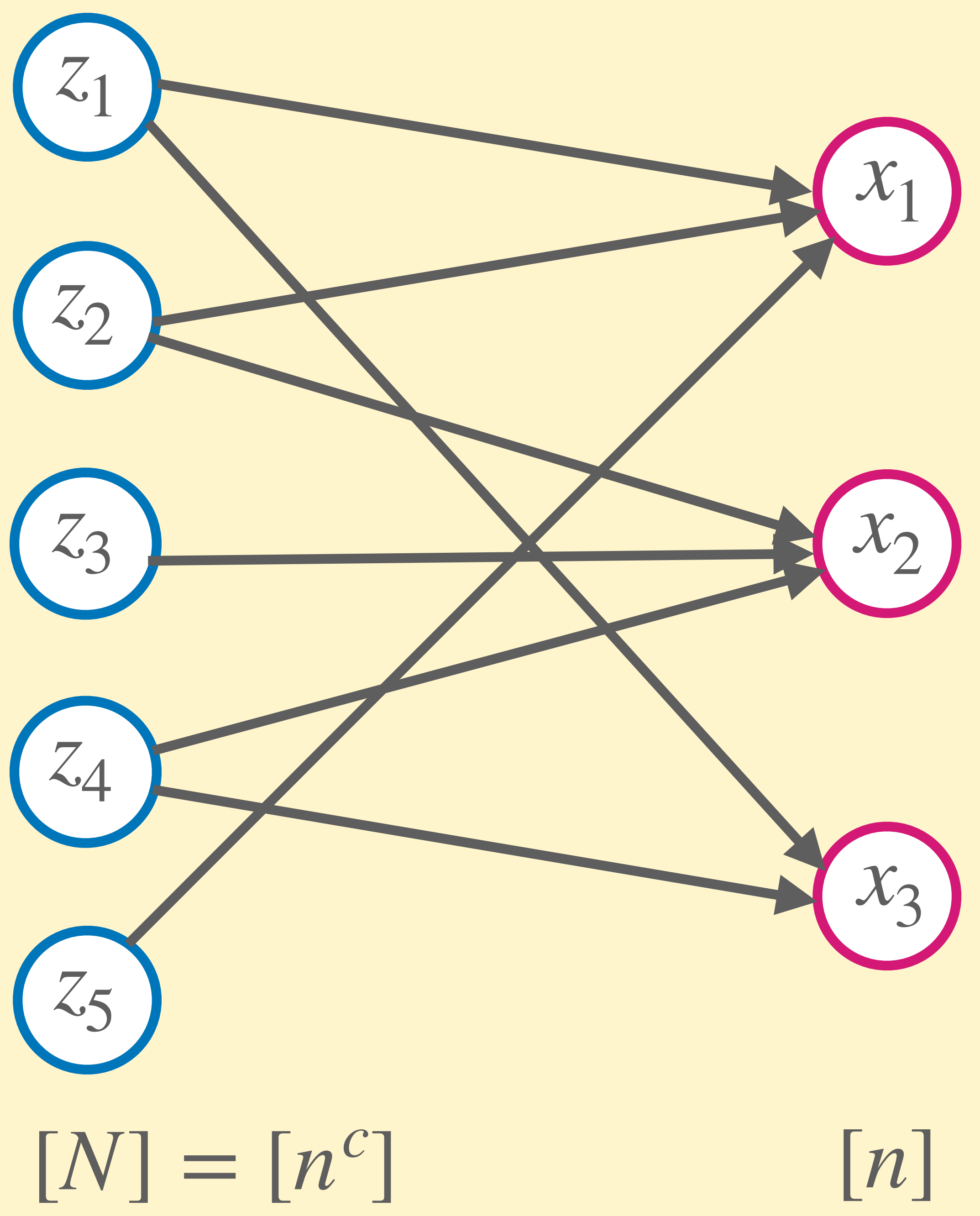

Adversary strategy for $F \circ \text{XOR}_G$:

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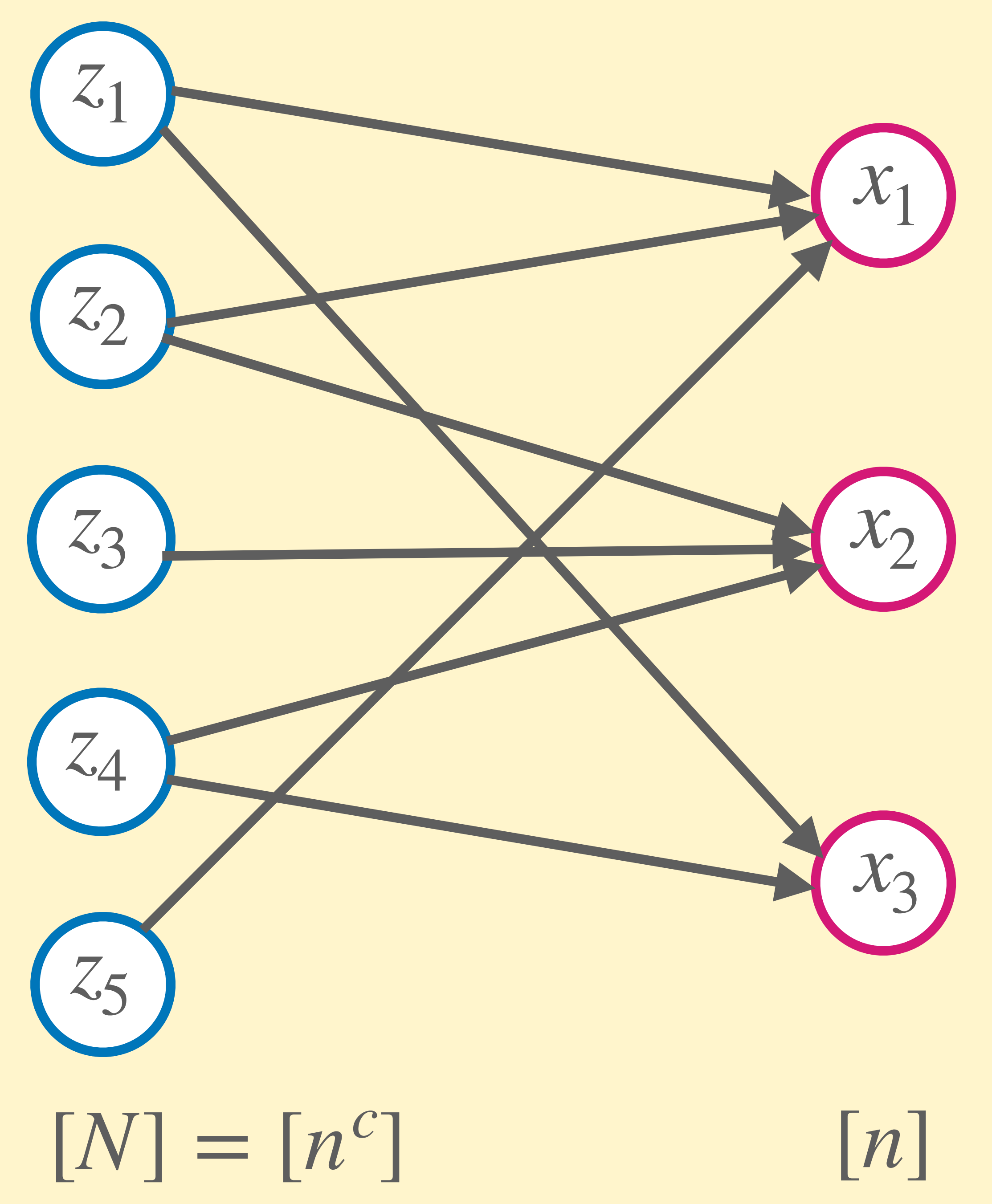

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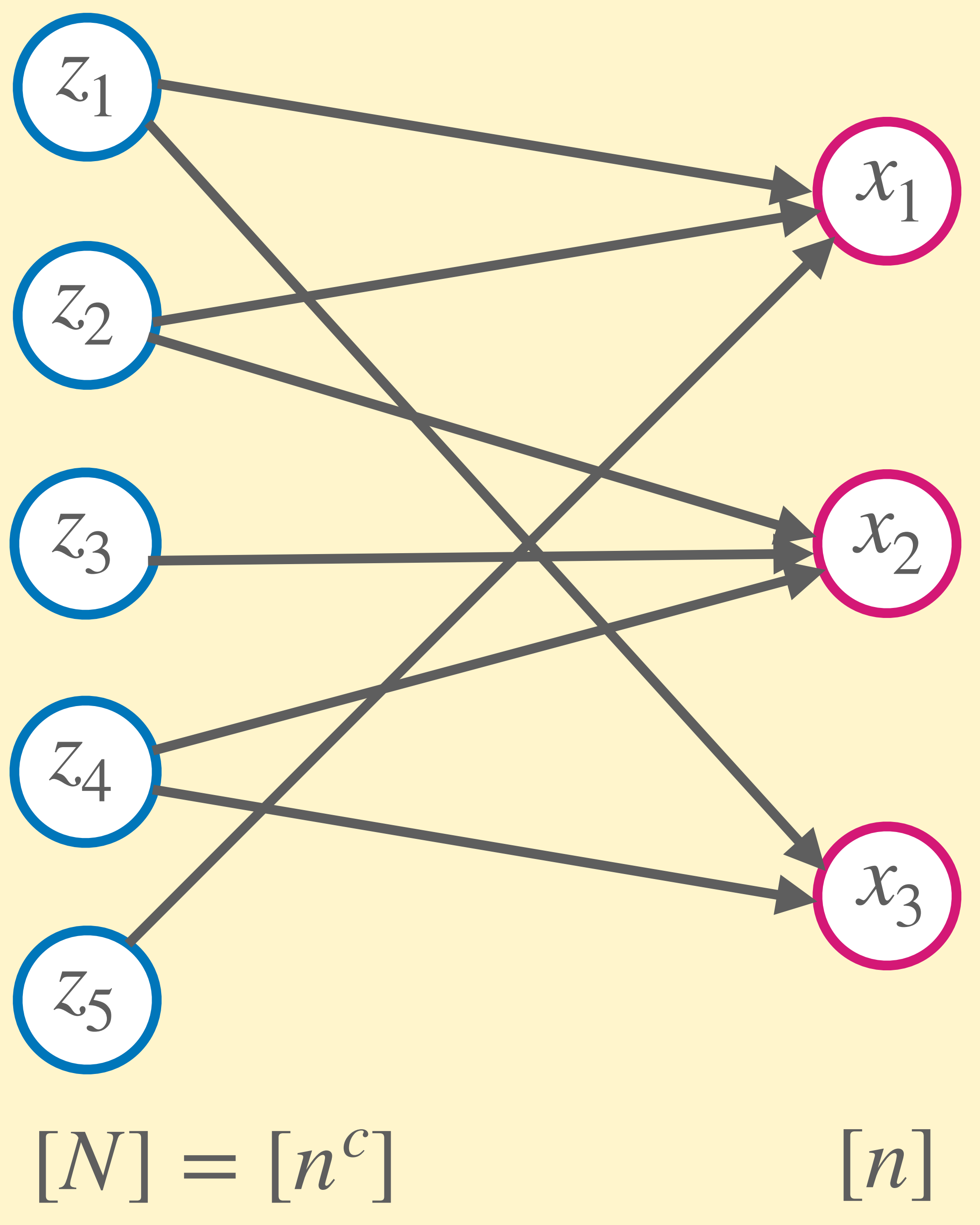
- If there are ≥ 2 variables in $\mathrm{N}(z_j)$ for every $z_j \in \mathrm{N}(x_i)$: set x_i arbitrarily



