An Introduction to Causal Graphical Models

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Simons Institute Causality Bootcamp

Handout available at
`https://tinyurl.com/causalitybootcamp`

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Our viewpoint

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- We will start with probabilistic causal models.
- We will (mostly) work with causal Bayesian networks.

Probabilistic Causal Models

A tuple $M = \langle U, V, F, P(U) \rangle$ where

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Together $P(U)$ and F induce a distribution on V, $P(V)$.

$$P(v) = \sum_{u \in D_U} \prod_{i=1}^{n} P(x_i \mid \mathrm{parents}(x_i)) P(u)$$

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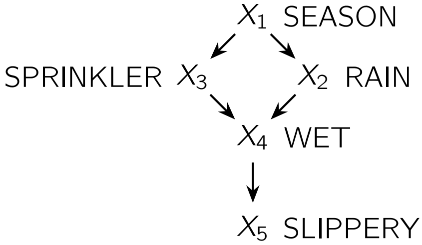
Is there any better way to understand this?

An example, continued



Each model induces a graph.

The graph has a vertex for each $X \in V$, an edge $X \to Y$ if f_Y depends on X.

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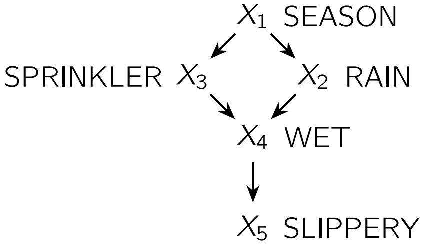

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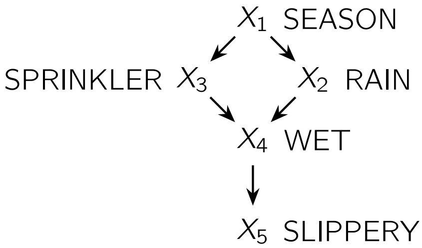

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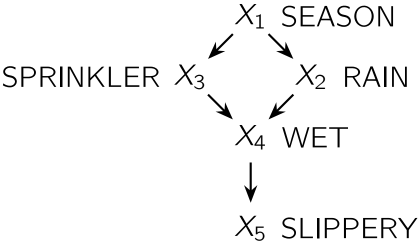

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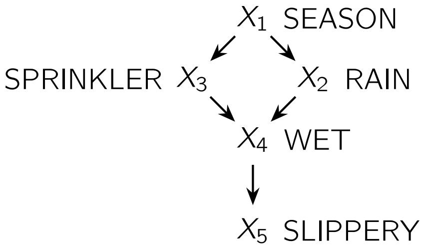
- We will only be interested in models that induce *acyclic* graphs!
- What about confounders? If f_X, f_Y depend on a common U, we represent this with

$$X \leftarrow\dashrightarrow Y$$

Factorization



With no confounders the $P(V)$ induced by $P(U)$ factors according to G:

$$
\begin{aligned}
&P(X_1, X_2, X_3, X_4, X_5) \\
&\qquad = P(X_1)P(X_2 \mid X_1)P(X_3 \mid X_1)P(X_4 \mid X_2, X_3)P(X_5 \mid X_4)
\end{aligned}
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Interventions correspond to changing the mechanism determining some X_i

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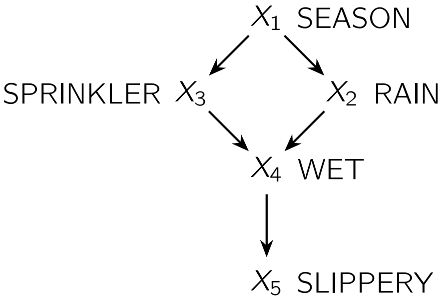
The induced graph and $P(V)$ change as well.

We write $P_x(V)$ for the distribution obtained by intervening to set $X := x$.

Interventions, continued



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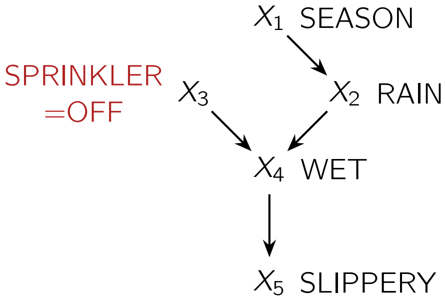

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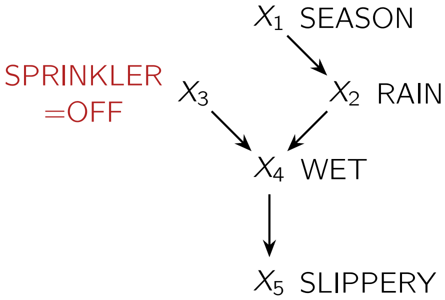

Let v be an assignment to V such that $X_3 = \text{OFF}$. Then

$$P_{X_3=\text{OFF}}(v)$$
$$= P(x_1)P(x_2 \mid x_1)P(x_4 \mid x_2, X_3 = \text{OFF})P(x_5 \mid x_4)$$

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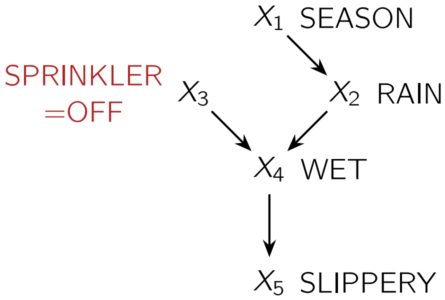

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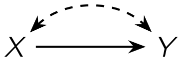
We can compute this from $P(V)$ alone. We don't need $P(U)$.

