CDCL vs Resolution

Marc Vinyals

DPLL

$y \vee z \quad y \vee \bar{z} \quad x \vee \bar{y} \vee z \quad x \vee \bar{y} \vee \bar{z} \quad \bar{x} \vee \bar{y}$

Algorithm 1: DPLL

while *not solved* **do**
  if *conflict* **then** backtrack()
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  else branch()

State: partial assignment

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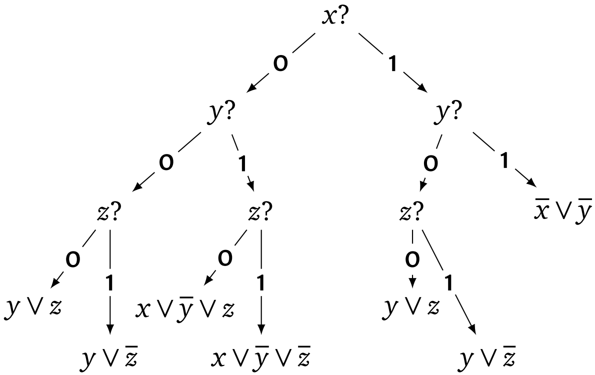
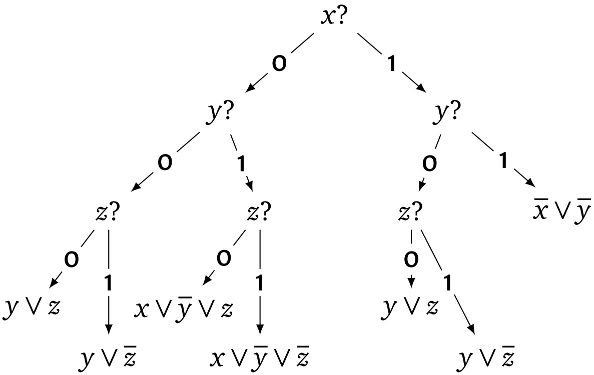


State: partial assignment

Resolution

- Interpret DPLL run as resolution proof

$$\frac{C \lor v \qquad D \lor \overline{v}}{C \lor D}$$



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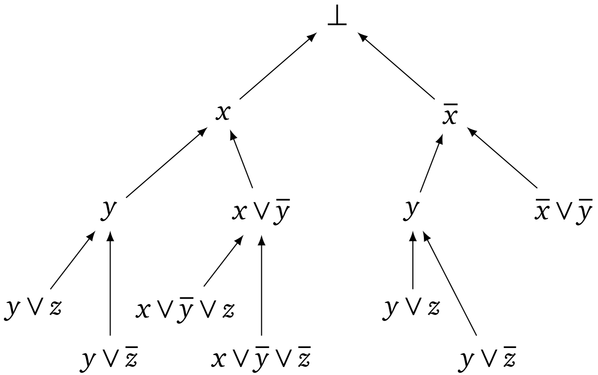

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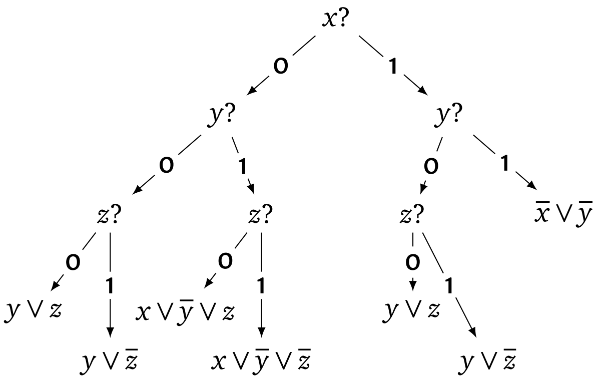
- And Resolution → DPLL?



Resolution to DPLL

Algorithm 1: DPLL

```
while not solved do
    if conflict then backtrack()
    else if unit then
        propagate()
    else branch on topmost
        available variable
```

DPLL can reproduce tree-like resolution proofs with at most $O(n)$ overhead

- # branches in search tree $\leq$ # branches in proof
- branch length $\leq n$
- $\Rightarrow$ proof size $L \rightarrow$ search tree size $\leq nL$

Resolution to DPLL

Sometimes $\Omega(n)$ overhead is needed.