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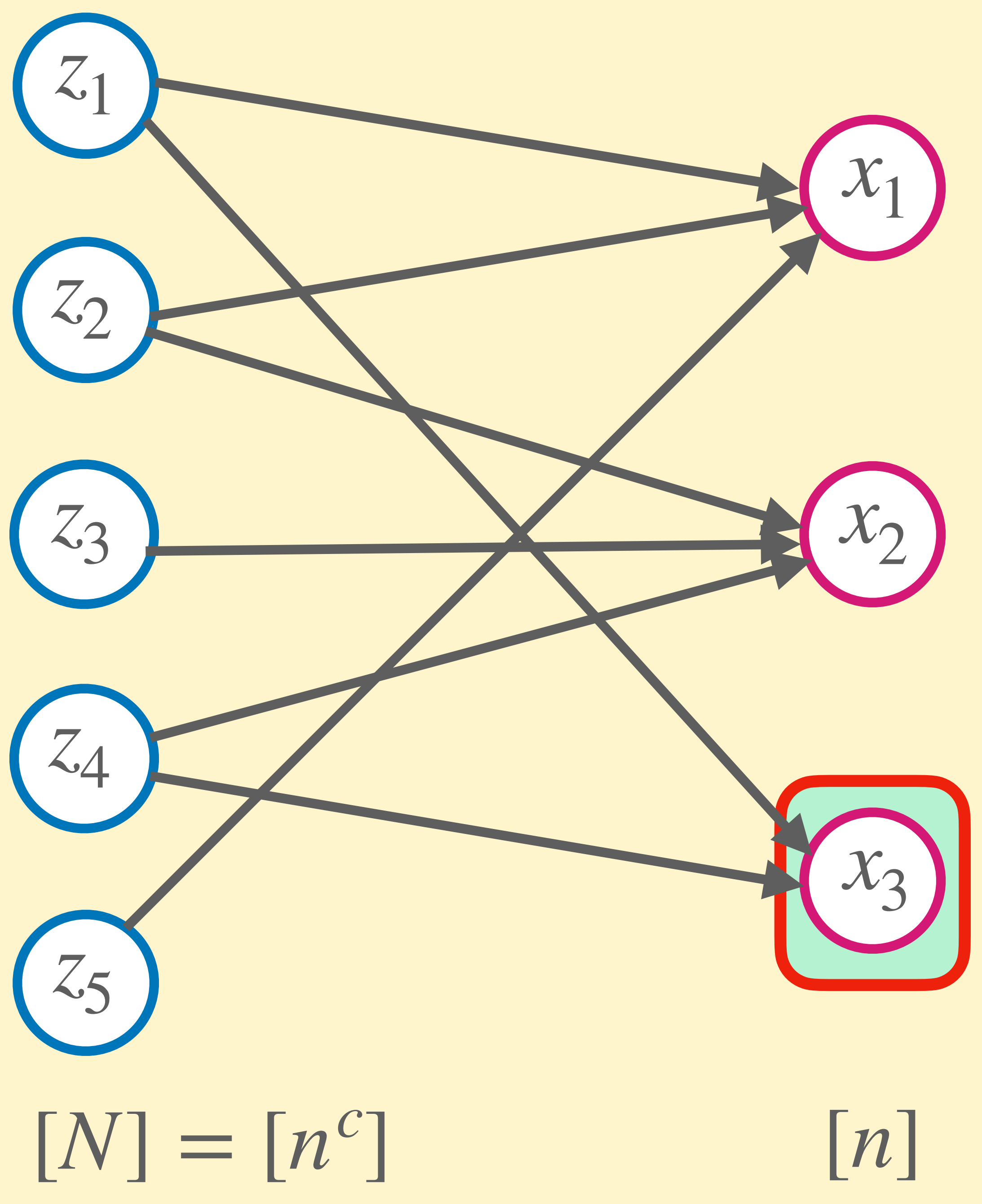

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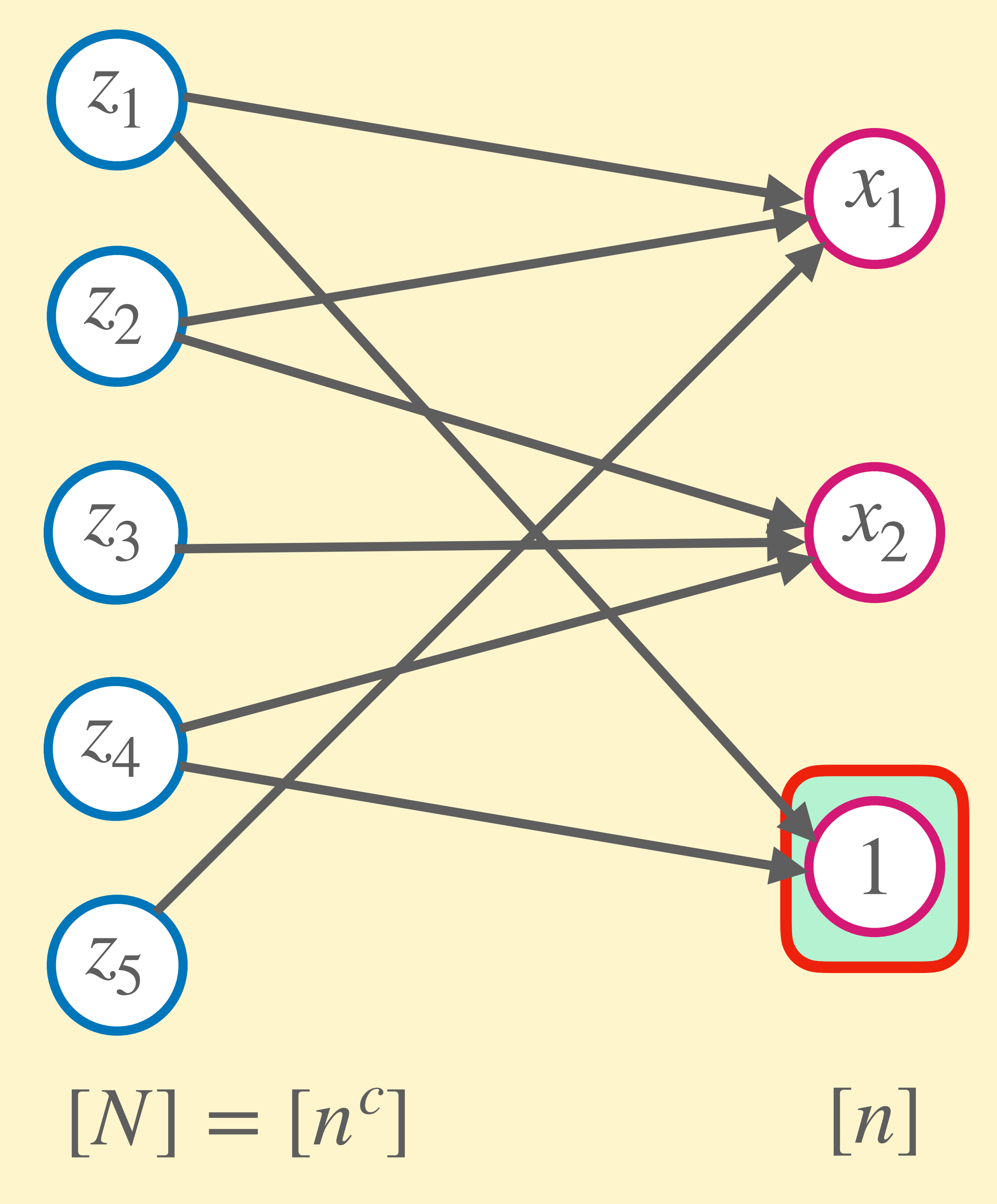

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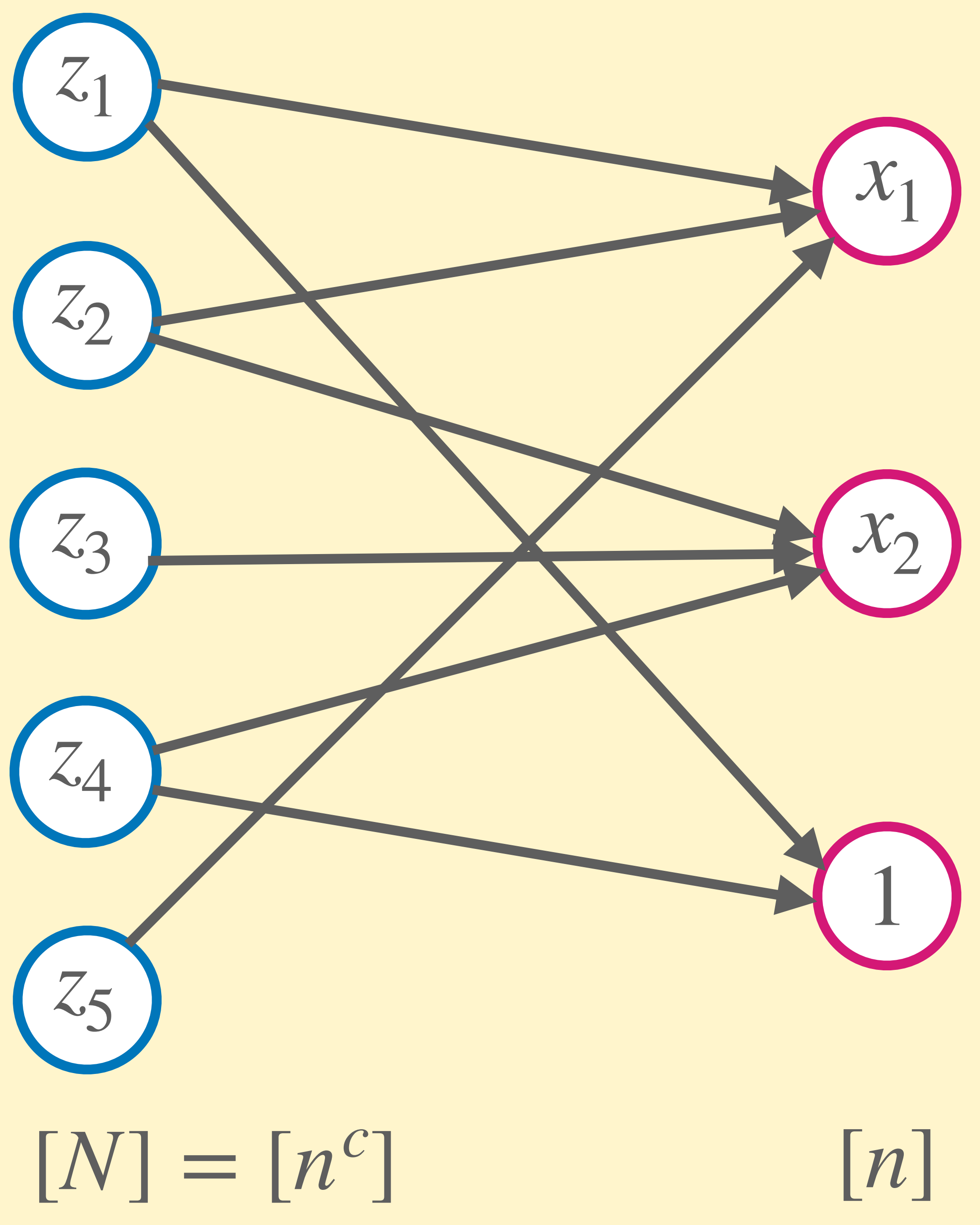

