The Planted Matching Problem: Sharp Threshold and Infinite-order Phase Transition

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Duke $\Longrightarrow$ Simons $\Longrightarrow$ Cornell

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The Planted Matching Problem

Joint work with **Jian Ding** [1], **Yihong Wu** [2] and **Jiaming Xu** [3].

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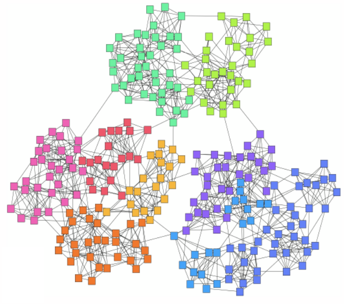
[3] The Fuqua School of Business, Duke University

Planted models and recovery

Model : planted structure + noise.
Question : When/how can one recover the planted structure from its noisy observation ?



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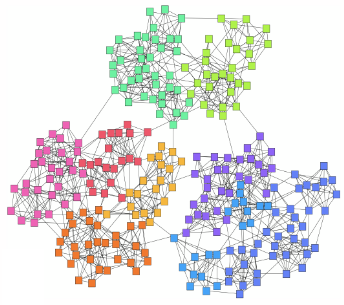
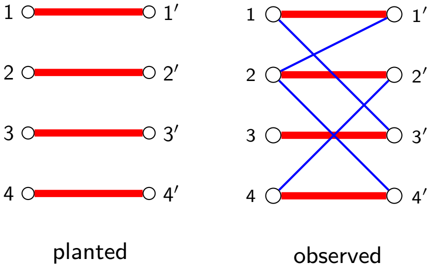


Examples

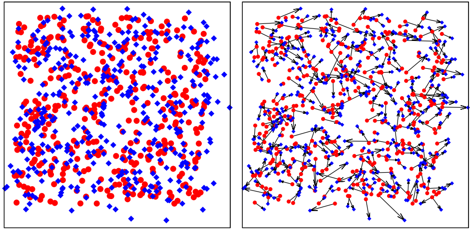
- Recovery of planted clique in Erdős-Rényi graphs.
- Community detection under the Stochastic Block Model.

The planted matching problem

Planted structure : perfect matching in bipartite graph.



Motivating application : particle tracking



[Chertkov-Kroc-Krzakala-Vergassola-Zdeborová PNAS'10]

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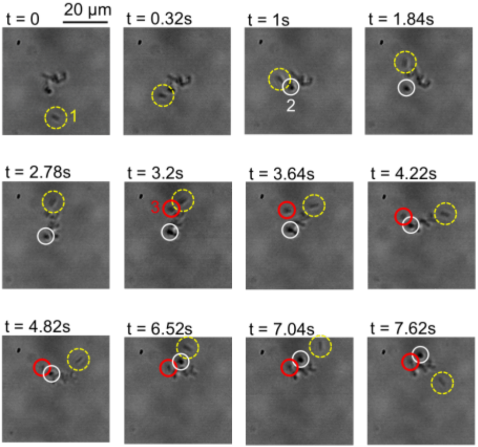

[Sinibaldi-Iebba-Chinappi MicrobiologyOpen'18]

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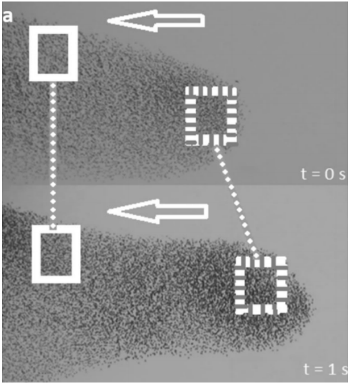

[Hemelrijk-Costanzo-Hildenbrandt-Carere Behavioral Ecology and Sociobiology'19]