- Take complete tautology over $x_1, \ldots, x_{\log n}$.
- Replace two variables in every clause with $y_{i,1}$.
- Add implications $y_{i,j} \to y_{i,j+1}$.
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Notation

$\mathcal{C}(S) = \{\bigvee_{i \in S} x_i^{b_i} \mid b \in \{0,1\}^S\}$ = all $2^{|S|}$ full-width clauses over variables in $\{x_i \mid i \in S\}$

$\ell = \log n$

Formula

C	for $C \in \mathcal{C}([\ell])$
$C \vee y_{i,1}$	for $C \in \mathcal{C}(S)$, $S \in \binom{[\ell]}{\ell-2}$, $i \in [\ell]$
$y_{i,j} \to y_{i,j+1}$	for $i \in [\ell]$, $j \in [n]$

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- Tree-like proof: branch on variables $x_1, \ldots, x_{\log n}$.
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- Tree-like proof: branch on variables $x_1, \ldots, x_{\log n}$.
  Size $2^{\log n} = n$.
- DPLL run: branch on variables $x_1, \ldots, x_{\log n-2}$, propagate all $y_{i,j}$, branch on $x_{\log n-1}, x_{\log n}$.
  Size $2^{\log n} \cdot n \log n \simeq n^2$.

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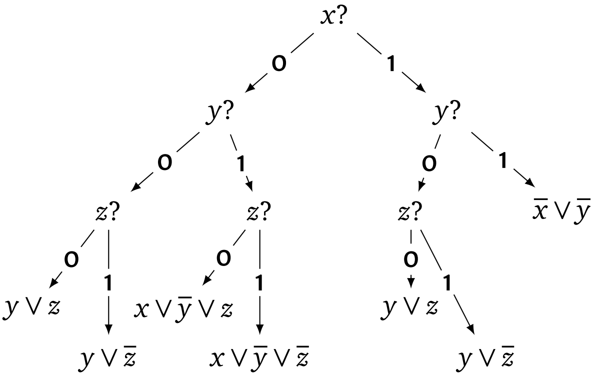

State: partial assignment

CDCL

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Algorithm 2: CDCL

while *not solved* **do**

  if *conflict* **then** learn()

  else if *unit* **then** propagate()

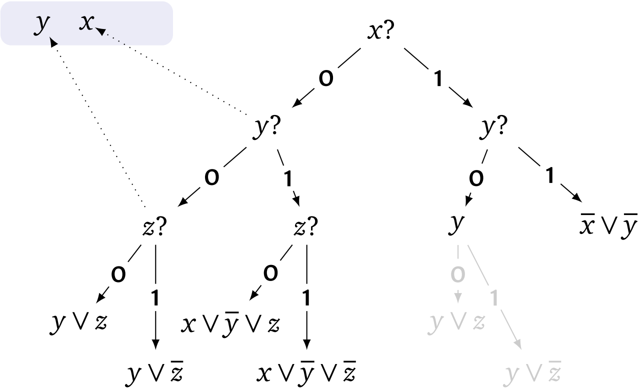
  else

    maybe forget()

    maybe restart()

    branch()

State: partial assignment
& learned clauses



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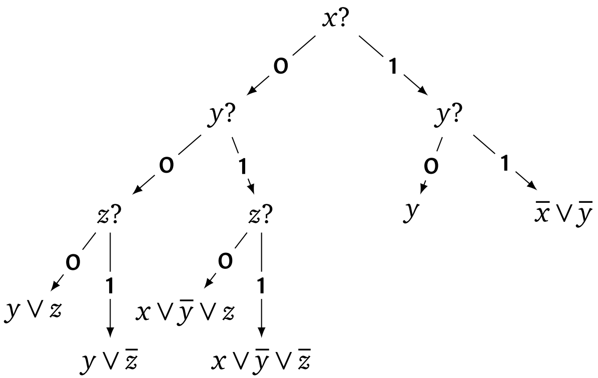

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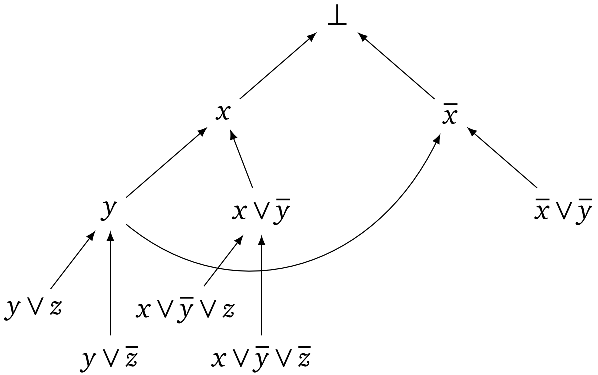

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- CDCL implicit proofs are in resolution form
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  - There are formulas with polynomial resolution proofs but all tree-like proofs are exponential
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- Yes (under natural model)

[Pipatsrisawat, Darwiche '09]
[Atserias, Fichte, Thurley '09]
[Beyersdorff, Böhm '21]

CDCL equivalent to Resolution: Results

Theorem [Pipatsrisawat, Darwiche '09]

With **non-deterministic** variable decisions,
CDCL can efficiently find resolution proofs

Theorem [Atserias, Fichte, Thurley '09]

With **random** variable decisions,
CDCL can efficiently find **bounded-width** resolution proofs

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CDCL equivalent to Resolution: Simulation

- Derivation $\pi = C_1, \ldots, C_t$.
- Goal: learn every clause $C_i \in \pi$.