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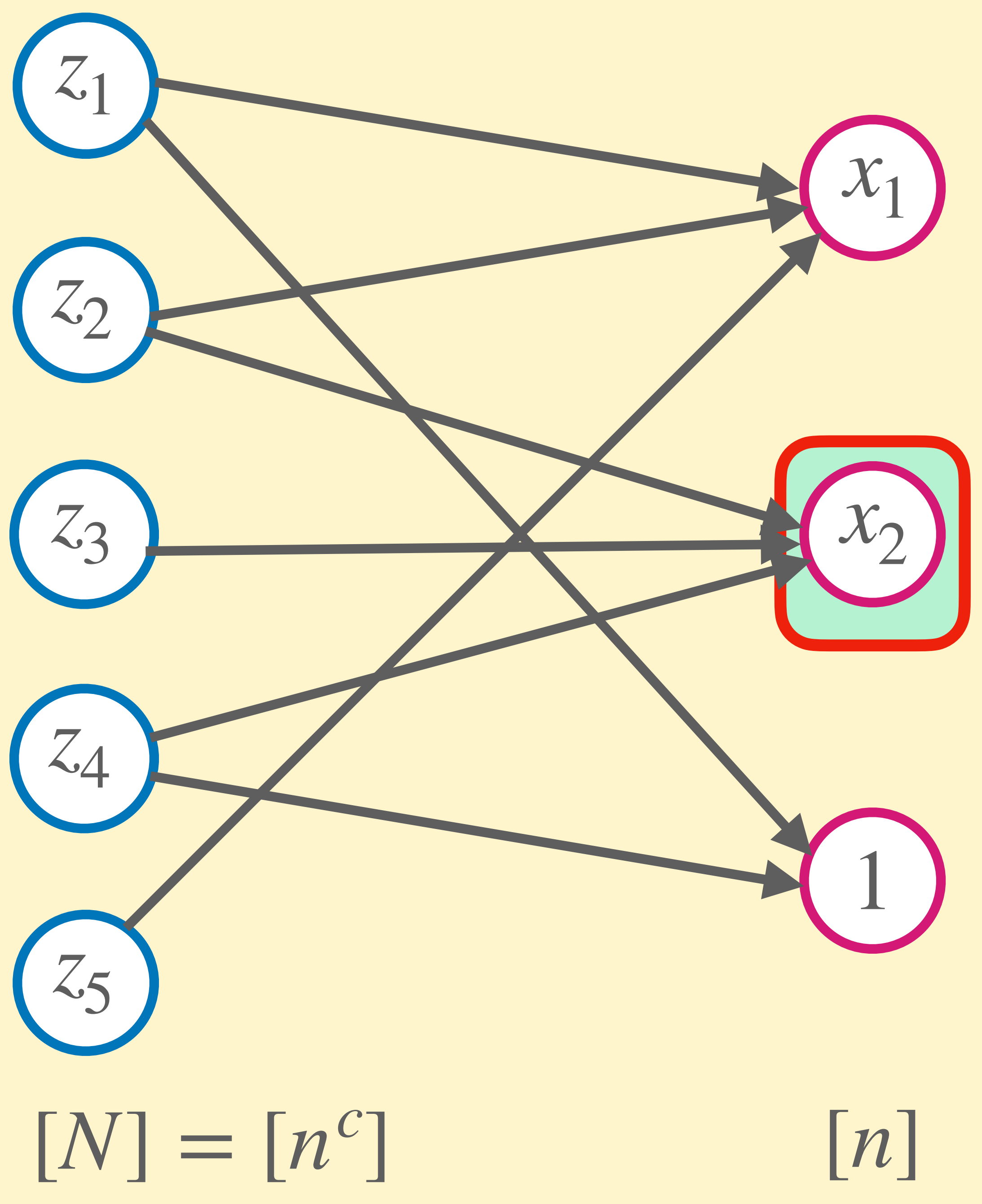

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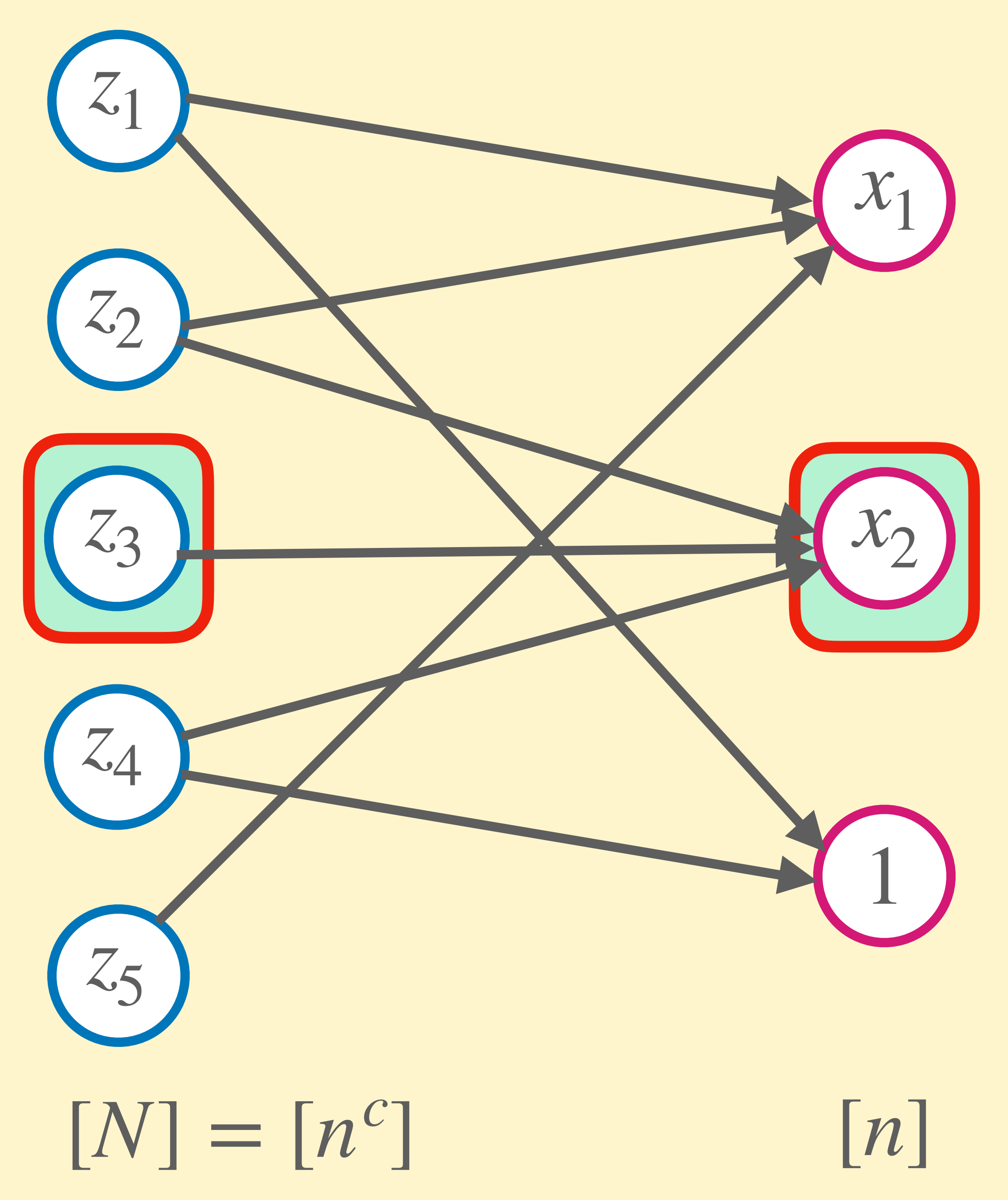

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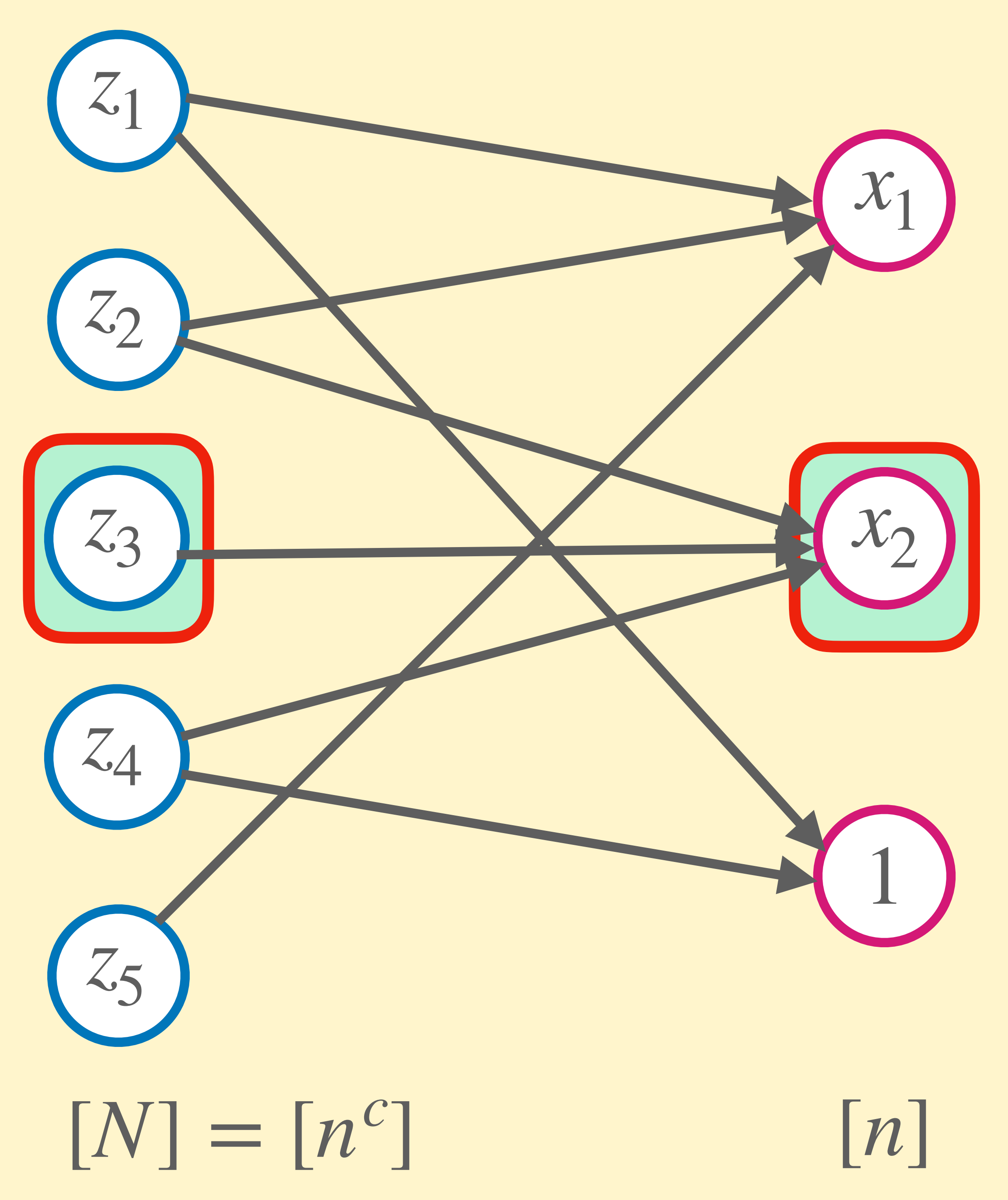