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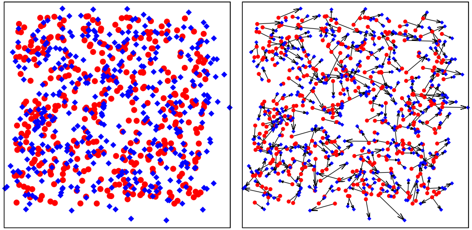

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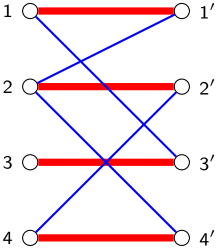
Planted matching model



- Planted matching M^* uniformly distributed in all perfect matchings.
- Edges not in M^* appear independently w.p. $\frac{d}{n}$.
- Edge weight

$$W_e \overset{\text{ind.}}{\sim} \begin{cases} P & e \in M^* \\ Q & e \notin M^* \end{cases}$$

Main results

Theorem (Ding-Wu-Xu-Y '21)

Sharp threshold for almost perfect recovery :

- If $\sqrt{d}B(\mathcal{P},\mathcal{Q}) \leq 1$, then some $\widehat{M}$ achieves

$$\frac{1}{n}\mathbb{E}\left|\widehat{M}\triangle M^*\right| \to 0 \text{ as } n \to \infty.$$

- If $\sqrt{d}B(\mathcal{P},\mathcal{Q}) \geq 1+\epsilon$ for $\epsilon > 0$, then for all $\widehat{M}$ and some constant c,

$$\frac{1}{n}\mathbb{E}\left|\widehat{M}\triangle M^*\right| \geq c.$$

Bhattacharyya coefficient (Hellinger affinity) $B(\mathcal{P},\mathcal{Q}) \triangleq \int \sqrt{d\mathcal{P}d\mathcal{Q}}$.

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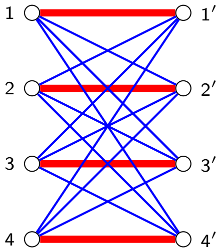
- Result works for both sparse (d bounded) and dense ($d \to \infty$) graphs.
- In the dense $d \to \infty$ regime, need certain scaling assumptions on Q.

Examples

Exponential model (dense regime)[Maharrami-Moore-Xu '19, Semerjian-Sicuro-Zdeborová '20] :

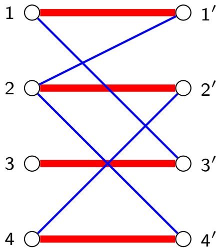
- $d = n$, $P = \text{Exp}(\lambda)$, $Q = \text{Exp}(1/n)$.
- Sharp threshold : $\lambda = 4$.



Examples

Unweighted model (sparse regime) :

- d =const, $P = Q$.
- The edge weights do not offer any information for recovery.
- Sharp threshold : $d = 1$.



Analysis (unweighted model)

1. Establish positive direction ($d \leq 1$) by analyzing the maximum likelihood estimator;
2. Establish negative direction ($d \geq 1 + \epsilon$) by analyzing the posterior distribution.

Analysis for $d \leq 1$

1. Analyze the MLE :

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2. First moment analysis :

$$\mathbb{P}\left\{\left|\widehat{M}_{\mathrm{ML}} \triangle M^*\right| \geq \beta n\right\}$$
$$\leq \mathbb{P}\left\{\exists \text{ perfect matching } M \text{ in } G,\ s.t.\ |M \triangle M^*| \geq \beta n\right\}$$
$$\leq \sum_{t \geq \beta n/2} \binom{n}{t} t! \left(\frac{d}{n}\right)^t$$
$$\to 0 \text{ for some } \beta = o(1) \text{ when } d \leq 1.$$

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Crucial observations

1. Sampling from the posterior distribution is optimal within a factor of two.

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Proof : Let $\widetilde{M}$ be sampled from the posterior distribution, then for any estimator $\widehat{M}$,

$$\mathbb{E}\left|\widetilde{M} \triangle M^*\right| \leq \mathbb{E}\left|\widetilde{M} \triangle \widehat{M}\right| + \mathbb{E}\left|\widetilde{M} \triangle M^*\right| = 2\mathbb{E}\left|\widehat{M} \triangle M^*\right|,$$

where the equality is because $\mathcal{L}(G, \widetilde{M}) = \mathcal{L}(G, M^*)$.