Interventions and confounders

Consider a model that induces this graph:



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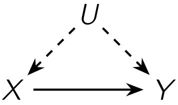

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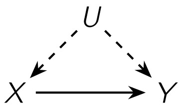

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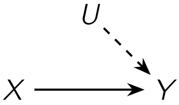

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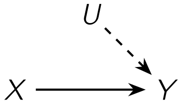

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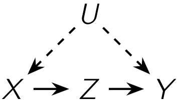
We can't compute $P_x(Y)$ with knowledge only of $P(V)$.

Causal inference with unobserved confounders

Consider a slightly different example:



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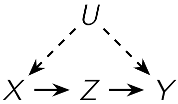

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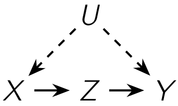

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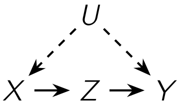

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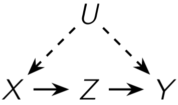

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Here $P(V)$ uniquely determines $P_x(y)$ in any causal model that induces G. In this case we say that $P_x(y)$ is identifiable.

The big picture

The Shpitser-Pearl ID algorithm takes a graph G induced by a causal model, a distribution $P(V)$ for that model, and a target intervention $X, Y \subseteq V$, and returns

- a formula for $P_x(y)$ if it is identifiable from $P(V)$, or
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The agenda

- Understand the relationship between DAGs and distributions.
  - When do G_1 and G_2 correspond to the same set of possible distributions?
  - What conditional independencies are implied by a graph G?
- Understand the do-calculus, rules for manipulating interventional distributions.
- Understand the Shpitser-Pearl ID algorithm.

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Probability review

- X and Y are independent conditioned on Z if $\forall x \in D_X, y \in D_Y, z \in D_Z$,

  $$P(x \mid y, z) = P(x \mid z) \quad \text{if } P(y, z) > 0.$$

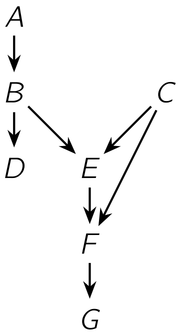
  Alternatively,

  $$P(x, y \mid z) = P(x \mid z) P(y \mid z).$$

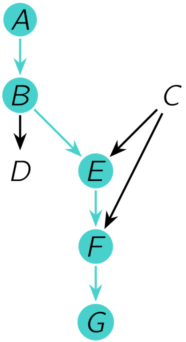
  We write:

  $$(X \perp\!\!\!\perp Y \mid Z)_P$$

Graph preliminaries



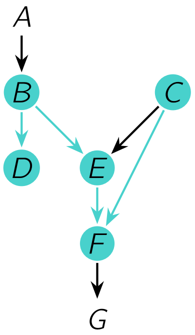
Graph preliminaries



- Directed paths

$$A \to B \to E \to F \to G \quad \text{(written } A \rightsquigarrow G\text{)}$$

Graph preliminaries



- Directed paths
- Trails

$$D \leftarrow B \rightarrow E \rightarrow F \leftarrow C \quad (\text{written } D \;\circ\!\!\leadsto\!\!\circ\; C)$$

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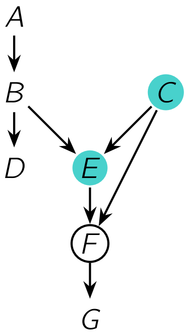

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- Parents, $\mathrm{Pa}(X)$.

$$\mathrm{Pa}(F) = \{C, E\}$$

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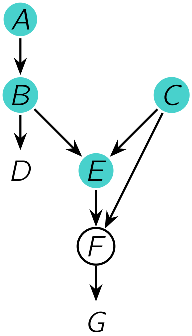

- Directed paths
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- Parents, $\mathsf{Pa}(X)$.
- Ancestors, $\mathsf{An}(X)$.

$$\mathsf{An}(F) = \{A, B, C, E, F\}$$

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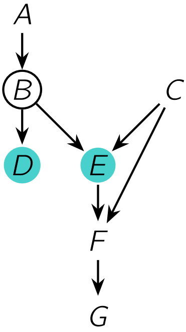

- Directed paths
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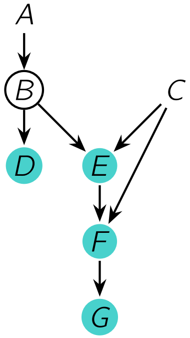
- Parents, $\mathsf{Pa}(X)$.
- Ancestors, $\mathsf{An}(X)$.
- Children, $\mathsf{Ch}(X)$.

$$\mathsf{Ch}(B) = \{D, E\}$$

Graph preliminaries



- Directed paths
- Trails
- Parents, $\mathsf{Pa}(X)$.
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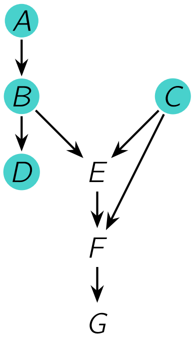

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$$\{A, B, C, D\}$$

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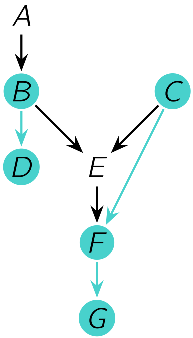

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- Induced subgraph, $G[V']$

$$G[\{B, C, D, F, G\}]$$

Bayesian networks

A DAG $G = (V, E)$ along with a distribution $P(V)$ factoring as

$$P(V) = \prod_{X \in V} P(X \mid \mathsf{pa}(X)).$$

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Markov conditions

Ordered Markov Condition

P is compatible with $G \Leftrightarrow$ in any topological ordering $X_1, \ldots, X_n$, each X_i is independent of its predecessors given its parents.