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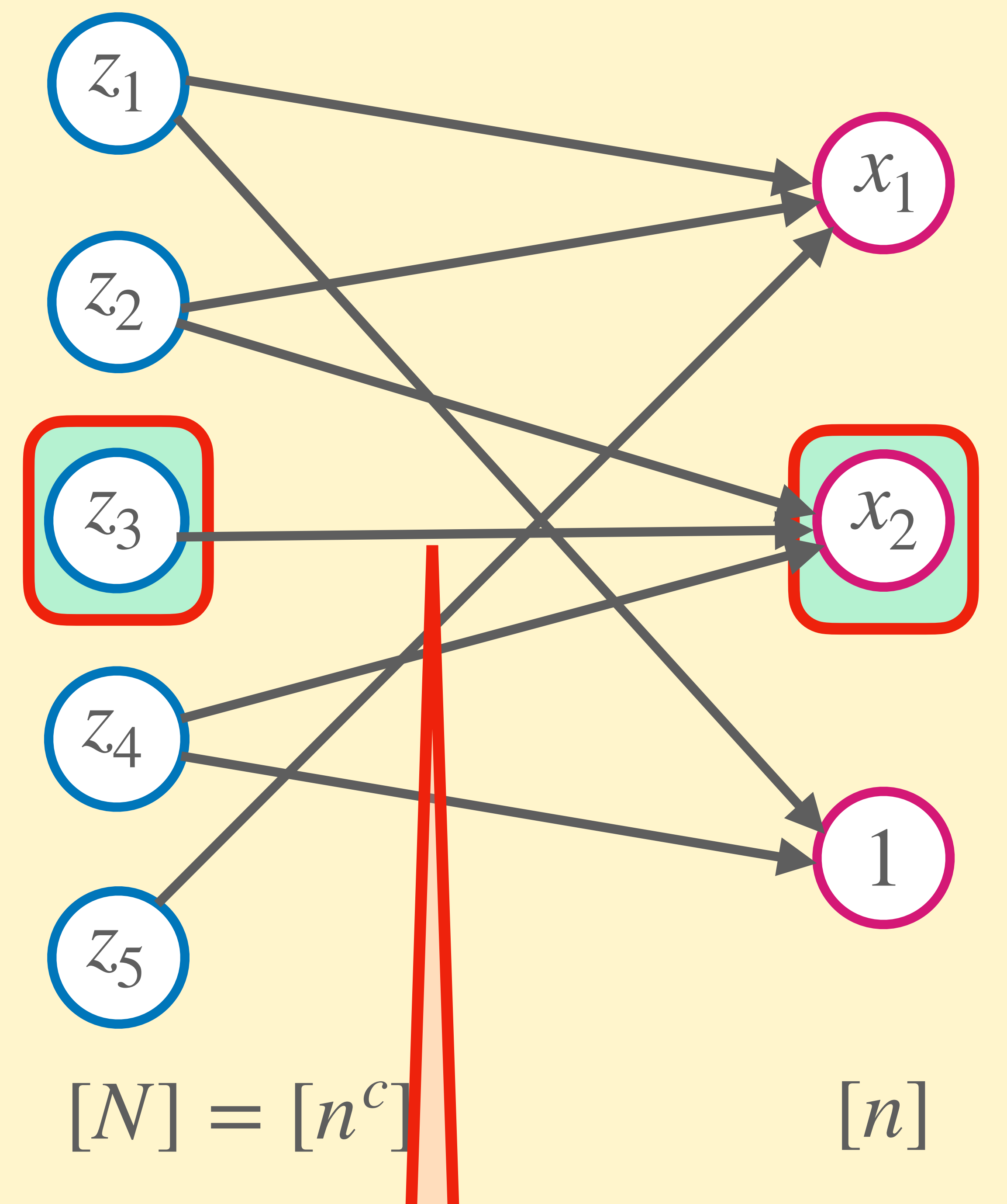

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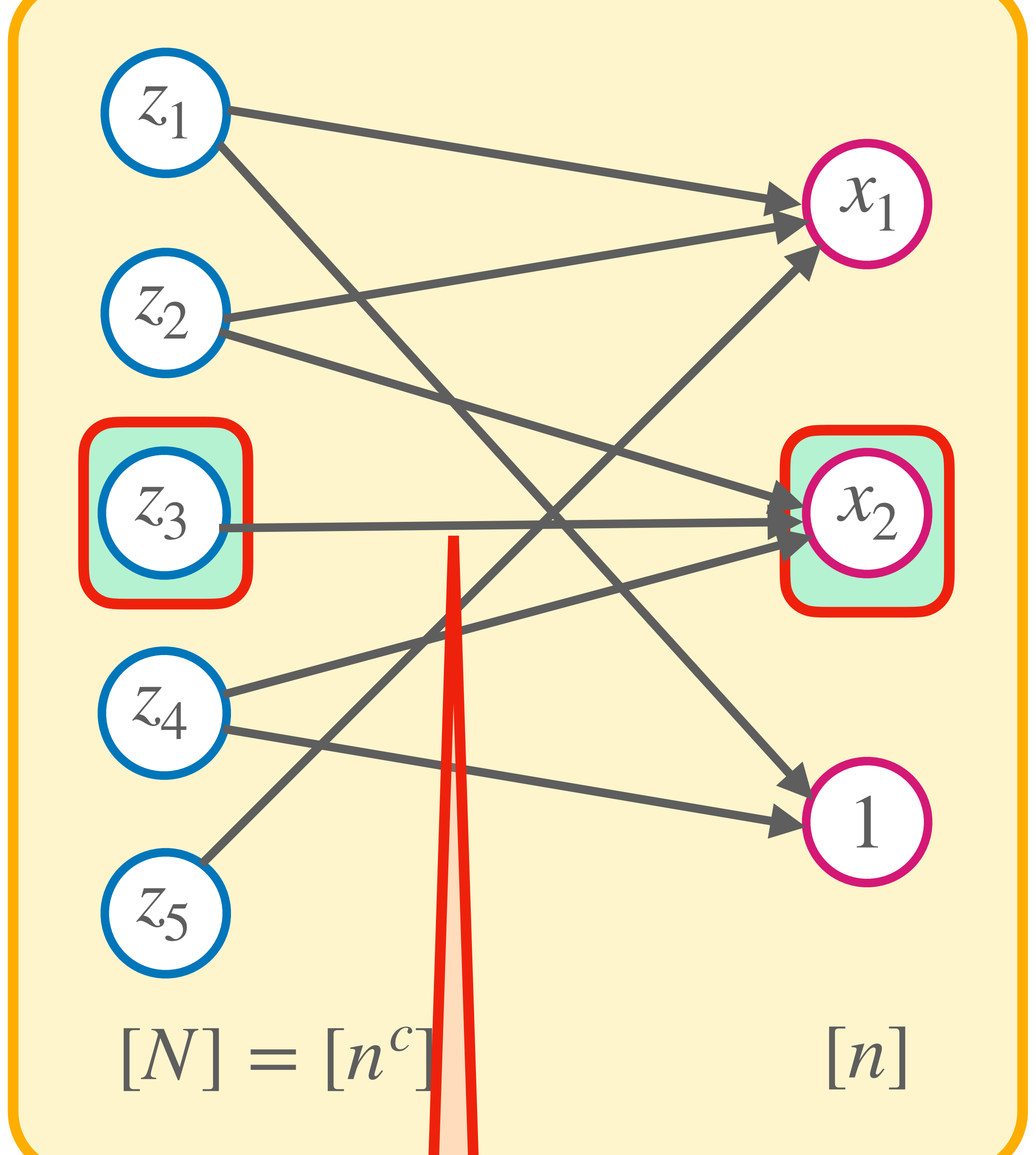

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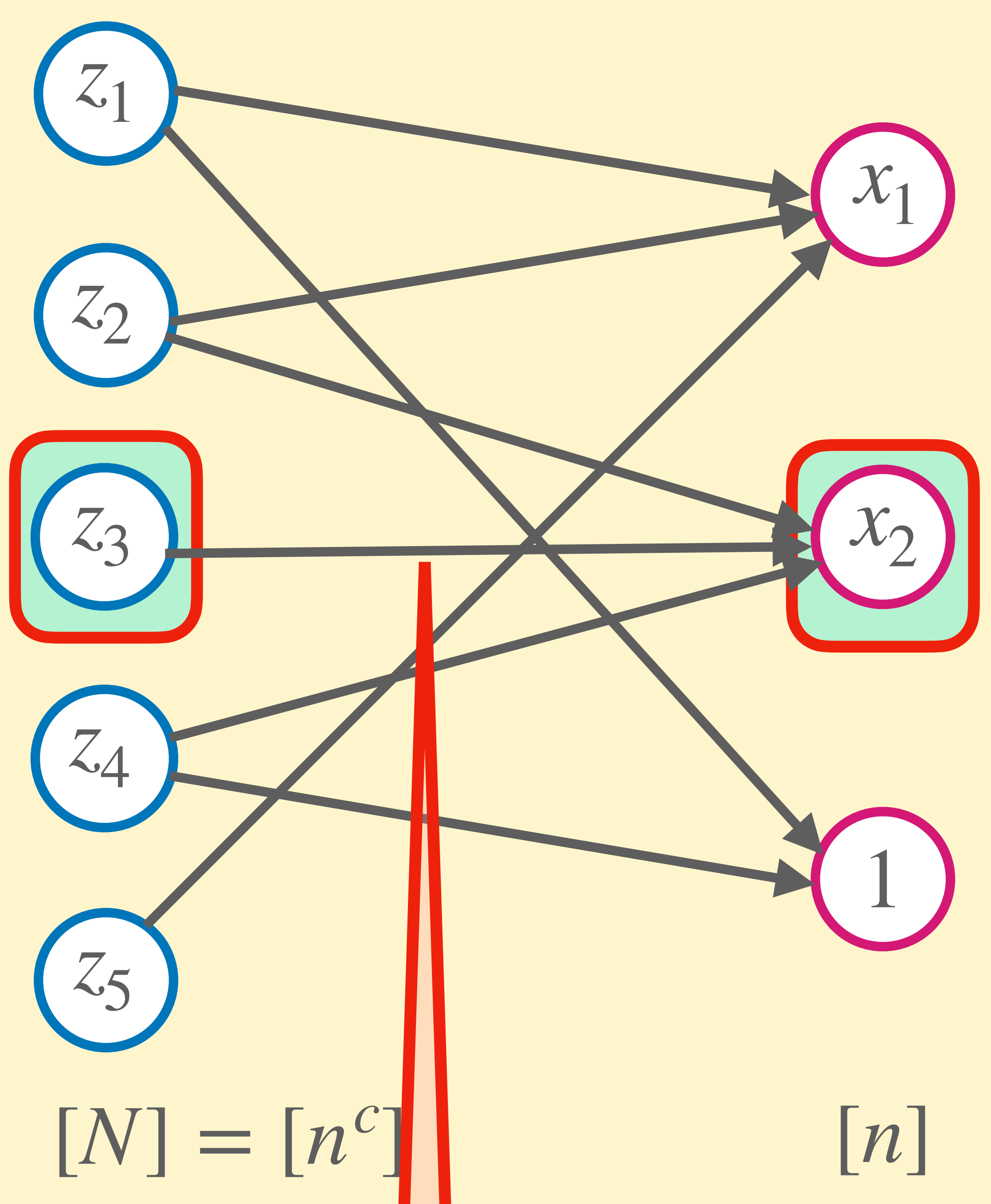