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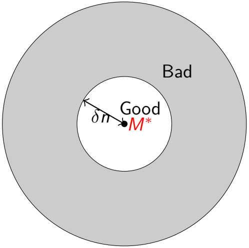

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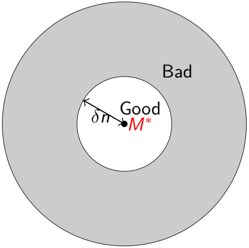

Claim : number of "bad" solutions is $\exp(\Omega(n))$.

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3. For all perfect matching M, $M \triangle M^*$ consists of a disjoint union of alternating cycles.

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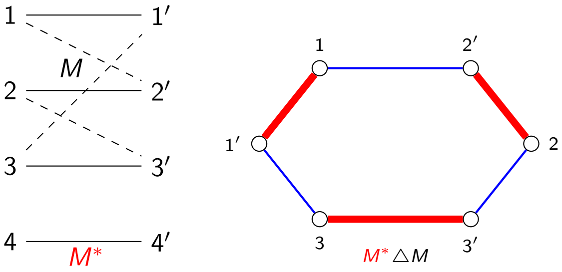

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Conclusion : It suffices to show the existence of $e^{\Omega(n)}$ distinct alternating cycles in G of length $\Omega(n)$.

Existence of many long alternating cycles

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- However, $\mathbb{E}(|S|^2) \gg (\mathbb{E}|S|)^2$ due to the excessive correlation between long cycles.

Key idea : First find many disjoint short paths, then connect the paths into long cycles [Aldous '98, Ding-Goswami '15, ...].

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Reserve a set V of γn vertices for some small $\gamma > 0$.

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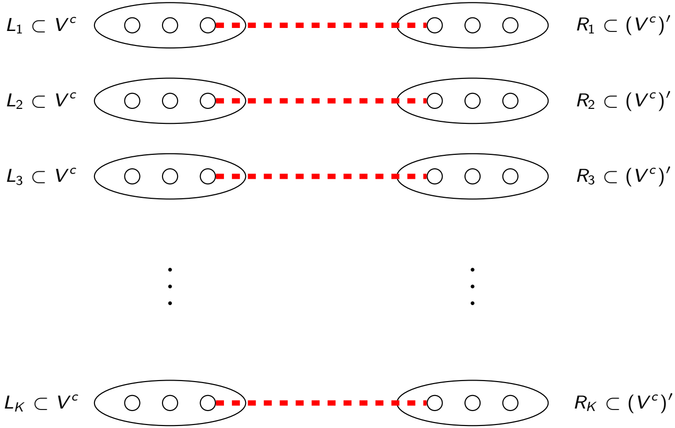

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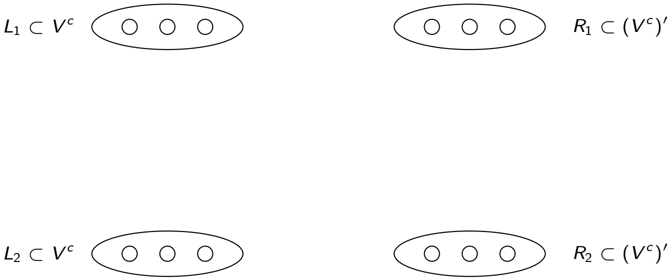
Stage 2 (sprinkling) :