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Conditioning on common ancestors

Lemma

Fix any G and disjoint $X, Y, Z \subseteq V$. If $\mathsf{An}(X) \cap \mathsf{An}(Y) \subseteq Z$ and $\mathsf{An}(Z) \subseteq Z$, then

$$P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z)$$

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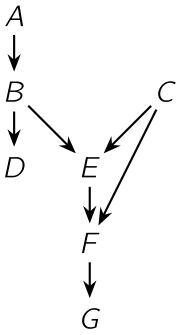

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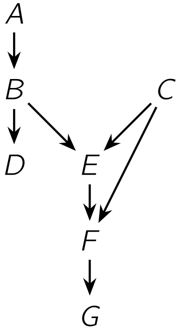

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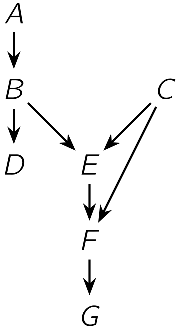

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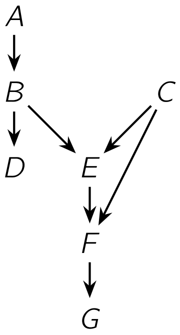

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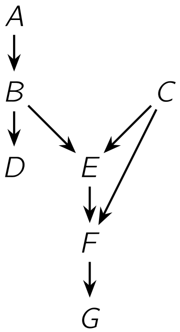

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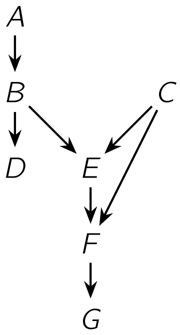
Let $\mathcal{I}_{\text{prob}}(P) := \{(X, Y, Z) : (X \perp\!\!\!\perp Y \mid Z)_P\}$.

Blocked trails

A trail in G is blocked by a set Z if it contains three consecutive vertices such that



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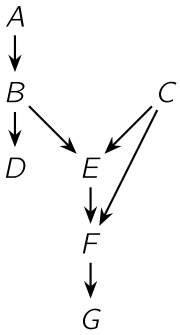

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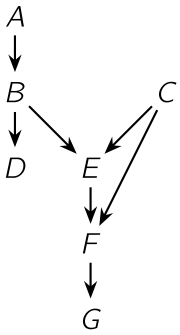

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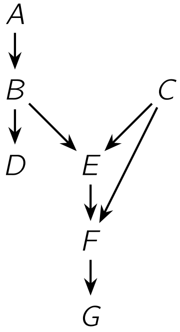

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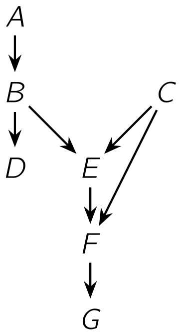

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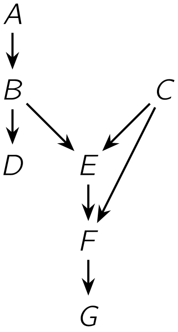

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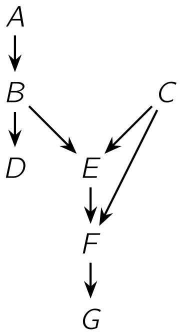

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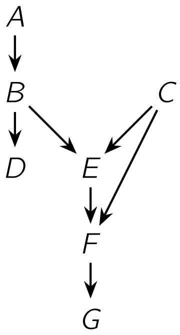

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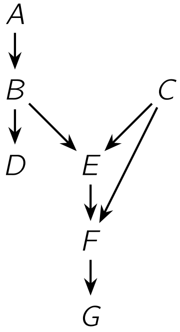

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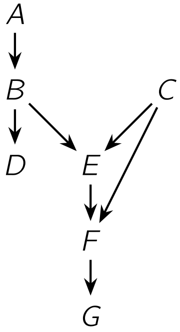

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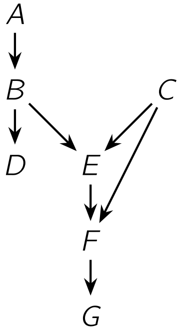

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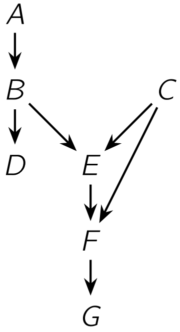

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d-Separation

Let $X, Y, Z \subseteq V$ be disjoint. Then X is d-separated from Y by Z if every trail between any vertex in X and any vertex Y in G is blocked. We write

$$(X \perp\!\!\!\perp Y \mid Z)_G.$$

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If there is a trail from a vertex in X to a vertex in Y that is not blocked, we say that X and Y are d-connected given Z.

