Primal-Dual Symmetric Interior-Point Methods from SDP to Hyperbolic Cone Programming and Beyond

Tor Myklebust Levent Tunçel

September 26, 2014

Convex Optimization in Conic Form

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$F_*(s) := -\inf_{x \in \text{int}(K)} \{ \langle s, x \rangle + F(x) \}$ Legendre-Fenchel Conjugate

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 & & \mathcal{A}^*(y) & + & s = c, \\
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Moreover, these solutions define a smooth path, called *central path*, and each point (x_μ, y_μ, s_μ) on the central path can be characterized as a unique solution of a system of equations (and being in the interior of the underlying cone).

We assume, we are given $x^{(0)}$, $s^{(0)}$ both strictly feasible in the problems (P) and (D), respectively. Define $\mu_k := \frac{\langle x^{(k)}, s^{(k)} \rangle}{\vartheta}$, we will compute $x^{(k)}$ and $s^{(k)}$ by an interior-point algorithm, which follows the central path approximately, such that both vectors are feasible and for a given desired accuracy $\epsilon \in (0, 1)$, we have $\mu_k \leq \epsilon \mu_0$.

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Such algorithms (for LP, SDP and Symmetric Cone Programming) with current best complexity compute an ϵ-solution $\left(x^{(k)}, s^{(k)}\right)$ in $O\left(\sqrt{\vartheta} \ln\left(\frac{1}{\epsilon}\right)\right)$ iterations.

A Hierarchical view of conic optimization

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Yet in an another sense,

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See, Ben-Tal and Nemirovski; Chua; Faybusovich; Nesterov and Nemirovski; Vinnikov; Helton and Vinnikov; Lewis, Parrilo and Ramana; Gurvits; Gouveia, Parrilo and Thomas; Netzer and Sanyal; Netzer, Plaumann and Schweighofer; Plaumann, Sturmfels and Vinzant; ...

Pinnacle of Symmetric Primal-Dual IPMs

Symmetric primal-dual interior-point algorithms

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Nesterov-Todd [1997-1998]: Primal-dual interior-point methods for self-scaled cones.

Hyperbolic programming and primal-dual interior-point algorithms?

A key property of self-scaled barriers is "the Long-step Hessian Estimation property" which hinges on the following "compatibility" property of the underlying barrier

$$\langle -F'(x), y \rangle \text{ is convex for every } y \in K.$$

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Nesterov [1997] showed that we can't have this property for both F and F_* unless we are in the self-scaled case.

v-space

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- Extension of v-space based primal-dual ipms maintaining the long-step Hessian estimation property (New for HomCP, HypCP)
- Software (some of the Primal-Dual metrics T^2 utilized by these algorithms are new even for LP!)
- Connections to other research areas in mathematics and mathematical sciences.

Lemma

For every pointed closed convex cone K with nonempty interior, there is an associated function $F : \text{int}(K) \to \mathbb{R}$ with the following properties:

Towards a symmetric primal-dual scaling

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1. *F is continuously differentiable on $\text{int}(K)$;*
2. *F is strictly convex on $\text{int}(K)$;*
3. *F is a barrier function for K (for every sequence $\{x^{(k)}\} \subset \text{int}(K)$ converging to a boundary point of K, $F\left(x^{(k)}\right) \to +\infty$).*

A huge family of symmetric primal-dual scalings

Theorem

(T. [2001]) Let $K \subset \mathbb{E}$ *be a pointed closed convex cone with nonempty interior and let* $F : \text{int}(K) \to \mathbb{R}$ *be a function with the properties listed in the lemma (*$+\ \vartheta$*-log.-homogeneity). Then for every* $x \in \text{int}(K)$*,* $s \in \text{int}(K^*)$*, there exists* $T : \mathbb{E} \to \mathbb{E}^{\frac{*}{2}}$ *linear, such that*

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The set of solutions T^2 of the above problems are denoted by $\mathcal{T}_0(x, s)$ and $\mathcal{T}_1(x, s)$.

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This theorem allows very wide generalizations of symmetric primal-dual interior-point algorithms to general convex (conic) optimization setting (linear operator T generalizes the notion of primal-dual symmetric local metric—primal-dual scaling).

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$$\tilde{x} := -F_*^{'}(s) \text{ and } \tilde{s} := -F^{'}(x).$$

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$$\mu\tilde{x} = x \text{ iff } \mu\tilde{s} = s \text{ iff } (x, s) \text{ lies on the central path.}$$