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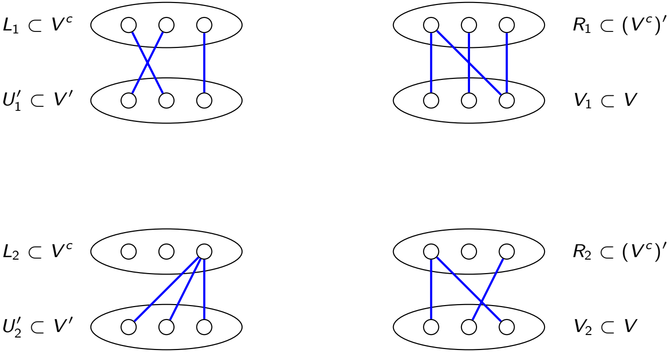

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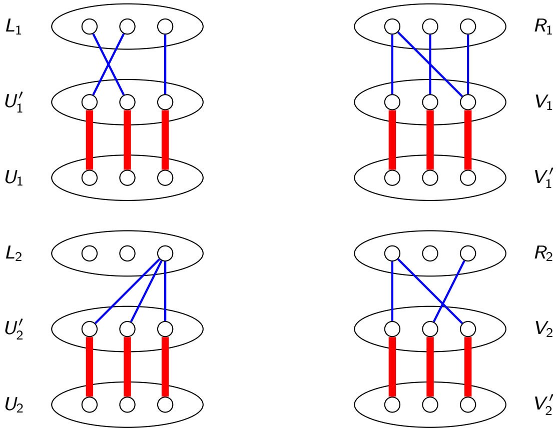

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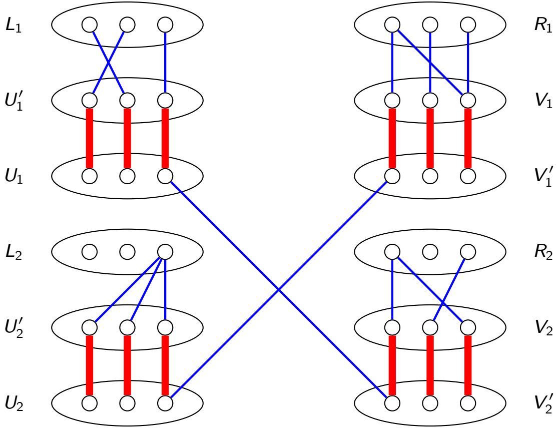

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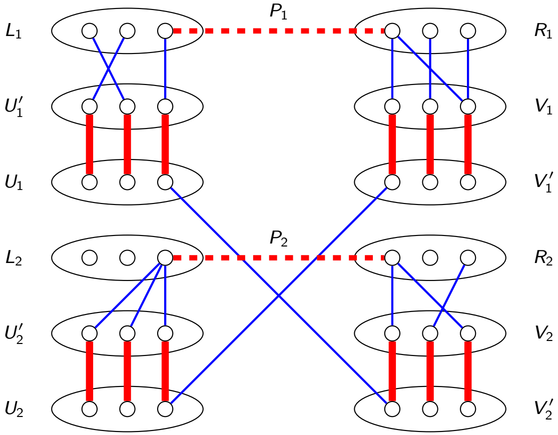
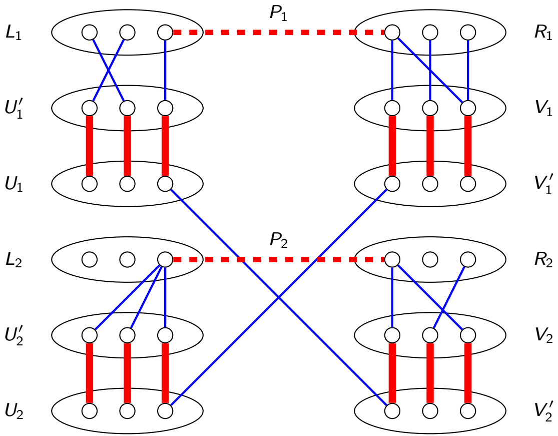
2. Look for blue edges connecting $\{U_k\}_{k \le K}, \{V'_k\}_{k \le K}$.