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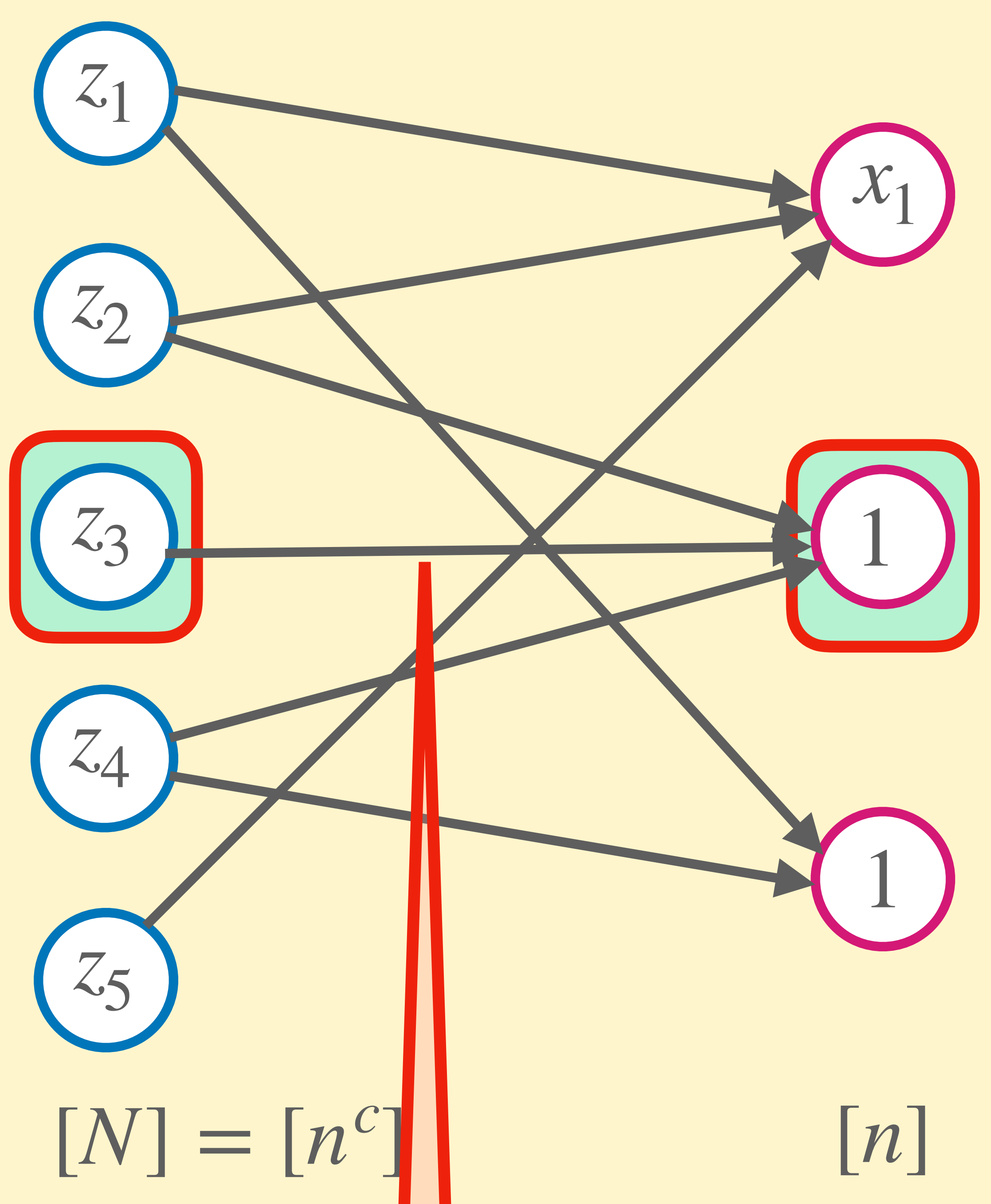

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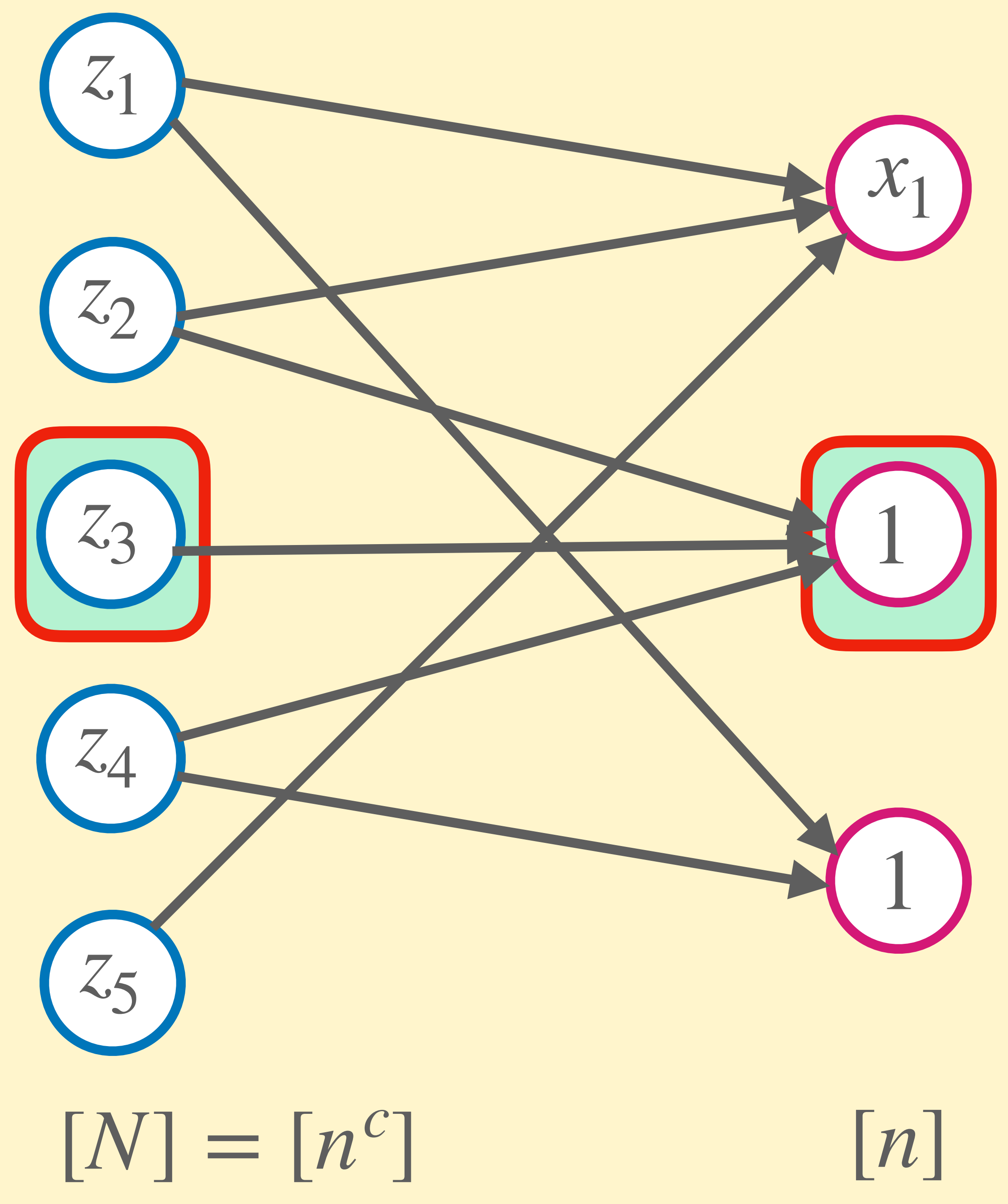

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Problem!

This forces $z_4 = 0$

What if A sets $z_4 = 1$?

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Problem: z-variables are correlated

→Setting one can x-variable can force several z-variables

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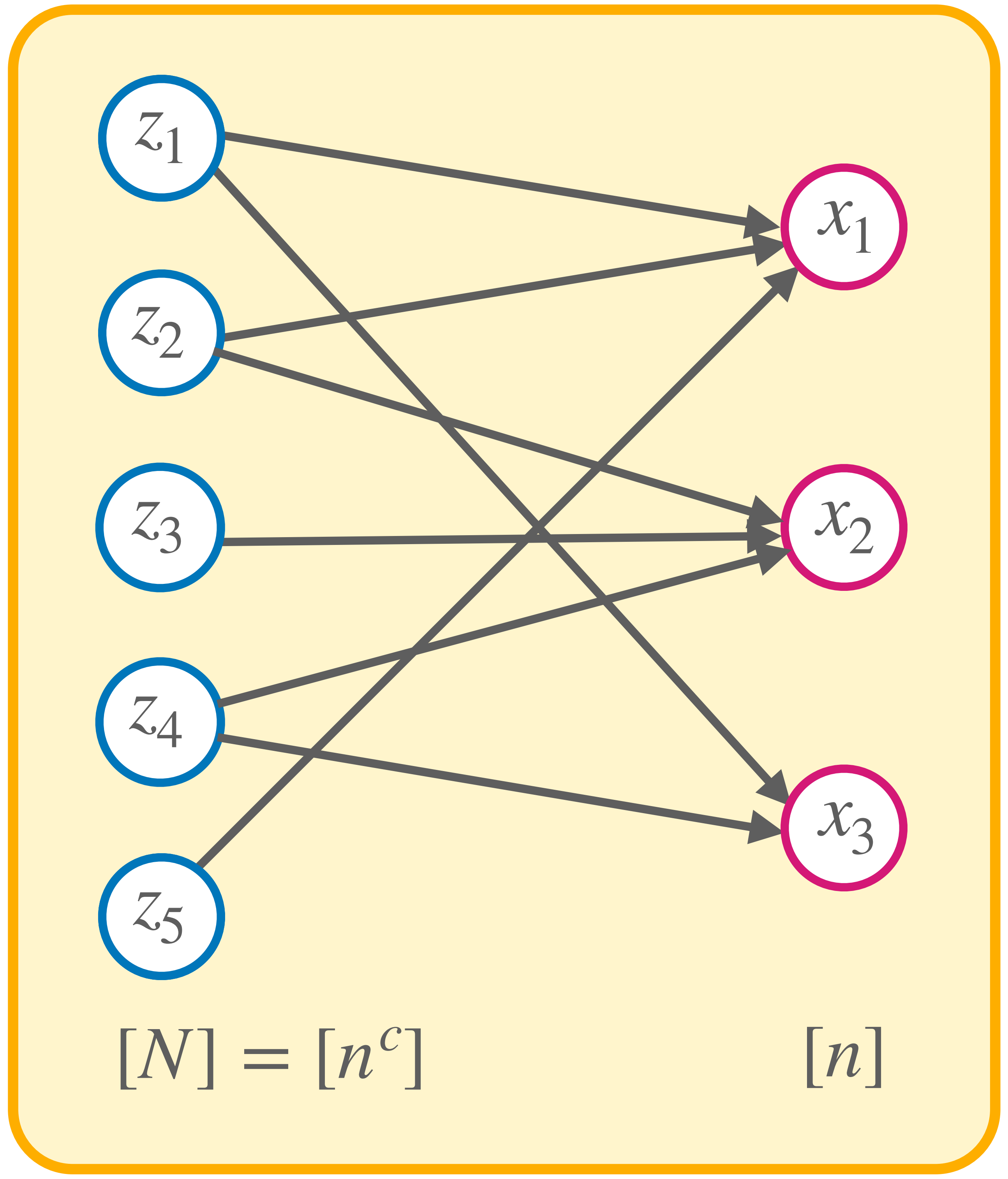