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We also denote

$$\tilde{\mu} := \frac{\langle \tilde{x}, \tilde{s} \rangle}{\vartheta}.$$

Explicit formulas for symmetric primal-dual local metrics

"Proof" (for every H symmetric, positive definite):

$$T_H^2 := H + a_1 x x^\top + g_1 H s s^\top H + \tilde{a}_1 \tilde{x}\tilde{x}^\top + \tilde{g}_1 H\tilde{s}\tilde{s}^\top H + a_2 \left(x\tilde{x}^\top + \tilde{x} x^\top \right) + g_2 \left(Hs\tilde{s}^\top H + H\tilde{s}s^\top H \right),$$

where

$$a_1 := \frac{\tilde{\mu}}{\vartheta(\mu\tilde{\mu} - 1)}, \tilde{a}_1 := \frac{\mu}{\vartheta(\mu\tilde{\mu} - 1)}, a_2 := -\frac{1}{\vartheta(\mu\tilde{\mu} - 1)},$$

$$g_1 := -\frac{\tilde{s}^\top H\tilde{s}}{(s^\top Hs)(\tilde{s}^\top H\tilde{s}) - (\tilde{s}^\top Hs)^2}, \tilde{g}_1 := -\frac{s^\top Hs}{(s^\top Hs)(\tilde{s}^\top H\tilde{s}) - (\tilde{s}^\top Hs)^2}, g_2 := \frac{\tilde{s}^\top Hs}{(s^\top Hs)(\tilde{s}^\top H\tilde{s}) - (\tilde{s}^\top Hs)^2}.$$

□

Nicer Hessian update formulas

Take

$$\begin{aligned} \delta_D &:= s - \mu\tilde{s} \\ \delta_P &:= x - \mu\tilde{x}. \end{aligned}$$

Consider two consecutive DFP/BFGS-like updates:

$$\begin{aligned} H_1 &:= H + \frac{1}{\langle s, x\rangle} xx^\top - \frac{1}{\langle s, Hs\rangle} Hss^\top H \\ H_2 &:= H_1 + \frac{1}{\langle \delta_P, \delta_D\rangle} \delta_P \delta_P^\top - \frac{1}{\langle \delta_D, H_1\delta_D\rangle} H_1 \delta_D \delta_D^\top H_1 \end{aligned}$$

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Theorem

They're equivalent! I.e., $H_2 = T^2$.

However, this new form is much more revealing!

What will it reveal?

Complexity Analysis via Hessian update formulas

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$$\begin{array}{llll}
\inf & \xi & & \\
& T^2(s) & = & x, \\
& T^2(-F'(x)) & = & -F'_*(s), \\
& \frac{1}{\xi h(x,s)} F''_*(s) & \preceq & T^2 \quad \preceq \quad \xi h(x,s) \left[F''(x)\right]^{-1}, \\
& \xi \geq 1, & & T \in \mathbb{S}^n,
\end{array}$$

where $h(x, s)$ is a certain proximity measure for the central path.

Bounds on ξ^*

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Any upper bound by an absolute constant in a constant-sized neighbourhood of the central path leads to an iteration complexity bound of $O\left(\sqrt{\vartheta}\ln\frac{1}{\epsilon}\right)$.

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In our language today,

Theorem

(Nesterov and Todd [1997, 1998]) Let $K \subseteq \mathbb{E}$ *be a symmetric (homogeneous self-dual) cone. Further let* F *be a* ϑ*-self-scaled barrier for* K*. Then for every* $x \in \text{int}(K)$, $s \in \text{int}(K^*)$,

$$\xi^* \leq \frac{4}{3}.$$

We now have:

Theorem

Let $K \subseteq \mathbb{E}$ be a convex cone. Further let F be a ϑ-self-concordant barrier for K. There are absolute constants C_1 and C_2 such that, for every $x \in \text{int}(K)$ and $s \in \text{int}(K^*)$ lying in a constant size (C_1) neighbourhood of the central path, we have

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Is there any hope for maintain this long-step Hessian estimation property in a primal-dual v-space based algorithm?

Theorem

Let F be a LHSCB for K and $(x, s) \in \text{int}(K) \oplus \text{int}(K^*)$. Then, the linear transformation

$$T_D^2 := \mu \int_0^1 F_*''(s - t\delta_D)dt$$

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is self-adjoint, positive definite, maps s to x, and maps $\tilde{s}$ to $\tilde{x}$. Therefore, its unique self-adjoint, positive definite square root T_D is in $\mathcal{T}_1(x, s)$.

Proof

Using the fundamental theorem of calculus (for the second equation below) followed by the property $-F_*'(-F'(x)) = x$ (for the third equation below), we obtain

$$\begin{aligned} T_D^2 \delta_D &= \mu \int_0^1 F_*''(s - t\delta_D)\delta_D dt \\ &= \mu \left(F_*'(s - \delta_D) - F_*'(s) \right) \\ &= \mu \left(x/\mu - \tilde{x} \right) \\ &= \delta_P. \end{aligned}$$

Proof ... continued

We next compute, using the substitution $u = 1/t$,

$$\begin{aligned}
T_D^2 s &= \mu \int_0^1 F_*''(s - t\delta_D) s dt \\
&= \mu \int_0^1 \frac{1}{t^2} F_*''(s/t - \delta_D) s dt \\
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Hence, any specific choice of T^2 (which may not be primal-dual symmetric) from any one this sets can be made into a primal-dual symmetric local metric via taking the operator geometric mean with the inverse of its counterpart.

Conclusion: What are the new results?

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- Connections to other research areas in mathematics and mathematical sciences.