Two-stage cycle-finding scheme

Super graph : Define G_{super} on $[K] \times [K]'$, such that (k, k') is a red edge for all k, and (i, j') is a blue edge iff U_i and V_j' is connected by at least one blue edge.



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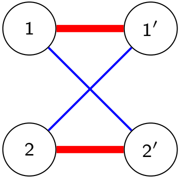

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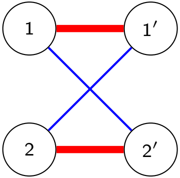

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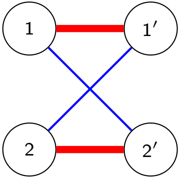

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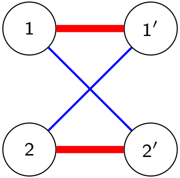

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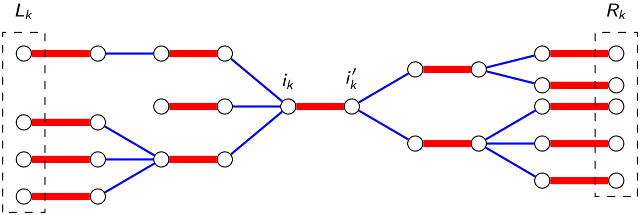
3. G_{super} contains $e^{\Omega(n)}$ alternating cycles of length $\Omega(n)$.

Path construction via breadth-first search

Two-sided tree (“bush") :



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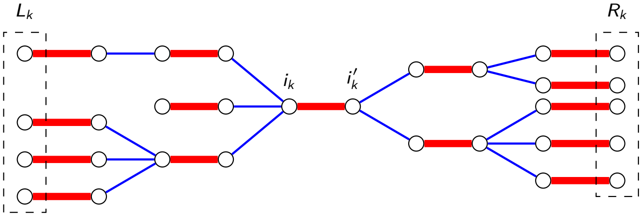

When $d \geq 1 + \epsilon$, the branching processes survive with constant probability.

Exponential model

Unplanted version (Random assignment model) [Mézard-Parisi 87', Aldous 01', ...] :

- Observe W_e on the complete bipartite graph, where $W_e \stackrel{i.i.d.}{\sim} \exp(1)$.
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- Minimum weight matching achieves almost perfect recovery iff $\lambda \geq 4$[Maharrami-Moore-Xu '19].
- Sharp threshold : $\lambda = 4$.
- For $\lambda = 4 - \epsilon$,

$$e^{-c_1/\sqrt{\epsilon}} \leq \inf_{\widehat{M}} \frac{1}{n}\mathbb{E}\left|\widehat{M}\triangle M^*\right| \leq e^{-c_2/\sqrt{\epsilon}},$$

  revealing an infinite-order phase transition, conjectured in [Semerjian-Sicuro-Zdeborová '20].
- For the lower bound proof, we resort to ~~"bushes"~~ paths.

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2. Error characterization in entire parameter range ?
3. Extension to planted k-factor model ?
   Conjecture[Sicuro-Zdeborová '20] : $\sqrt{kd}B(\mathcal{P}, \mathcal{Q}) = 1$.

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1. Phase transition for general distributions ?
2. Error characterization in entire parameter range ?
3. Extension to planted k-factor model ?
   Conjecture[Sicuro-Zdeborová '20] : $\sqrt{kd}B(\mathcal{P}, \mathcal{Q}) = 1$.

Thank you !

General model

Planted matching model (general)

True matching $M^* \sim \text{Unif}\{\text{all perfect matchings on } [n] \times [n]'\}$. Observed graph G contains all the edges in M^*, and for each $e \notin M^*$, $e \in G$ independently with probability d/n. Observe $(W_e)_{e \in G}$, where

$$W_e \overset{\text{indep.}}{\sim} \begin{cases} \mathcal{P} & \text{if } e \in M^* \\ \mathcal{Q} & \text{if } e \notin M^*. \end{cases}$$

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Sharp threshold : $\sqrt{d} B(\mathcal{P}, \mathcal{Q}) = 1$.