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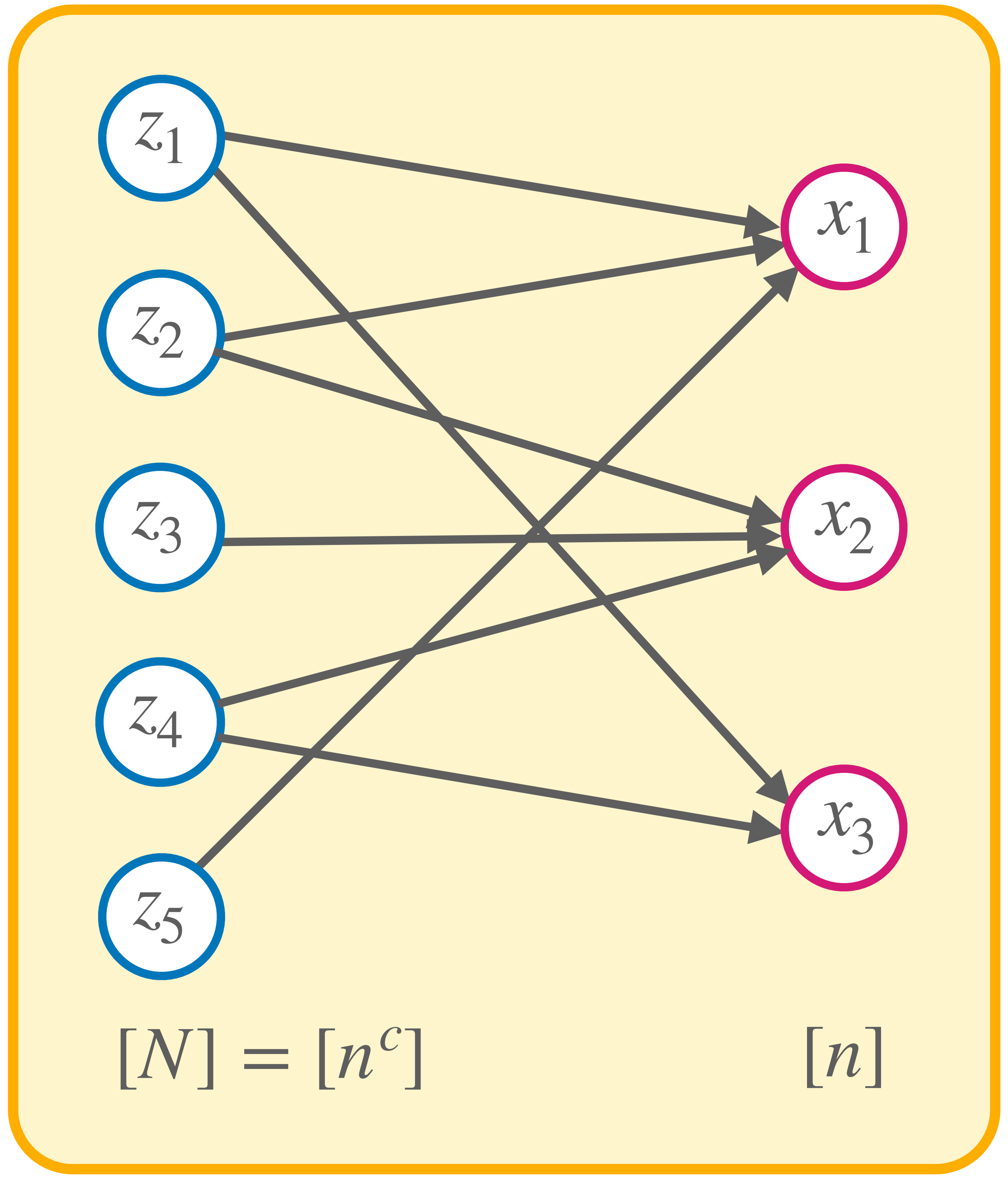

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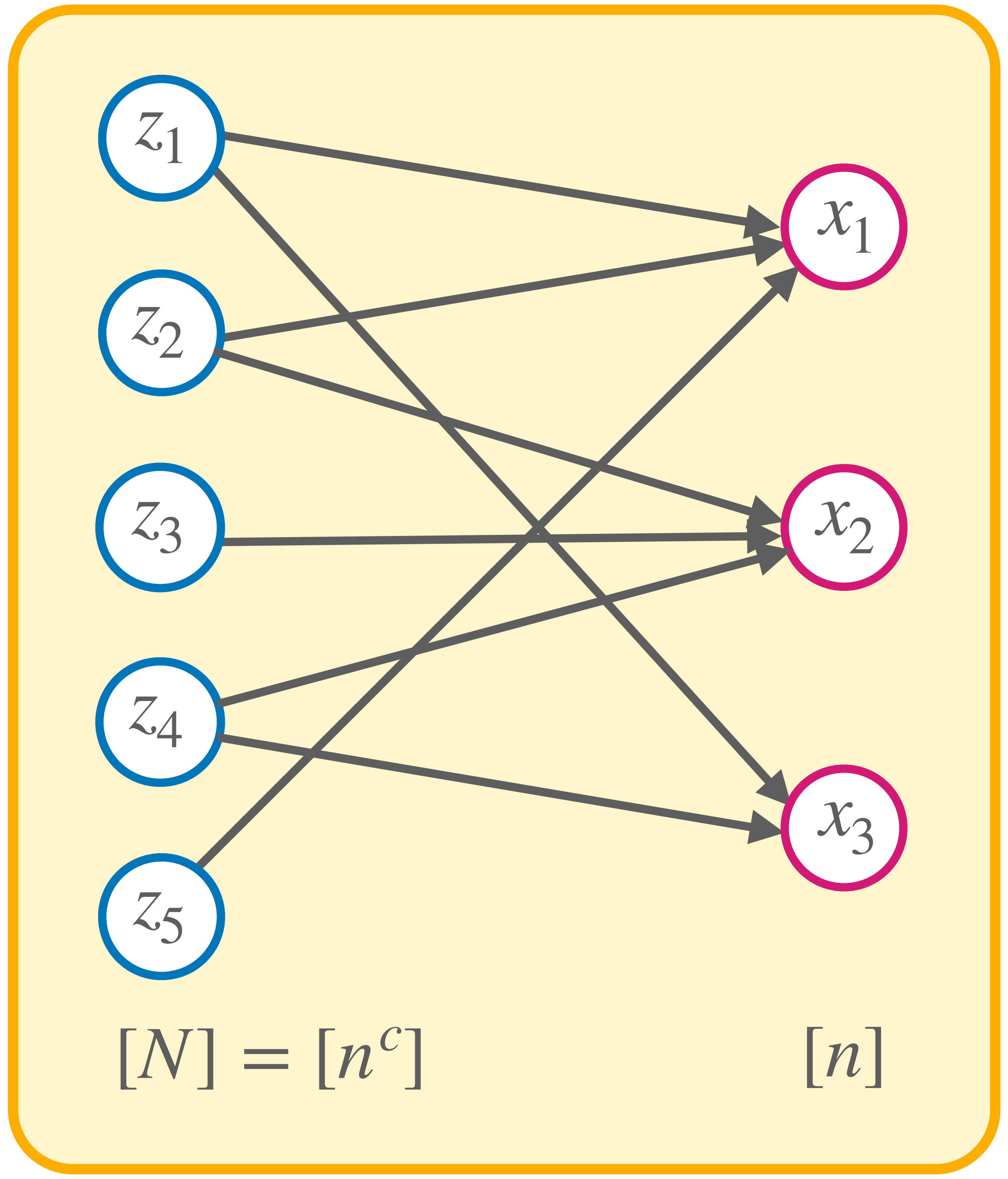

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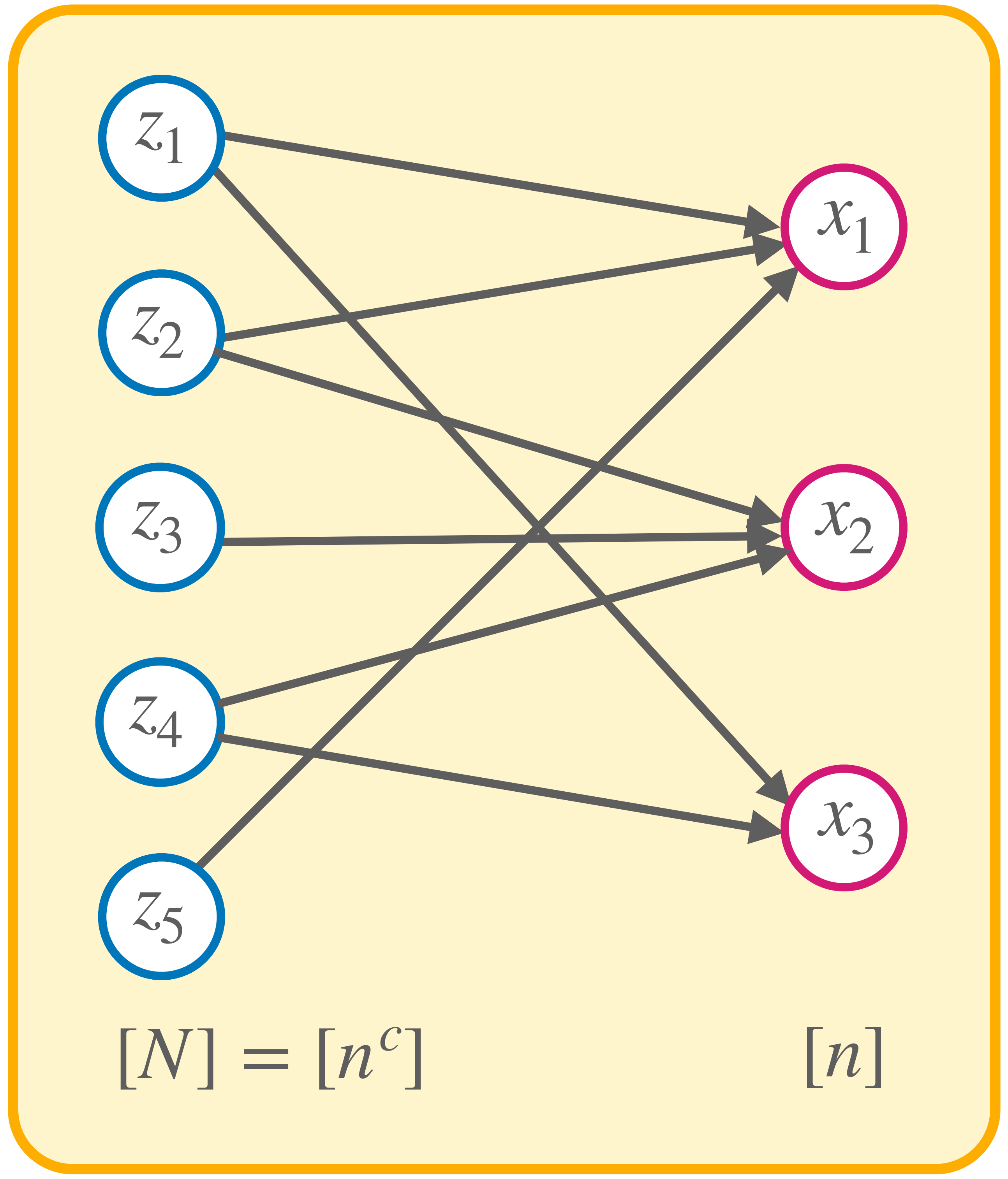