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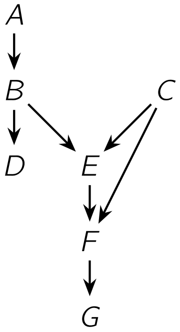
Any trail that is not blocked is an active trail.

d-Separation examples

What d-separations hold in G?



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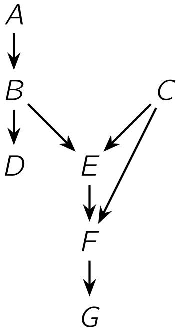

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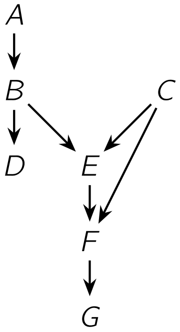

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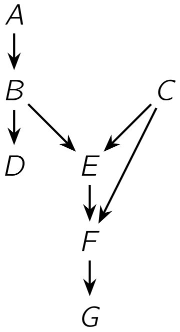

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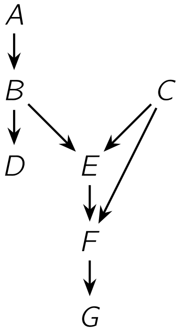

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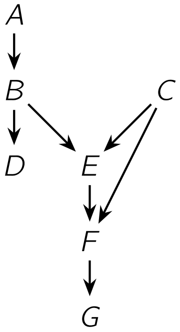

- $(A \perp\!\!\!\perp F \mid C, E)_G$.
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- $(B \perp\!\!\!\perp F \mid E)_G$? **No!**

Let $\mathcal{I}_{\textsf{d-sep}}(G) \coloneqq \{(X, Y, Z) : (X \perp\!\!\!\perp Y \mid Z)_G\}$.

d-Separation and conditional independence

Theorem

$(X \perp\!\!\!\perp Y \mid Z)_G \implies (X \perp\!\!\!\perp Y \mid Z)_P$ in every distribution P compatible with G.

d-Separation and conditional independence

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$(X \perp\!\!\!\perp Y \mid Z)_G \implies (X \perp\!\!\!\perp Y \mid Z)_P$ *in every distribution P compatible with G.*

We'll prove this later.

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Theorem

If $\neg(X \perp\!\!\!\perp Y \mid Z)_G$, then there exists a distribution P compatible with G in which $\neg(X \perp\!\!\!\perp Y \mid Z)_P$.

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$(X \perp\!\!\!\perp Y \mid Z)_G \implies (X \perp\!\!\!\perp Y \mid Z)_P$ *in every distribution* P *compatible with* G*.*

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If $\neg(X \perp\!\!\!\perp Y \mid Z)_G$*, then there exists a distribution* P *compatible with* G *in which* $\neg(X \perp\!\!\!\perp Y \mid Z)_P$*.*

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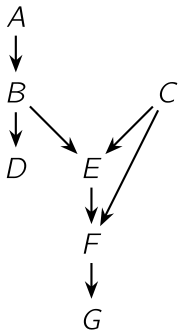

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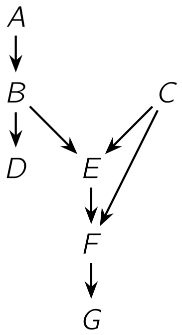

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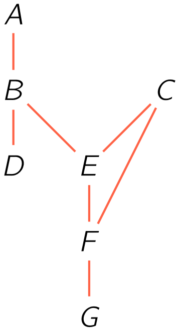

- The skeleton of a graph G is an undirected graph with the same adjacencies as in G.

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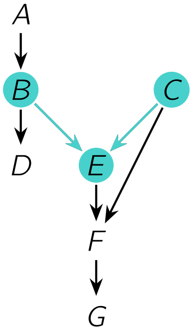

- The skeleton of a graph G is an undirected graph with the same adjacencies as in G.
- An immorality is a collider $X \to Y \leftarrow Z$ in which X and Z are not adjacent.

Markov equivalence

Which graphs have the same d-separations?

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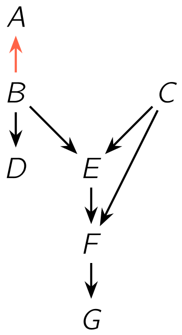
G_1 and G_2 have the same d-separations if and only if they have the same skeleton and the same immoralities.



- The skeleton of a graph G is an undirected graph with the same adjacencies as in G.
- An immorality is a collider $X \to Y \leftarrow Z$ in which X and Z are not adjacent.
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Markov equivalence

Which graphs have the same d-separations?

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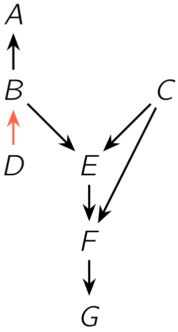

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Proving Markov equivalence

We need a preliminary lemma

Lemma

If X_i and X_j are not adjacent in G, then $(X_i \perp\!\!\!\perp X_j \mid \mathrm{Pa}_i, \mathrm{Pa}_j)_G$.

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Proof.

On board... □

Proving Markov equivalence, continued

Lemma

$\mathcal{I}_{\textsf{d-sep}}(G_1) = \mathcal{I}_{\textsf{d-sep}}(G_2) \implies G_1$ and G_2 have the same skeleton and immoralities.