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Example

$x \vee y \vee z \quad x \vee y \vee \overline{z}$

$x \vee y$ not absorbed:

- if $x = 0$ then would propagate y, but DB does not.

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$x \vee y$ is absorbed:

- if $x = 0$ then propagate $z = 1$ and $y = 1$;
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Algorithm 3: Simulation

for $C_i \in \pi$ **do**
  while C_i *not absorbed* **do**
    if *conflict* **then**
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- Optimal variable choices
- Clauses not thrown away
- Frequent restarts
- Standard learning

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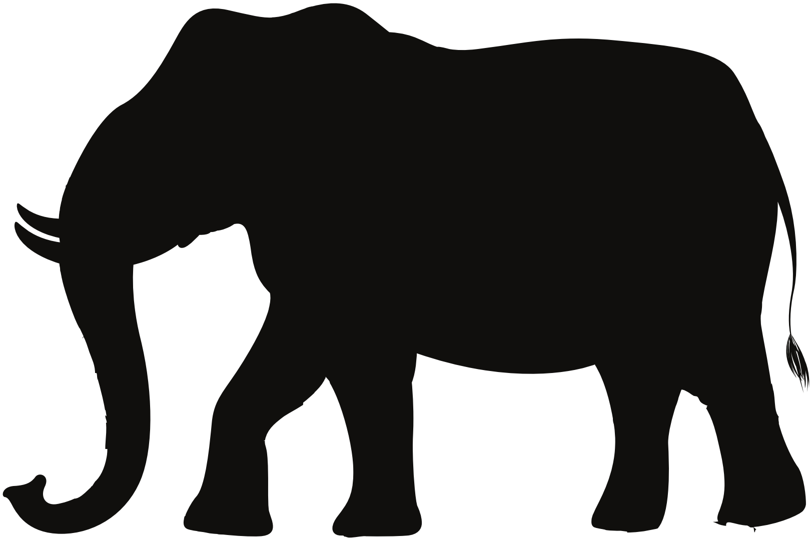

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Optimal variable choices are needed

- No deterministic algorithm simulates resolution unless FPT hierarchy collapses.
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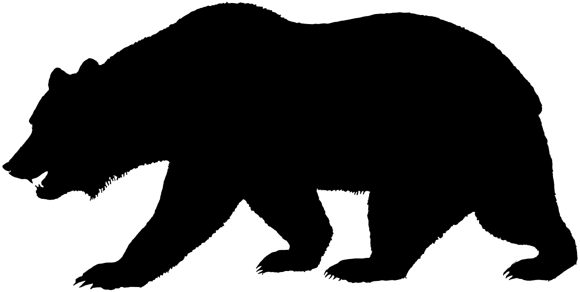
- CDCL with any static order exponentially worse than resolution.
  [Mull, Pang, Razborov '19]
- CDCL with VSIDS and similar heuristics exponentially worse than resolution.
  [V '20]



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Throwing Clauses Away

- With nondeterministic erasures enough to keep only $n \ll L$ clauses in memory.
  [Esteban, Torán '01]
- But more are needed to simulate resolution:
- Keeping $\ll n$ clauses can exponentially blow-up runtime.
  [Ben Sasson, Nordström '11]
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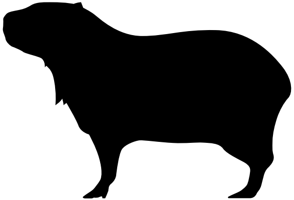
- Keeping only narrow clauses can exponentially blow-up runtime.
  [Thapen '16]
- What about clauses with low LBD?



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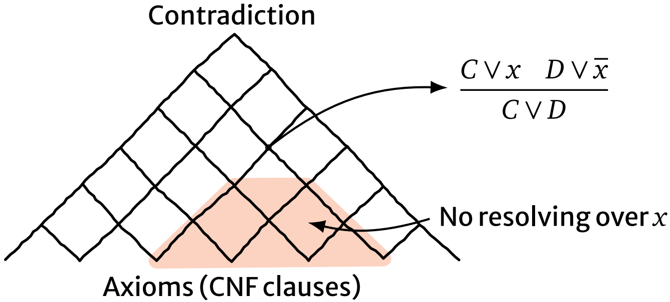
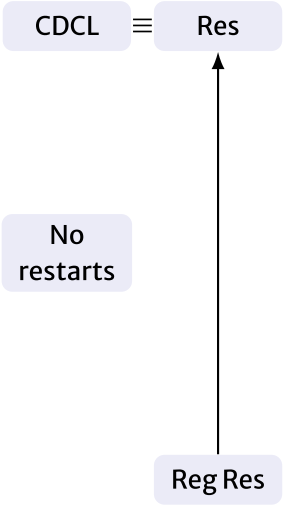
- CDCL without restarts between *regular* and *standard* resolution.