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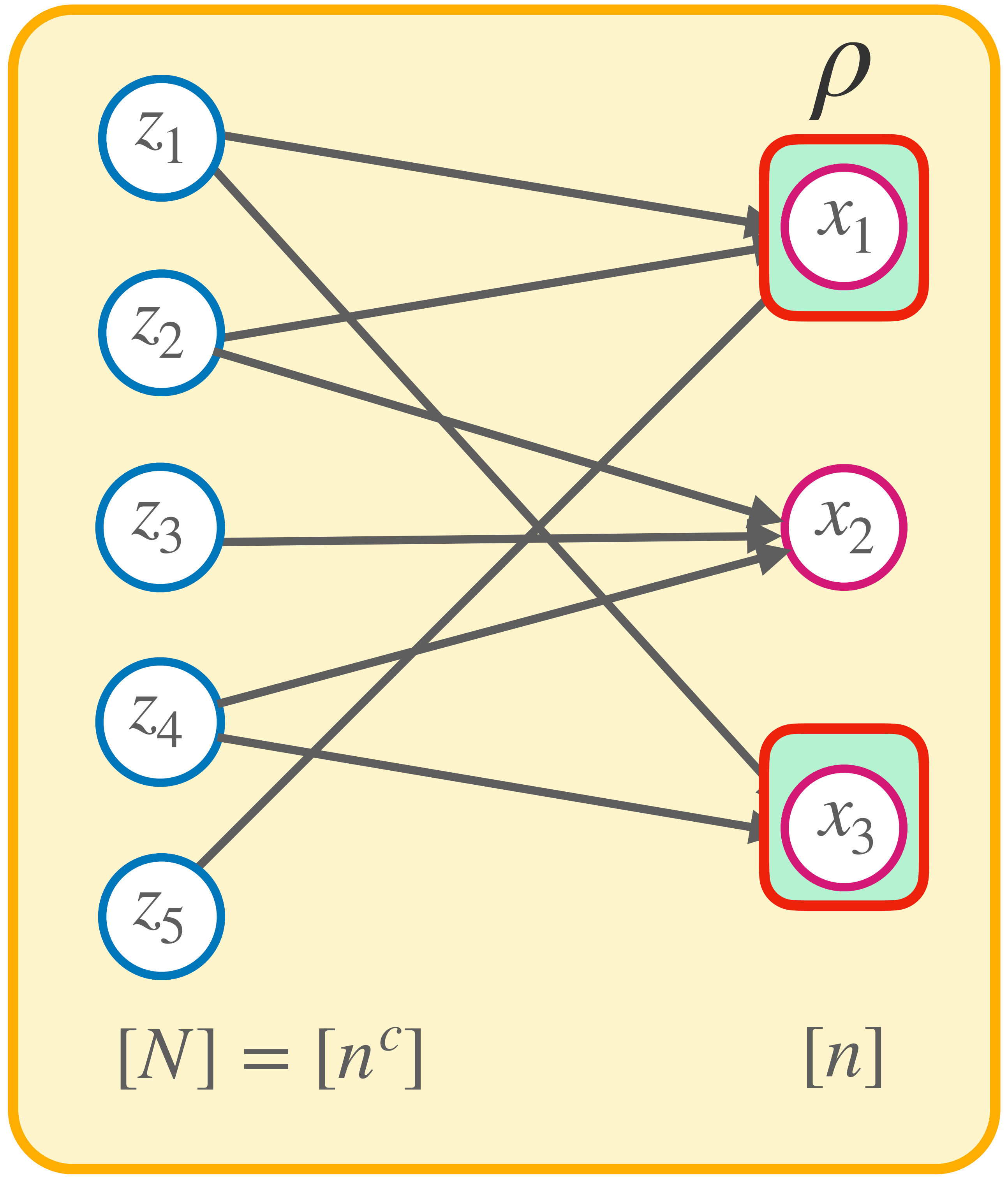

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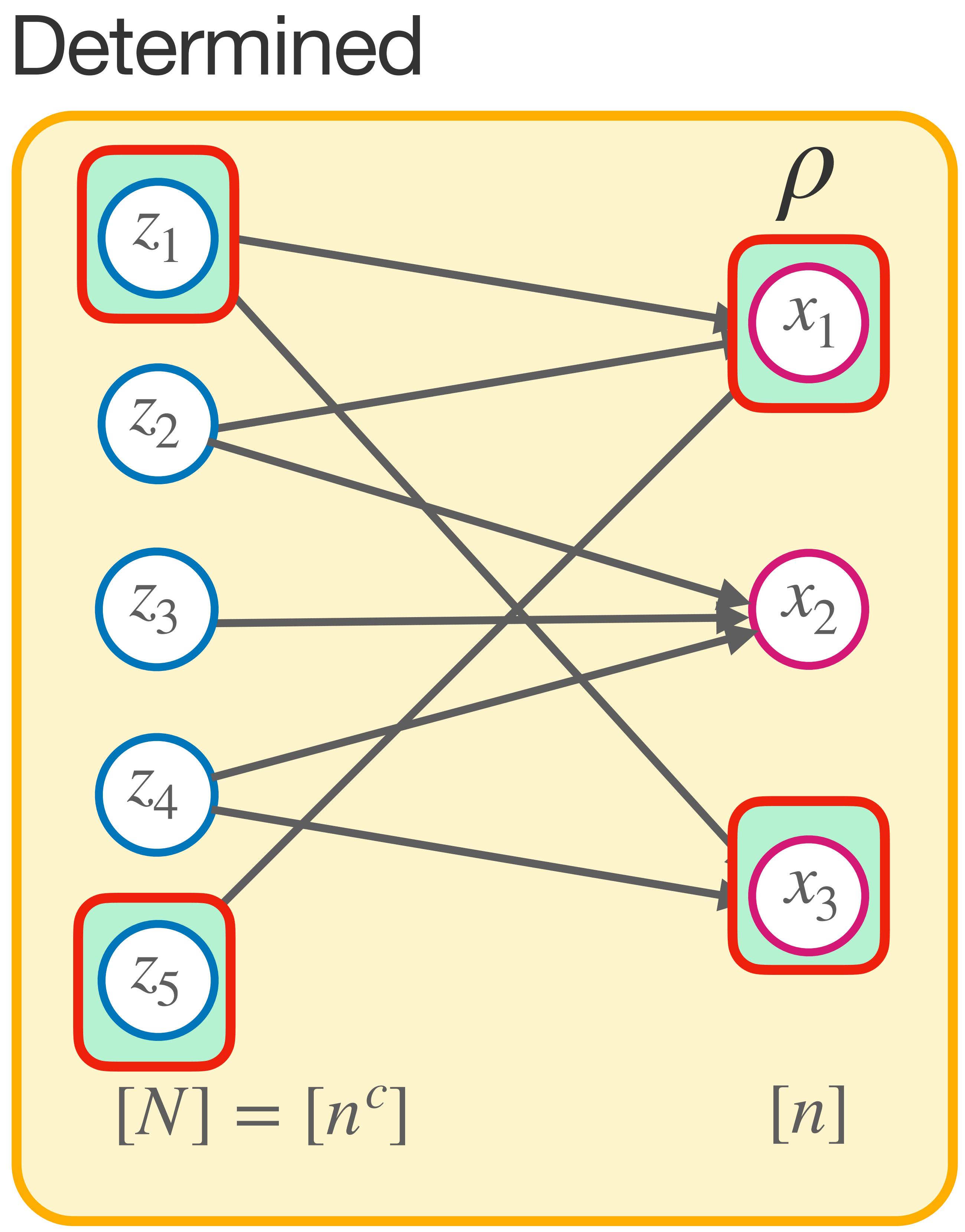

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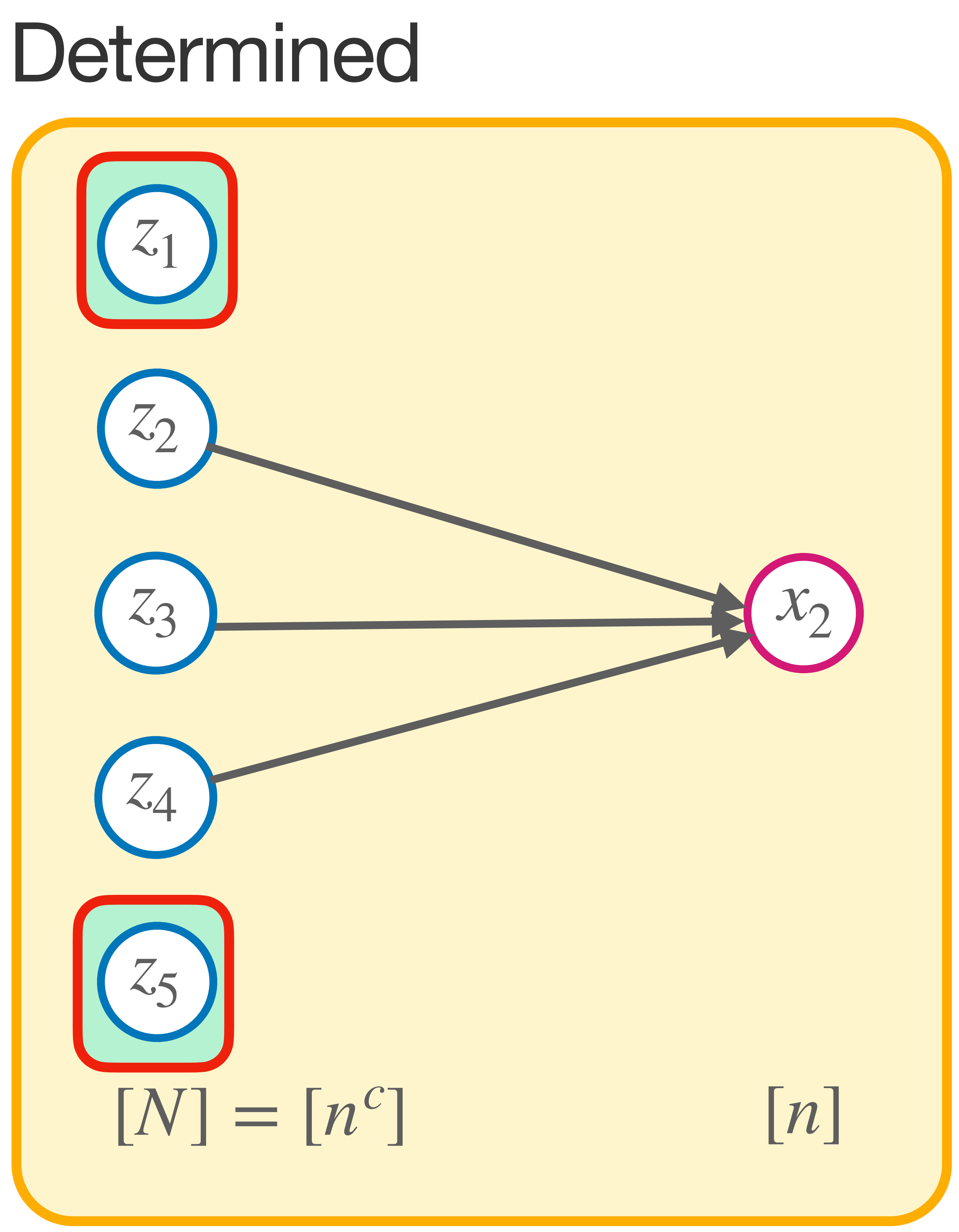