Proof of the positive result

1. Analyze the MLE :

$$\widehat{M}_{\mathrm{ML}} \in \{\text{perfect matchings in } G\}\,.$$

2. Union bound :

$$\begin{aligned}
&\mathbb{P}\left\{\left|\widehat{M}_{\mathrm{ML}} \triangle M^*\right| \geq \beta n\right\} \\
\leq &\mathbb{P}\left\{\exists \text{ perfect matching } M \text{ in } G,\ s.t.\ |M \triangle M^*| \geq \beta n\right\} \\
\leq &\sum_{t \geq \beta n/2} \binom{n}{t} t! \left(\frac{d}{n}\right)^t \\
\to &\ 0 \text{ for some } \beta = o(1) \text{ when } d \leq 1.
\end{aligned}$$

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$\to 0$ for some $\beta = o(1)$ when $\sqrt{d} B(\mathcal{P}, \mathcal{Q}) \leq 1$.

Proof of the negative result

Crucial observations

1. Sampling from the posterior distribution is close to optimal.
2. The posterior distribution is uniform over the set of all perfect matchings in G.
3. For all perfect matching M, $M \triangle M^*$ consists of a disjoint union of alternating cycles.

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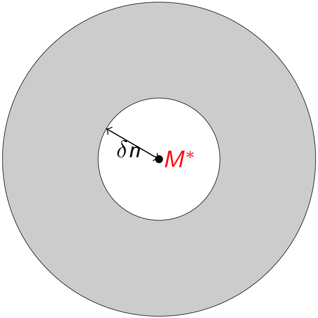
Analysis of the posterior distribution



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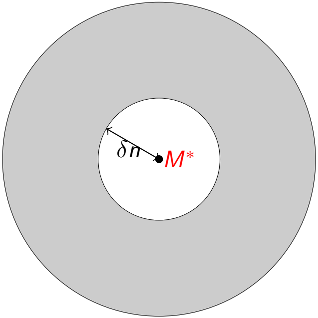

Additional challenges for weighted graphs :

- Control the posterior mass of matchings close to M^*.

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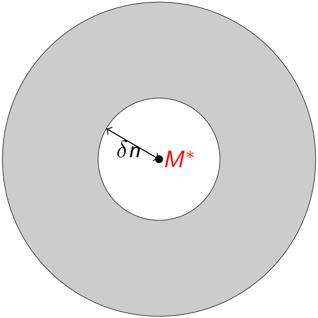


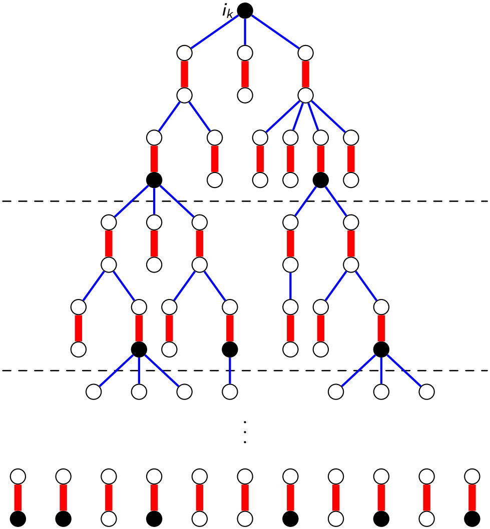
Additional challenges for weighted graphs :

- Control the posterior mass of matchings close to M^*.
- Find exponentially many long alternating cycles C that are augmenting :

$$\sum_{e \in E_{\text{blue}}(C)} \log \frac{\mathcal{P}}{\mathcal{Q}}(W_e) \geq \sum_{e \in E_{\text{red}}(C)} \log \frac{\mathcal{P}}{\mathcal{Q}}(W_e).$$

Exploration + selection

Construction of augmenting paths :



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Path construction (exponential model)

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2. Extract a large collection of vertex-disjoint paths via Turán's Theorem [Ding-Goswami '15].