CDCL and Regular Resolution

- Regular resolution: do not resolve a variable twice on same path.





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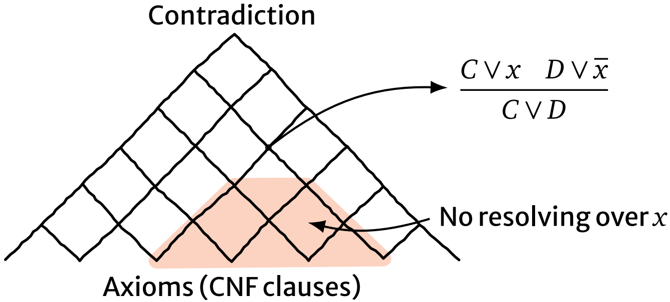

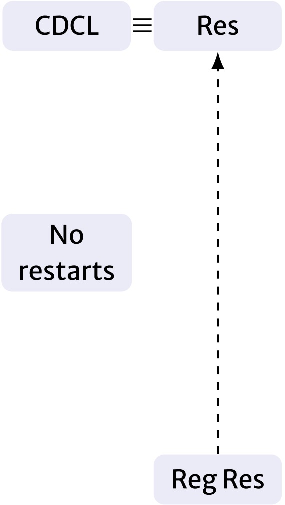

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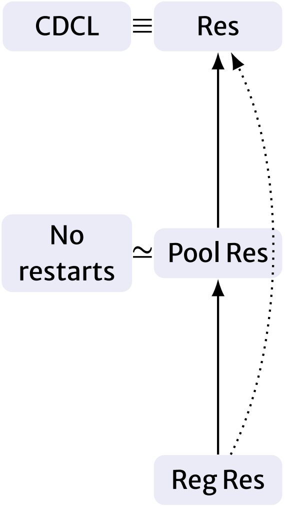

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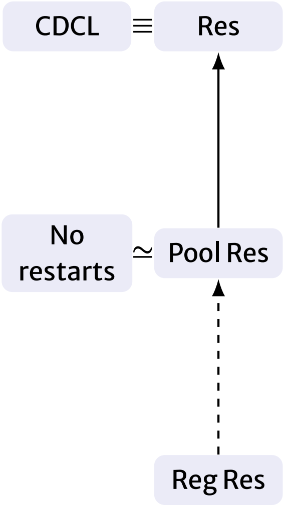

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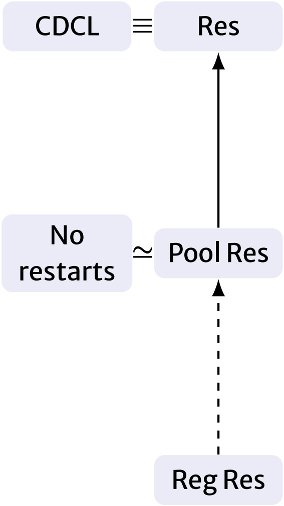
- Formula with CDCL proof of length L but requires $L+1$ w/o restarts?



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for Ci ∈ π do
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Learning

- Any **asserting** learning scheme works.
- C asserting if unit after backtracking.
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- Less overhead with decision learning scheme.
- Is decision faster than 1UIP?
- How much overhead is needed?

Merge Resolution

- A resolution step is a merge if C and D share a literal.

Merge

$$\frac{x \lor y \lor z \qquad x \lor y \lor \overline{z}}{x \lor y}$$

Not a merge

$$\frac{x \lor z \qquad y \lor \overline{z}}{x \lor y}$$

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- Merge resolution: at least one premise either axiom or merge. [Andrews '68]
- Merge resolution 2.0: only reuse merges.
- 1UIP produces merge resolution proofs.
- Merge resolution can simulate standard resolution with $O(n)$ overhead.
- And $\Omega(n)$ overhead sometimes needed. [Fleming, Ganesh, Kolokolova, Li, V]

Take Home

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- But only under assumptions, not all reasonable

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Open Problems

- CDCL-specific results about space?
- Are restarts important?
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Images: Vecteezy.com

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Thanks!