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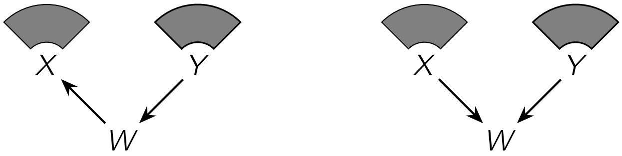

Tight active trails

An active trail is tight if...

Proposition

If X and Y are d-connected by Z, there is a tight active trail witnessing the connection.

Tight active trails, continued

Lemma

Let $T = (X = X_1 \circ\!\!-\!\!\circ \cdots \circ\!\!-\!\!\circ X_k = Y)$ be a tight active trail with observation set Z. Then for $i = 2, \ldots, k-1$, if X_{i-1} is adjacent to X_{i+1}, then $X_{i-1} \leftarrow X_i \rightarrow X_{i+1}$ and at least one of X_{i-1} or X_{i+1} is a collider in T.

Corollary

If X_i is a collider in T, then $X_{i-1} \rightarrow X_i \leftarrow X_{i+1}$ is an immorality in G.

Proving Markov equivalence, continued

Lemma

If G_1 and G_2 with common vertex set V have the same skeleton and immoralities then $\mathcal{I}_{\text{d-sep}}(G_1) = \mathcal{I}_{\text{d-sep}}(G_2)$.

Proof.

On board... □

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d-Separation and conditional independence

Theorem

- *Completeness* *If* $\neg(X \perp\!\!\!\perp Y \mid Z)_G$ *then there exists a distribution* P *compatible with* G *such that* $\neg(X \perp\!\!\!\perp Y \mid Z)_P$.
- *Soundness* *If* $(X \perp\!\!\!\perp Y \mid Z)_G$ *then* $(X \perp\!\!\!\perp Y \mid Z)_P$ *in any distribution* P *compatible with* G.

Proof.

On board... □

Completeness of d-separation

Lemma

If $\neg(X \perp\!\!\!\perp Y \mid Z)_G$ then there exists a distribution P compatible with G such that $\neg(X \perp\!\!\!\perp Y \mid Z)_P$.

Proof.

Let $T = (X = V_1 \multimap\!\!\circ \cdots \multimap\!\!\circ V_k = Y)$ be an active path given Z.

Continued on board... □

Soundness of d-separation

Lemma

If $(X \perp\!\!\!\perp Y \mid Z)_G$ then $(X \perp\!\!\!\perp Y \mid Z)_P$ in any distribution P compatible with G.

Soundness of d-separation

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If $(X \perp\!\!\!\perp Y \mid Z)_G$ then $(X \perp\!\!\!\perp Y \mid Z)_P$ in any distribution P compatible with G.

Proof.

Let $(X \perp\!\!\!\perp Y \mid Z)_G$.

- Let $Z_1, \ldots, Z_k$ be a topological order of Z.
- Define $Z(j) \coloneqq \left\{Z_1, \ldots, Z_j\right\}$.

Continued... □

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Continued... □

We complete a DAG G by picking a topological order and adding all edges consistent with the order.

The modification procedure

We'll define a sequence of graphs: $G_0, G_1, \ldots, G_k$.

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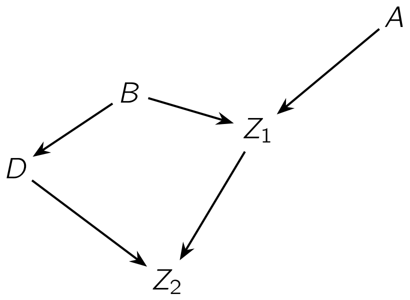

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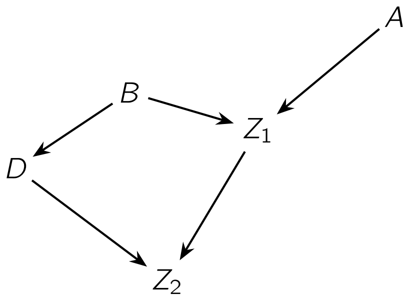

G_0

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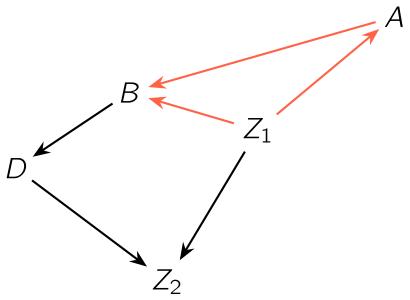

G_1

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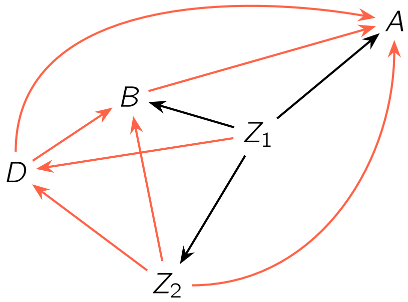

G_2

Soundness of d-separation

Proposition

In G_j:

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3. G_j is acyclic.

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5. $(X \perp\!\!\!\perp Y \mid Z)_{G_j} \iff (X \perp\!\!\!\perp Y \mid Z)_G$.

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Now we can finish the proof!

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Back to causal models

Recall: We model interventions in a causal model by swapping the mechanism used to set X with a constant function of our choice.