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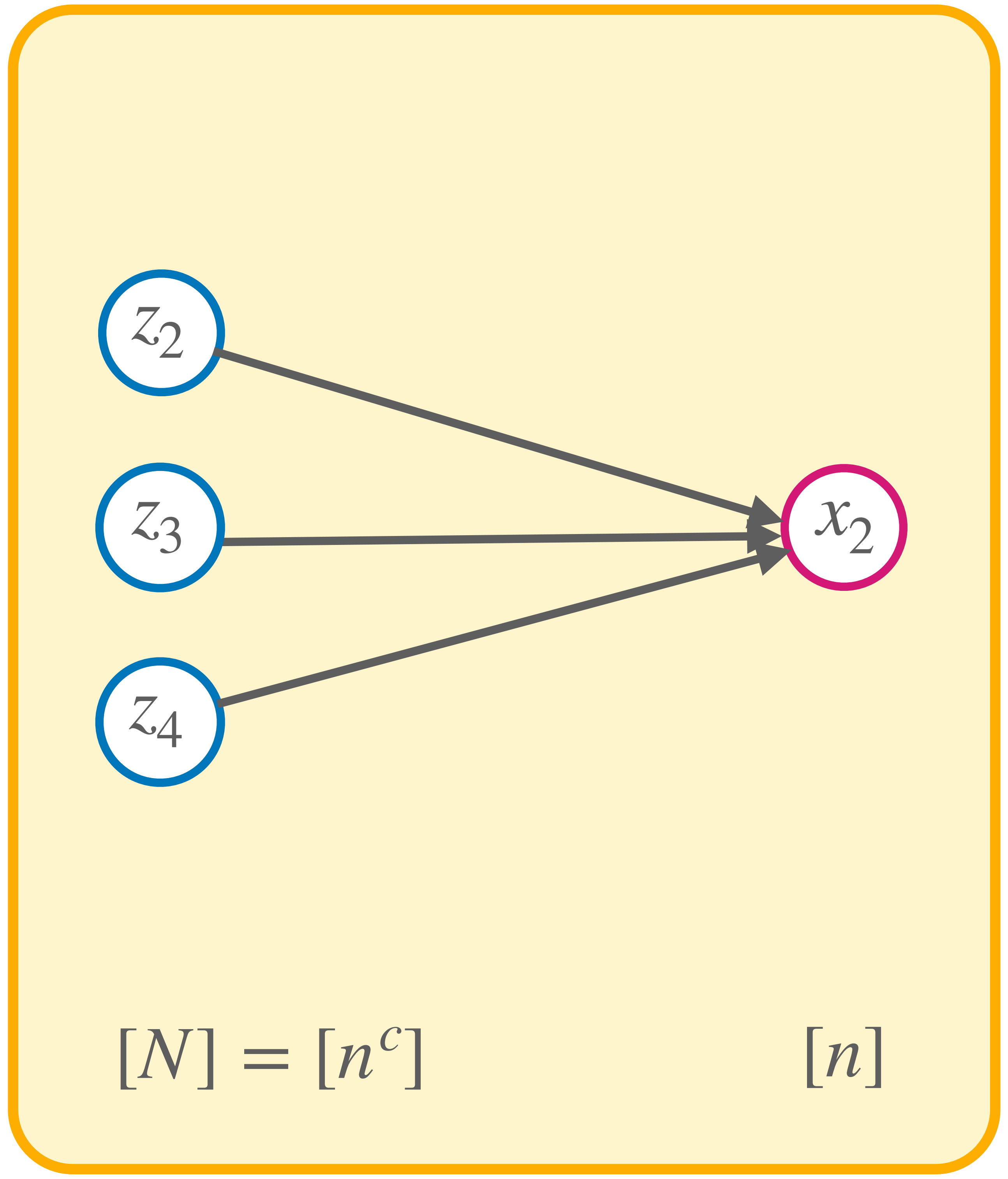

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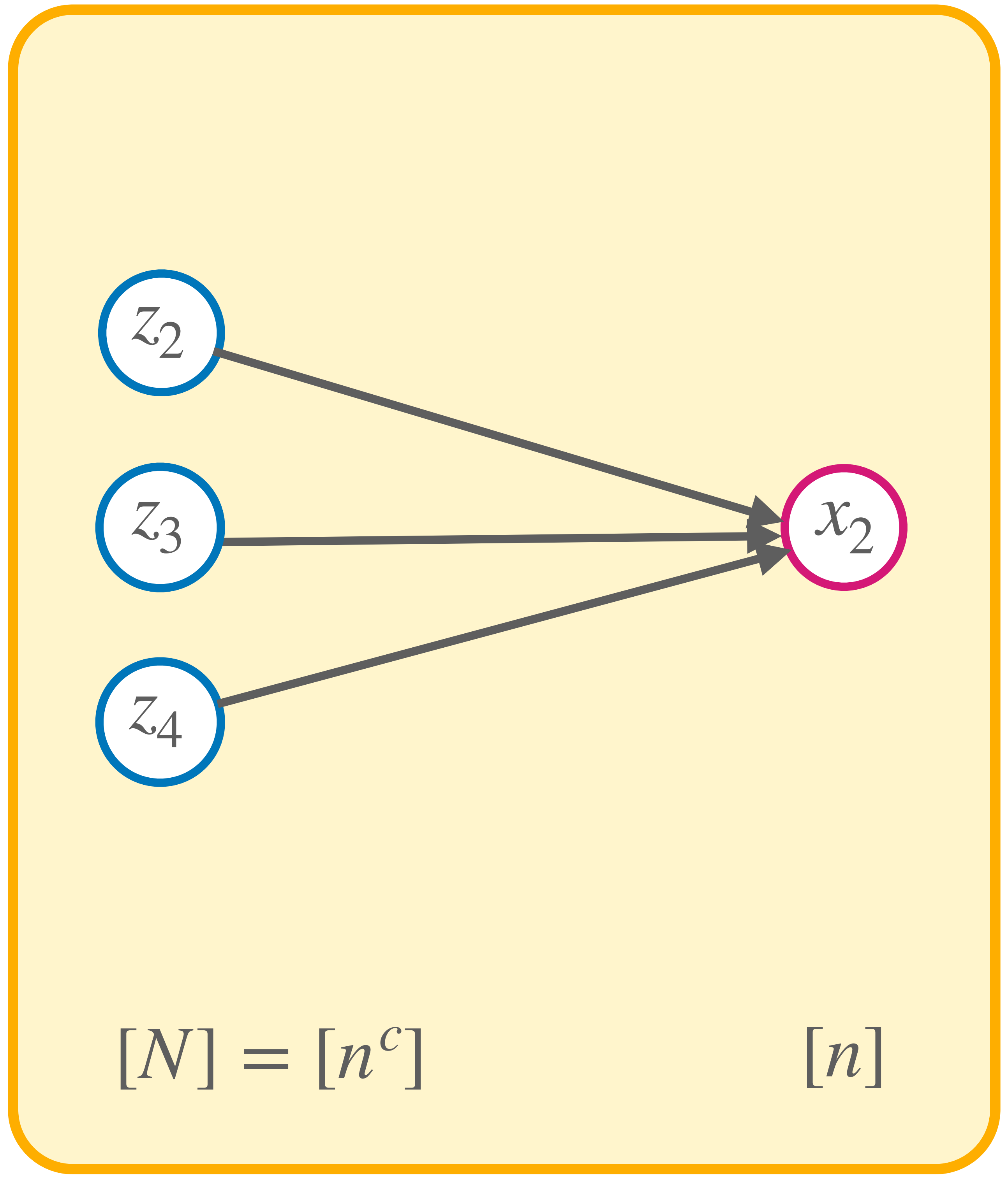

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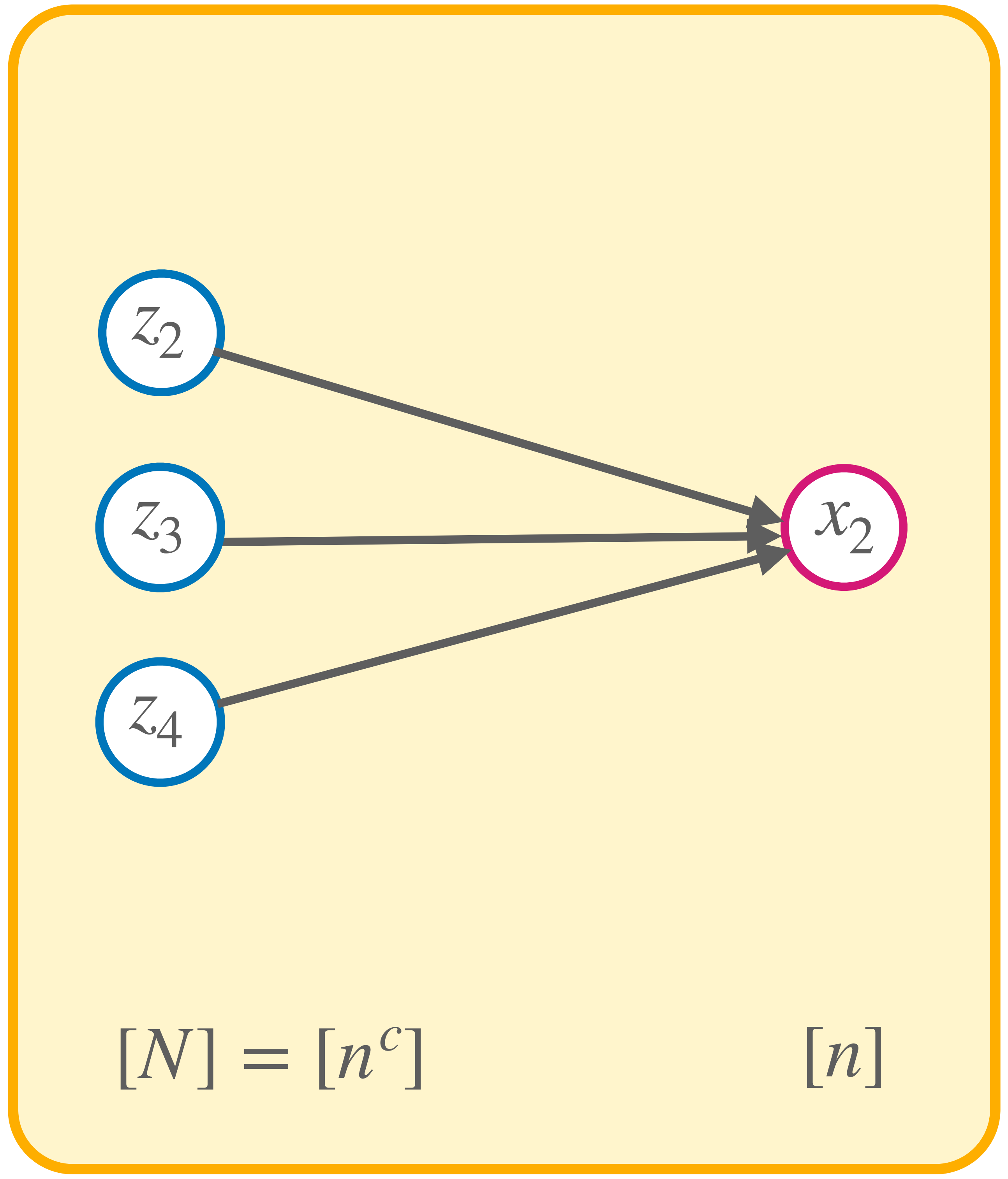

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Upshot: any width w proof of $F \circ \mathsf{XOR}_G$ requires depth $\Omega(d/w)$

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What about supercritical size/depth tradeoffs for other models of computation?

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Q. Supercritical size/depth tradeoffs for non-monotone circuits?