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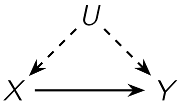

$$P(x, y) = \sum_u P(x \mid u) P(y \mid x, u) P(u)$$

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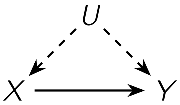

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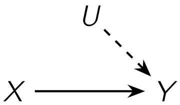

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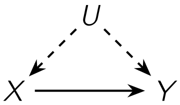

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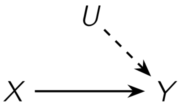

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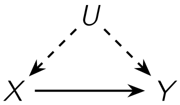

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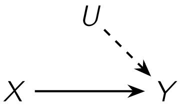

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$$P(Y \mid \mathsf{do}(x)) \coloneqq P_x(Y).$$

The graph induced by $\mathsf{do}(x)$ is $G_{\overline{X}}$, obtained by removing all edges from $\mathsf{Pa}(X)$ to X.

The do-calculus

Rules for manipulating interventional distributions.

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We can use d-separation to reason about interventional distributions!

Rule 1: Insertion/deletion of observations

Theorem (Insertion/deletion of observations)

$$P(y \mid \mathrm{do}(x), z, w) = P(y \mid \mathrm{do}(x), w)$$

if $(Y \perp\!\!\!\perp Z \mid X, W)_{G_{\overline{X}}}$.

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Proof.

$(Y \perp\!\!\!\perp Z \mid X, W)_{G_{\overline{X}}} \implies (Y \perp\!\!\!\perp Z \mid X, W)_{P_x}$ since P_x is compatible with $G_{\overline{X}}$. □

Rule 2: Action/observation exchange

Theorem (Action/observation exchange)

Let $X, Y, Z, W \subseteq V$ be disjoint. Then

$$P(y \mid \mathrm{do}(x), \mathrm{do}(z), w) = P(y \mid \mathrm{do}(x), z, w)$$

if $(Y \perp\!\!\!\perp Z \mid X, W)_{G_{\overline{X}\underline{Z}}}$.

Lemma

Let $H = G_{\overline{X}\underline{Z}}$. Then

$$(Y \perp\!\!\!\perp Z \mid X, W)_H \iff (\hat{Z} \perp\!\!\!\perp Y \mid X, Z, W)_{\mathbf{Aug}(H,Z)}.$$

Rule 3: Insertion/deletion of actions

Theorem (Insertion/deletion of actions)

$$P(y \mid \mathrm{do}(x), \mathrm{do}(z), W) = P(y \mid \mathrm{do}(x), w)$$

if $(Y \perp\!\!\!\perp Z | X, W)_{G_{\overline{XZ(W)}}}$, *where* $Z(W) := Z \setminus \mathrm{An}_{G_{\overline{X}}}(W)$.

Lemma

Any trail in $\mathrm{Aug}(G_{\overline{X}}, Z)$ that is active given X, W and uses only edges present in $G_{\overline{XZ(W)}}$ is also active in $G_{\overline{XZ(W)}}$ given X, W, where $Z(W) = Z \setminus \mathrm{An}_{G_{\overline{X}}}(W)$.

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Identifiability

Which causal effects can be determined from the observed variables only?

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Definition (Identifiability)

The causal effect of an intervention $\mathsf{do}(x)$ on a set of variables $Y \subseteq V$ (for $Y \subseteq V \setminus X$) is *identifiable* from P in a DAG G if $P_x(y)$ is uniquely computable from $P(V)$ in any causal model that induces G.

The ID algorithm theorem

Theorem (Shpitser-Pearl)

The algorithm **ID** *will return an expression for* $P_x(Y)$ *whenever it is identifiable from a graph* G*, and will return a witness to nonidentifiability whenever* $P_x(Y)$ *is not identifiable.*

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Every line of the algorithm is an application of a rule of the do-calculus!

The ID algorithm

function $\mathbf{ID}(\mathbf{y}, \mathbf{x}, P, G)$

1: if $\mathbf{x} = \varnothing$, return $\sum_{\mathrm{v} \setminus \mathrm{y}} P(\mathbf{v})$.

2: if $\mathbf{V} \neq \mathrm{An}(\mathbf{Y})_G$,
return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap \mathrm{An}(\mathbf{Y})_G, P(\mathrm{An}(\mathbf{Y})), \mathrm{An}(\mathbf{Y})_G)$.

3: let $\mathbf{W} = (\mathbf{V} \setminus \mathbf{X}) \setminus \mathrm{An}(\mathbf{Y})_{G_{\overline{\mathrm{X}}}}$.
if $\mathbf{W} \neq \varnothing$, return $\mathbf{ID}(\mathbf{y}, \mathbf{x} \cup \mathbf{w}, P, G)$.

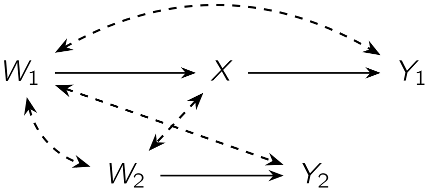
4: if $C(G \setminus \mathbf{X}) = \{S_1, \ldots, S_k\}$ (for $k \geq 2$),
return $\sum_{v \setminus (\mathrm{y} \cup \mathrm{x})} \prod_i \mathbf{ID}(s_i, \mathbf{v} \setminus s_i, P, G)$.
else if $C(G \setminus \mathbf{X}) = \{S\}$,

5: if $C(G) = \{G\}$, throw $\mathbf{FAIL}(G, S)$.

6: if $S \in C(G)$, return $\sum_{s \setminus \mathrm{y}} \prod_{V_i \in S} P(v_i \mid v_\pi^{(i-1)})$.

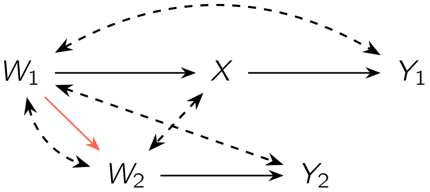
7: if $\exists S', S \subseteq S' \in C(G)$,
return
$\mathbf{ID}(\mathbf{y}, \mathbf{x} \cap S', \prod_{V_i \in S'} P(V_i \mid V_\pi^{(i-1)} \cap S', v_\pi^{(i-1)} \setminus S', S')$.

Two examples



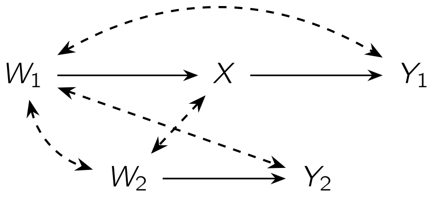
Is $P_x(y_1, y_2)$ identifiable?

Two examples



Is $P_x(y_1, y_2)$ identifiable? How about now?

A positive example



$$P_x(y_1, y_2) = \sum_{w_2} \left(\sum_{w_1} P(y_1|w_1, x)P(w_1) \right) P(y_2|w_2)P(w_2).$$

Hedges

Definition (C-component)

Let G be a semi-Markovian graph such that a subset of its bidirected edges form a spanning tree of V. Then G is a *C-component (confounded component)*.

Definition (Decomposition into C-components)

Any graph can be uniquely partitioned into a collection of subgraphs $C(G)$, each of which is a maximal C-component. (If G is itself a C-component, the partition is trivial.)

Definition (C-forest)

Let Y be the set of all sinks in a semi-Markovian graph G. Then G is a Y-rooted C-forest if G is a C-component and all vertices have at most one child.

Hedges and identifiability

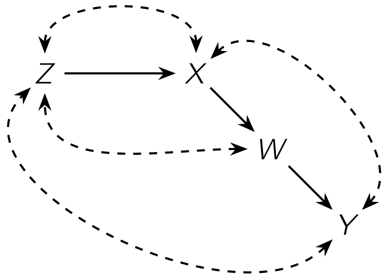
Definition (Hedge)

Let $X, Y \subseteq V$ in a graph G. Let F, F' be R-rooted C-forests such that $F \cap X \neq \varnothing$, $F' \cap X = \varnothing$, $F' \subseteq F$, and $R \subseteq \mathrm{An}(Y)_{G_{\overline{X}}}$. Then (F, F') form a *hedge* for $P_x(y)$ in G.

Theorem (Hedge Criterion for Identifiability)

$P_x(y)$ is identifiable if and only if there does not exists a hedge for $P_{x'}(y')$ in G for any $X' \subseteq X$, $Y' \subseteq Y$.

Hedges



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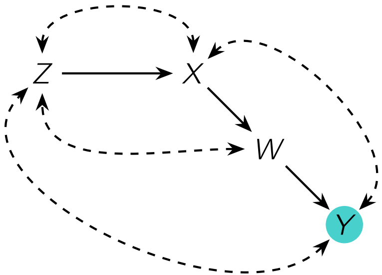
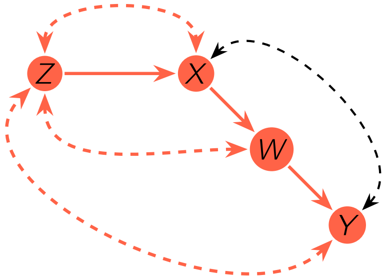
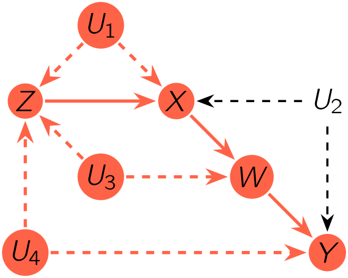


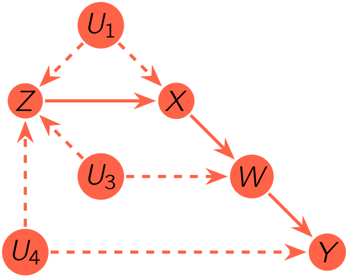
Hedges



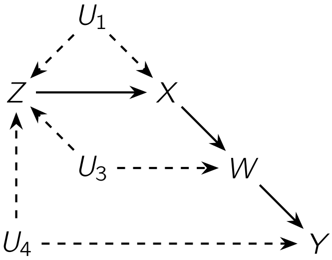
Hedges



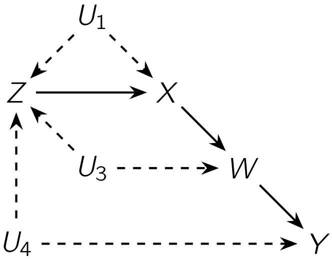
Hedges



Non-identifiability in hedges



Non-identifiability in hedges



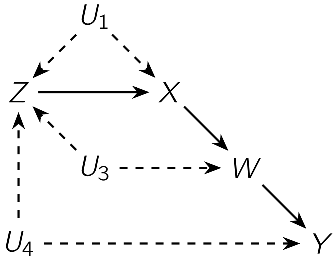
$\mathbf{M}^1$:		$\mathbf{M}^2$:	
U_i	$:= \text{Unif}(\{0,1\})$	U_i	$:= \text{Unif}(\{0,1\})$
Z	$:= U_1 \oplus U_3 \oplus U_4$	Z	$:= U_1 \oplus U_3 \oplus U_4$
X	$:= Z \oplus U_1$	X	$:= Z \oplus U_1$
W	$:= X \oplus U_3$	W	$:= X \oplus U_3$
Y	$:= W \oplus U_4$	Y	$:= 0$

Non-identifiability in hedges

In M_1 we also have $P^1(Y=0)=1$:

$$\begin{aligned} Y &= W \oplus U_4 \\ &= (X \oplus U_3) \oplus U_4 \\ &= (Z \oplus U_1) \oplus U_3 \oplus U_4 \\ &= (U_1 \oplus U_3 \oplus U_4) \oplus (U_1 \oplus U_3 \oplus U_4) \\ &= 0 \end{aligned}$$

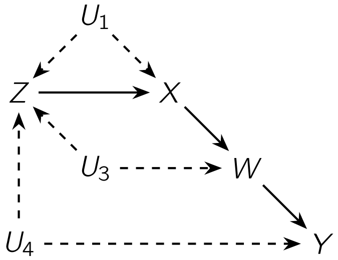
so $P^1(V) = P^2(V)$.



$\mathbf{M}^1$:		$\mathbf{M}^2$:	
U_i	$:= \text{Unif}(\{0,1\})$	U_i	$:= \text{Unif}(\{0,1\})$
Z	$:= U_1 \oplus U_3 \oplus U_4$	Z	$:= U_1 \oplus U_3 \oplus U_4$
X	$:= Z \oplus U_1$	X	$:= Z \oplus U_1$
W	$:= X \oplus U_3$	W	$:= X \oplus U_3$
Y	$:= W \oplus U_4$	Y	$:= 0$

Non-identifiability in hedges

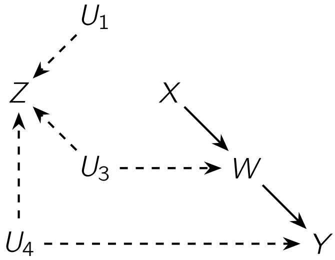
What happens when we intervene on X?



$$
\begin{array}{lll}
\mathbf{M}^1: & & \\
U_i & := \text{Unif}(\{0,1\}) \\
Z & := U_1 \oplus U_3 \oplus U_4 \\
X & := Z \oplus U_1 \\
W & := X \oplus U_3 \\
Y & := W \oplus U_4
\end{array}
\qquad
\begin{array}{lll}
\mathbf{M}^2: & & \\
U_i & := \text{Unif}(\{0,1\}) \\
Z & := U_1 \oplus U_3 \oplus U_4 \\
X & := Z \oplus U_1 \\
W & := X \oplus U_3 \\
Y & := 0
\end{array}
$$

Non-identifiability in hedges

What happens when we intervene on X?

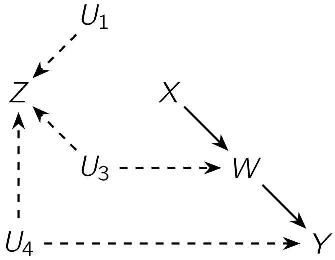


$\mathbf{M}^1$:		$\mathbf{M}^2$:	
U_i	$:= \text{Unif}(\{0,1\})$	U_i	$:= \text{Unif}(\{0,1\})$
Z	$:= U_1 \oplus U_3 \oplus U_4$	Z	$:= U_1 \oplus U_3 \oplus U_4$
X	$:= x$	X	$:= x$
W	$:= X \oplus U_3$	W	$:= X \oplus U_3$
Y	$:= W \oplus U_4$	Y	$:= 0$

Non-identifiability in hedges

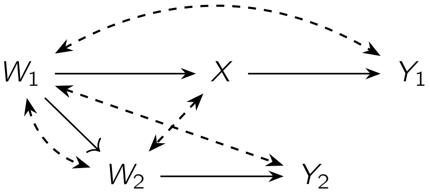
Then $Y = x \oplus U_3 \oplus U_4$. We have

$$P^1_x(Y) > 0, \quad P^2_x(Y = 1) = 0.$$

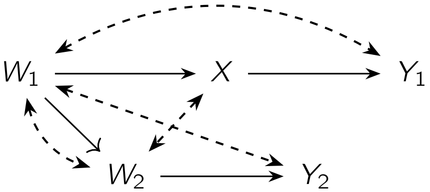


$\mathbf{M}^1$:		$\mathbf{M}^2$:	
U_i	$:= \text{Unif}(\{0, 1\})$	U_i	$:= \text{Unif}(\{0, 1\})$
Z	$:= U_1 \oplus U_3 \oplus U_4$	Z	$:= U_1 \oplus U_3 \oplus U_4$
X	$:= x$	X	$:= x$
W	$:= X \oplus U_3$	W	$:= X \oplus U_3$
Y	$:= W \oplus U_4$	Y	$:= 0$

Non-identifiability for the earlier example



Non-identifiability for the earlier example



In this example, $P_x(y_1, y_2)$ is unidentifiable because $\{W_1, W_2, Y_1, Y_2\}$ and $\{W_1, W_2, Y_1, Y_2, X\}$ form a hedge.